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RESEARCH ARTICLE

On Automorphisms of Transformation Semigroups

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This paper addresses the problem of describing automorphisms of semigroups of transformations. In [2] we were involved in characterizing all automorphisms of Croisot-Teissier semigroups. The semigroups of transformations that belong to this large family generally consist of many-to-one transformations whose restrictions to range sets are one-to-one. Here we consider enlargements of Croisot-Teissier semigroups whose elements, restricted to range-sets, are no longer one-to-one. We show that such semigroups contain a maximal Croisot-Teissier semigroup, which in turn is used to present a complete description of automorphisms of these semigroups. Moreover we describe the Green's relations on these enlargements of Croisot-Teissier semigroups, and show that they are in fact simple semigroups, whose regular elements form a bisimple subsemigroup. We start by recalling the definition of Croisot-Teissier semigroups.

Let p and q be infinite cardinals with $p \geq q$, and let X be a set with $|X| \geq p$. Let $\mathcal{E} = \{\mathcal{E}_i \mid i \in I\}$ be a set of distinct equivalences on X such that $|X/\mathcal{E}_i| = p$ for all $i \in I$. A subset A of X is said to be well-separated (w.s.) by $\mathcal{E}$ if $|A| = p$ and $\mathcal{E}_i \cap (A \times A)$ is the identity relation on A for all $i \in I$. For a cardinal t, with $q \leq t \leq p$, let $\mathcal{C}_t = \{\text{w.s. } A \mid \text{for some w.s. } B,\ A \subseteq B \text{ and } |B - A| = t\}$. When X contains a w.s. set, the *Croisot-Teissier semigroup on* $X, \mathcal{E}$ *of type* (p, q) is $CT(X, \mathcal{E}, p, q) = \{f : X \to X \mid \pi(f) \in \mathcal{E},\ R(f) \in \mathcal{C}_q\}$ with the operation of function composition [1]. Recall that for a transformation f of X, $R(f) = f(X)$ denotes the range of f, and $\pi(f)$ denotes the partition of X determined by f such that x and y are in the same class of $\pi(f)$ if and only if $f(x) = f(y)$.

A Croisot-Teissier semigroup $CT(X, \mathcal{E}, p, q)$ is idempotent-free and either simple (when $p = q$) or has a minimal ideal $CT(X, \mathcal{E}, p, p)$ that itself is a Croisot-Teissier semigroup. A simple Croisot-Teissier semigroup $CT(X, \mathcal{E}, p, p)$ is the disjoint union of its minimal left ideals, and any simple idempotent-free semigroup with a minimal left ideal can be embedded in a simple Croisot-Teissier semigroup $CT(X, \mathcal{E}, p, p)$. The Green's relations on these semigroups were described in [3], and their congruences were studied in [4], [5], [6], [7] and [8].

We construct the following generalization of a Croisot-Teissier semigroup. In view of the intimate connection between equivalences on X and partitions of X we write $[x] \in \mathcal{B}$ to indicate that $[x]$ is the equivalence class of the equivalence $\mathcal{B}$ containing $x \in X$. Given an infinite cardinal $r \leq p$, and an equivalence $\mathcal{A}$ on X let $\mathcal{A}^{(r)}$ be the family of all equivalences $\mathcal{B}$ on X such that $\mathcal{A} \subseteq \mathcal{B}$ and for every $[x] \in \mathcal{B}$, $|[x]/\mathcal{A}| < r$. Informally, such a $\mathcal{B}$ in $\mathcal{A}^{(r)}$ is formed by glueing together classes of $\mathcal{A}$, with each class in $\mathcal{B}$ made up of fewer than r classes of $\mathcal{A}$. The family $\mathcal{A}^{(r)}$ is referred to as the *family of* r *glueings of* $\mathcal{A}$. Let $\mathcal{E}^{(r)} = \cup\{\mathcal{E}_i^{(r)} \mid i \in I\}$ be the family of r glueings of $\mathcal{E}$ and

$$S = \{f : X \to X \mid R(f) \in \mathcal{C}_q \text{ and } \pi(f) \in \mathcal{E}^{(r)}\} .$$

The above semigroup S contains a maximal Croisot-Teissier subsemigroup $S^{\#}$ that generally does not coincide with the original $CT(X, \mathcal{E}, p, q)$. Let $\mathcal{E}^{\#} = \{\mathcal{A} \in \mathcal{E}^{(r)} \mid \pi(t) = \pi(ft), \text{for all } f, t \in S \text{ with } \pi(f) = \mathcal{A}\}$ and let $S^{\#} = CT(X, \mathcal{E}^{\#}, p, q)$. We show that $S^{\#}$ is a subsemigroup of S containing $CT(X, \mathcal{E}, p, q)$. Let $\mathcal{C}_q^{\#}$ be the set of ranges of all the mappings in $S^{\#}$. If $A \in \mathcal{C}_q$ and $\mathcal{B} \in \mathcal{E}^{\#}$ then $(A \times A) \cap \mathcal{B} = i_A$, else for $f,\ t \in S$ with $\pi(f) = \mathcal{B}$ and $R(t) = A$, $\pi(ft)$ and $\pi(t)$ are distinct, a contradiction. Therefore $\mathcal{C}_q$ is a subset of $\mathcal{C}_q^{\#}$. Moreover since $\mathcal{E}$ is a subset of $\mathcal{E}^{\#}$, $\mathcal{C}_q^{\#}$ is a subset of $\mathcal{C}_q$, and so the next result follows from the above and the observation that for any f and g in a Croisot-Teissier semigroup, $\pi(fg) = \pi(g)$.

Proposition 1. *$S^{\#}$ is a maximal Croisot-Teissier subsemigroup of S.* ■

In the following example we start with a specific Croisot-Teissier semigroup and construct the associated $\mathcal{E}^{(r)}$ and $\mathcal{E}^{\#}$. The example is based on [2, Example 4.2].

Example. Let $\mathbb{R}$ be the set of all real numbers, and $\mathbb{R}^+$ be the set of all positive reals. For each $a \in \mathbb{R}^+$ let $\mathcal{E}_a$ be the equivalence on $\mathbb{R}$ whose only non-singleton class is $[a] = \{a\} \cup (\mathbb{R} - \mathbb{R}^+)$. Let $\mathcal{E}_0$ be the equivalence on $\mathbb{R}$ having two non-singleton classes: $[-1] = \{b \in \mathbb{R} : -1 \leq b \leq 0\}$ and $[-2] = \{b \in \mathbb{R} : b < -1\}$. Let $\mathcal{E} = \{\mathcal{E}_b : b \in \mathbb{R},\ b \geq 0\}$, and $p = q = |\mathbb{R}|$. Note that the $\mathcal{C}_p$ sets are those subsets A of $\mathbb{R}^+$ having $|A| = |\mathbb{R}^+ - A|$, and that the semigroup $CT(\mathbb{R}, \mathcal{E}, p, p)$ consists of all transformations $f : \mathbb{R} \to \mathbb{R}$ having $\pi(f) \in \mathcal{E}$ and $R(f) \in \mathcal{C}_p$.

If $r = \aleph_0$, $\mathcal{E}^{(r)}$ is the set of all equivalences on $\mathbb{R}$ whose non-singleton classes are of the form $Y' \cup Y''$ where Y' is either $[-1], [-2], \mathbb{R} - \mathbb{R}^+$, or empty, and Y'' is either a finite subset of $\mathbb{R}^+$ or empty. Let $S = \{f : \mathbb{R} \to \mathbb{R} \mid R(f) \in \mathcal{C}_p,\ \pi(f) \in \mathcal{E}^{(r)}\}$. Since $\mathcal{E}^{\#}$ is just $\mathcal{E}$ together with all partitions in $\mathcal{E}^{(r)}$ for which every $\mathcal{C}_p$ set is a partial transversal, $\mathcal{E}^{\#}$ consists of all equivalences in $\mathcal{E}^{(r)}$ whose non-singleton classes are of the form $Y' \cup Y''$, where Y', Y'' are as above with $|Y''| \leq 1$. Thus we have that $CT(\mathbb{R}, \mathcal{E}, p, p) \subset S^{\#} = CT(X, \mathcal{E}^{\#}, p, p) \subset S$. ■

We show that the restriction $\phi^{\#}$ of an automorphism ϕ of S to $S^{\#}$ is a *range-preserving*, r *union-preserving* and r *glueing-preserving* automorphism of $S^{\#}$ (see Definitions 2,4, and 5 below), and that every such automorphism of $S^{\#}$ may be extended to an automorphism of S. Therefore using the characterization of automorphisms of Croisot-Teissier semigroups in [2] we are able to describe the automorphisms of S completely. The next definition was introduced in [2, p.228].

Definition 2. *An automorphism ϕ of a semigroup of transformations S is said to be* range-preserving *if for all $f, g \in S$, $R(f) \subseteq R(g)$ if and only if $R(\phi(f)) \subseteq R(\phi(g))$.*

The following decomposition of the union W of all well-separated sets, and the associated decomposition of the Croisot-Teissier semigroup into a union of its right ideals was first described in [2]. Here we present a brief account of these decompositions and some terminology introduced in [2], which we use to give a description of all range-preserving automorphisms of $S^{\#}$ and automorphisms of S.

Let K be an index set containing I such that $\mathcal{E}^{\#} = \{\mathcal{E}_i \mid i \in K\}$. A pair of $\mathcal{C}_q$ sets A and B are said to be *σ-related* if whenever A and B both meet a non-singleton class $[u]$ of $\delta = \cap\{\mathcal{E}_i : i \in K\}$ there exist $F_1 = A, F_2, \ldots, F_n = B \in \mathcal{C}_q$ such that $F_j \cap F_{j+1} \in \mathcal{C}_q$ and $F_j \cap [u] \neq \Phi$. Let $\{\mathcal{M}_\alpha \mid \alpha \in \Omega\}$ be the collection of all maximal families of σ-related $\mathcal{C}_q$ sets. For each $\alpha \in \Omega$ let $\mathcal{A}_\alpha = \cup\{A \mid A \in \mathcal{M}_\alpha\}$ and $\mathcal{I}_\alpha = \{f \in S^{\#} \mid R(f) \in \mathcal{M}_\alpha\}$, a right ideal of $S^{\#}$. A set $\{h_\alpha \mid \alpha \in \Omega\}$ of permutations of W is termed *compatible* if there exists a permutation k of W/ρ such that the equality of the ρ-classes $[h_\alpha(x)] = k([x])$ holds for all $\alpha \in \Omega$ and $x \in W$, k induces a permutation of the set $\{(\mathcal{E}_i|_{W\times W})/\rho : i \in K\}$ of the equivalences on W/ρ, and $h_\alpha f = h_\beta f$ for all $f \in \mathcal{I}_\alpha \cap \mathcal{I}_\beta$. For each $\mathcal{E}_i$ define $\mathcal{B}(\mathcal{E}_i) = \{[x] \in \mathcal{E}_i \mid [x] \cap W = \Phi\}$ and let $J = \{i \in K \mid \mathcal{B}(\mathcal{E}_i) \neq \Phi\}$. The following result describing range-preserving automorphisms of a Croisot-Teissier semigroup was proved in [2, Theorem 4.4]. The statement is in terms of the maximal Croisot-Teissier subsemigroup $S^{\#}$ of S.

Proposition 3. *Let $\phi^{\#}$ be a range-preserving automorphism of $S^{\#}$. There exists, uniquely,*

(i) a compatible set $\{h_\alpha \mid \alpha \in \Omega\}$ of permutations of W,

(ii) a permutation $z^{\#}$ of $\mathcal{E}^{\#}$ such that $z^{\#}(\mathcal{E}_i)|_W = h_\alpha(\mathcal{E}_i|_W)$ for any $\mathcal{E}_i \in \mathcal{E}^{\#}$ and $\alpha \in \Omega$, and

(iii) a family of bijections $\{y_i \mid i \in J\}$ where $y_i : \mathcal{B}(\mathcal{E}_i) \to \mathcal{B}(z^{\#}(\mathcal{E}_i))$, such that
  1) $\phi^{\#}(f)|_W = h_\alpha f h_\alpha^{-1}$ for all $f \in \mathcal{I}_\alpha$,
  2) $\pi(f) = z^{\#}(\pi(f))$, and
  3) $\phi^{\#}(f)(D) = h_\alpha f y_i^{-1}(D)$ for all $f \in \mathcal{I}_\alpha$ with $\pi(f) = \mathcal{E}_i$ and $D \in \mathcal{B}(z^{\#}(\mathcal{E}_i))$.

Conversely, given $S^{\#}$ and (i), (ii), and (iii), there exists a unique range-preserving automorphism $\phi^{\#}$ of $S^{\#}$ such that 1), 2), and 3) hold. ∎

Definition 4. *Given an automorphism $\phi^{\#}$ of $S^{\#}$ and an equivalence class A of $\mathcal{E}_i$ let $\tilde{A}$ be the equivalence class of $z^{\#}(\mathcal{E}_i)$ containing $h_\alpha(x)$, for some $\alpha \in \Omega$, if $x \in A \cap W \neq \Phi$, and $\tilde{A} = y_i(A)$ if $A \cap W$ is empty. An automorphism $\phi^{\#}$ of $S^{\#}$ is said to be r* union-preserving *if whenever $\mathcal{E}_i$, $\mathcal{E}_j \in \mathcal{E}^{\#}$ with $\mathcal{E}_i^{(r)} \cap \mathcal{E}_j^{(r)} \neq \Phi$ and $\mathcal{C}$, $\mathcal{D}$ are collections of fewer than r classes in $\mathcal{E}_i$ and $\mathcal{E}_j$ respectively, then $\cup\, \mathcal{C} = \cup\, \mathcal{D}$ if and only if $\cup\, \{\tilde{A} : A \in \mathcal{C}\} = \cup\, \{\tilde{B} : B \in \mathcal{D}\}$.*

Definition 5. *An automorphism $\phi^{\#}$ of $S^{\#}$ is said to be r* glueing-preserving *if for all $\mathcal{E}_i \in \mathcal{E}^{\#}$, $\mathcal{E}_i \in \mathcal{E}_j^{(r)}$ if and only if $z^{\#}(\mathcal{E}_i) \in z^{\#}(\mathcal{E}_j)^{(r)}$.*

We are now ready to present the main result of the paper describing automorphisms of S. The proof of the theorem below is the content of Lemmas 7 to 16 and Propositions 8 and 17.

Theorem 6. *An automorphism ϕ of S induces a range-preserving, r union-preserving, r glueing-preserving automorphism $\phi^{\#}$ of $S^{\#}$. Conversely every range-preserving, r union-preserving, r glueing-preserving automorphism of $S^{\#}$ can be extended uniquely to an automorphism of S.* ∎

Lemma 7. *Let $f, g \in S$. Then*

(i) $\pi(f) \in \pi(g)^{(r)}$ *if and only if* $f \in S^1 g$;

(ii) $\pi(f) = \pi(g)$ *if and only if* $S^1 f = S^1 g$;

(iii) $f \,\mathcal{L}\, g$ *if and only if* $\pi(f) = \pi(g)$.

Proof. Observe that (ii) follows directly from (i), while to prove (i) it suffices to show that if $\pi(f) \in \pi(g)^{(r)}$ then $f \in S^1 g$. For this choose any $\mathcal{E}_i \in \mathcal{E}^{\#}$ and let $\mathcal{D}$ be the set of all classes in $\mathcal{E}_i$ that have a non-empty intersection with $R(g)$. Define an equivalence relation μ on the classes of $\mathcal{D}$ via $(A, B) \in \mu$ if and only if $fg^{-1}(A) = fg^{-1}(B)$. Since $\pi(g) \in \mathcal{E}^{(r)}$ and $\pi(f) \in \pi(g)^{(r)}$, it follows that there are fewer than r classes of $\mathcal{D}$ in each μ-equivalence class. Let $\eta : \mathcal{E}_i - \mathcal{D} \to \mathcal{D}$ be a one-to-one mapping (it is readily checked that $|\mathcal{E}_i - \mathcal{D}| \leq |\mathcal{D}| = p$). Extend μ to $\mathcal{E}_i$ by adjoining to each μ-equivalence class the preimages of its elements under η. Fewer than r classes are adjoined, since η is one-to-one. The equivalence classes of μ on $\mathcal{E}_i$ naturally provide us with an r glueing $\mathcal{P}$ of $\mathcal{E}_i$. Note that $R(g)$ contains a transversal of $\mathcal{P}$ and let t be a transformation of X having $\pi(t) = \mathcal{P}$ and for every $y = g(x)$, $t(y) = f(x)$. Then $t \in S$ and $f = tg \in S^1 g$, as required. Finally note that (iii) is a restatement of (ii). ∎

Let ϕ be an automorphism of S. The following is a consequence of Lemma 7 and the definition of $S^{\#}$.

Proposition 8. *1. The correspondence* $z : \mathcal{E}^{(r)} \to \mathcal{E}^{(r)}$ *defined by* $z(\pi(f)) = \pi(\phi(f))$ *is a bijection such that* $\mathcal{P} \in \mathcal{E}_{\rangle}^{(\nabla)}$ *if and only if* $z(\mathcal{P}) \in \ddagger(\mathcal{E}_{\rangle})^{(\nabla)}$.

2. The restriction $\phi^{\#}$ *of* ϕ *to* $S^{\#}$ *is an* r *glueing-preserving automorphism of* $S^{\#}$. ∎

Lemma 9. *For every* $A \in C_q$ *and* $\mathcal{E}_i \in \mathcal{E}$ *there exists a* $\mathcal{P} \in \mathcal{E}_{\rangle}^{(\nabla)}$ *such that* A *is a total transversal of* $\mathcal{P}$.

Proof. Note that A is a partial transversal of $\mathcal{E}_i$ and let $\mathcal{D}$ be the set of all classes in $\mathcal{E}_i$ that have an empty intersection with A. Then $|\mathcal{D}| \leq p$, and there exists a one-to-one function $\eta : \mathcal{D} \to A$. Let $\mathcal{P}$ be a partition of X consisting of all classes of $\mathcal{E}_i$ that do not intersect $\eta(\mathcal{D})$ and all the sets of the form $F \cup [\eta(F)]$, where $[\eta(F)]$ is the $\mathcal{E}_i$-class of $\eta(F)$ and $F \in \mathcal{D}$. Then $\mathcal{P} \in \mathcal{E}_{\rangle}^{(\nabla)}$ as required. ∎

Lemma 10. *For every* $A \in C_q$ *and* $\mathcal{E}_i \in \mathcal{E}$ *there exists an idempotent* e *in* S *with* $R(e) = A$ *and* $\pi(e) \in \mathcal{E}_i^{(r)}$.

Proof. Using Lemma 9 choose $\mathcal{P} \in \mathcal{E}_{\rangle}^{(\nabla)}$ such that A is a total transversal of $\mathcal{P}$. Then the required idempotent is a transformation e of X with $\pi(e) = \mathcal{P}$, $\mathcal{R}(\rceil) = \mathcal{A}$ and $e(a) = a$, for every $a \in A$. ∎

Proposition 11. $S^2 = S$.

Proof. For an f in S let e be an idempotent in S with $R(e) = R(f)$ (Lemma 10). Then $f = ef \in S^2$. ∎

Lemma 12. *(i) For f and g in S, $R(f) \subseteq R(g)$ if and only if for every idempotent e in S, $eg = g$ implies $ef = f$.*
(ii) All automorphisms of S are range-preserving.

Proof. Observe that (ii) is an easy consequence of (i) and the fact that idempotents are preserved under automorphisms. To prove (i) note that if $R(f) \subseteq R(g)$ and e is an idempotent such that $eg = g$ then e is the identity on $R(g)$, hence e is the identity on $R(f)$, and so $ef = f$. Conversely assume $x \in R(f) - R(g)$ and let $x = f(y)$. Choose an idempotent e in $\mathbf{S}$ with $R(e) = R(g)$. Then $eg = g$ while $ef(y) = e(x) \neq x = f(y)$, so that $ef \neq f$. ∎

Note that the above result implies that the restriction $\phi^{\#}$ of ϕ to $S^{\#}$ is a range-preserving automorphism of $S^{\#}$, described in Proposition 3. We will use it to describe ϕ itself.

Lemma 13. *Let $f \in S$ with $R(f) \in \mathcal{M}_\alpha$, and take $x \in W$. Then $\phi(f)(x) = h_\alpha f h_\alpha^{-1}(x)$.*

Proof. We show that there exists a $g \in S^{\#}$ such that $fg \in S^{\#}$ and $x \in R(\phi(f))$. Let $[x]$ be the δ-class containing x and $V = h_\alpha^{-1}([x])$. Choose $A \in \mathcal{C}_q$ such that $A \cap V$ is non-empty and let $A \cap V = \{y\}$. Assume $\pi(f) \in \mathcal{E}_i^{(r)}$. Since A is a partial transversal of $\mathcal{E}_i$ and each class of $\pi(f)$ consists of fewer than r classes of $\mathcal{E}_i$, $r \leq p$, there exists a subset D of A of cardinality p such that $y \in D$ and D is a partial transversal of $\pi(f)$. Let $D \in \mathcal{M}_\beta$, for some $\beta \in \Omega$, and $h_\beta^{-1}(x) = w$. Note that $h_\beta^{-1}([x]) = h_\alpha^{-1}([x])$, so that $y \;\delta\; w$ (see [2, p.211] for details), and choose $g \in S^{\#}$ with $R(g) = (D - \{y\}) \cup \{w\}$ and $g(v) = w$ for some $v \in W$. Then $g \in \mathcal{I}_\beta$, and since $\pi(g) = \pi(fg)$ and $R(fg) \subseteq R(f)$ we have that $fg \in S^{\#} \cap \mathcal{I}_\alpha$. Let $u = h_\beta(v)$, then $u \in W$ and $\phi(g)(u) = h_\beta g h_\beta^{-1}(u) = h_\beta g h_\beta^{-1} h_\beta(v) = h_\beta g(v) = h_\beta(w) = x$; $\phi(fg)(u) = h_\alpha fg h_\alpha^{-1}(u)$; $\phi(f)(x) = \phi(f)\phi(g)(u) = \phi(fg)(u) = h_\alpha fg h_\alpha^{-1}(u) = h_\alpha fg h_\alpha^{-1} h_\beta(v) = h_\alpha fg(v) = h_\alpha f(w) = h_\alpha f h_\beta^{-1}(x) = h_\alpha f h_\alpha^{-1}(x)$, since $h_\alpha^{-1} h_\beta(v)$ and v, $h_\beta^{-1}(x)$ and $h_\alpha^{-1}(x)$ are pairwise δ-related. ∎

Recall (Proposition 8) that ϕ induces a permutation $z : \mathcal{E}^{(r)} \to \mathcal{E}^{(r)}$ defined by $z(\pi(f)) = \pi(\phi(f))$.

Lemma 14. *If $\mathcal{P} \in \mathcal{E}_i^{(\nabla)}$ and C and D are classes of $\mathcal{E}_i$, then $C \cup D$ is a subset of a $\mathcal{P}$-class if and only if $C \cup D$ is a subset of a class of $z(\mathcal{P})$.*

Proof. Let $f \in S$ with $\pi(f) = \mathcal{P}$. Using Lemma 7 choose $g, t \in S$, such that $f = tg$ and $\pi(g) = \mathcal{E}_i$. Assume $R(t) \in \mathcal{M}_\alpha$ (and so $R(f) \in \mathcal{M}_\alpha$), $R(g) \in \mathcal{M}_\beta$. If $C \in \mathcal{B}(\mathcal{E}_i)$ then $\tilde{C} = y_i(C)$, and by Proposition 3, $\phi(f)(\tilde{C}) = \phi(f)(y_i(C)) = \phi(t) h_\beta g y_i^{-1}(y_i(C)) = h_\alpha t h_\alpha^{-1} h_\beta g(C) = h_\alpha f(C)$, by Lemma 13, since $h_\beta g(C) \in W$. If $D \in \mathcal{B}(\mathcal{E}_i)$ then $\phi(f)(\tilde{C}) = \phi(f)(\tilde{D})$ iff $\phi(f)(y_i(C)) = \phi(f)(y_i(D))$ iff $f(C) = f(D)$, as required. If D is not in $\mathcal{B}(\mathcal{E}_i)$, then there is an $x \in D \cap W$ and $\phi(f)(\tilde{D}) = \phi(f)(h_\alpha(x)) = h_\alpha f h_\alpha^{-1} h_\alpha(x) = h_\alpha f(D)$, so again $\phi(f)(\tilde{C}) = \phi(f)(\tilde{D})$ iff $f(C) = f(D)$. The remaining case when C is not in $\mathcal{B}(\mathcal{E}_i)$ can be dealt with in a similar manner. ∎

Corollary 15. *The automorphism $\phi^{\#}$ is r union-preserving.* ∎

Lemma 16. *Let $f \in S$ with $R(f) \in \mathcal{M}_\alpha$, $\pi(f) \in \mathcal{E}_i^{(r)}$, and $D \in z(\pi(f))$ with $D \cap W = \Phi$. Then $\phi(f)(D) = h_\alpha f y_i^{-1}(C)$, where $C \subseteq D$, $C \in z(\mathcal{E}_i)$ and $C \cap W = \Phi$.*

Proof. Assume $R(g) \in \mathcal{M}_\beta$, $R(t) \in \mathcal{M}_\tau$. Since $D \in z(\pi(f)) \in z(\mathcal{E}_i^{(r)})$, there exists a subset C of D, $C \in z(\mathcal{E}_i)$. Then $\phi(f)(D) = \phi(f)(C) = \phi(t)\phi(g)(C) = h_\tau t h_\tau^{-1} h_\beta g y_i^{-1}(C)$, since $g \in S^\#$, and $h_\beta g y_i^{-1}(C) \in W$. Since for any $a \in X$, $h_\tau^{-1} h_\beta g(a)$ and $g(a)$ are $\mathcal{E}_j$-equivalent for any j, $h_\tau t h_\tau^{-1} h_\beta g y_i^{-1}(C) = h_\tau t g y_i^{-1}(C) = h_\tau f y_i^{-1}(C) = h_\alpha f y_i^{-1}(C)$, because $R(f)$ is a subset of $R(t)$. ∎

Proposition 17. *Let μ be a range-preserving, r union-preserving and r glueing-preserving automorphism of $S^\#$. Then μ can be extended uniquely to an automorphism τ of S.*

Proof. Let μ be as stated, and $\{h_\alpha \mid \alpha \in \Omega\}$, $z^\#$, $\{y_i \mid i \in I\}$ be the parameters describing μ as in Proposition 3. We extend $z^\#$ to a permutation z of $\mathcal{E}^{(r)}$ as follows. Define a mapping z from $\mathcal{E}^{(r)}$ to itself such that $z(\mathcal{E}_i) = z^\#(\mathcal{E}_i)$, $i \in K$, and for $\mathcal{P} \in \mathcal{E}_{\rangle}^{(\nabla)}$, $\ddagger(\mathcal{P}) \in (\ddagger^\#(\mathcal{E}_{\rangle}))^{(\nabla)}$ such that $\tilde{B} \cup \tilde{C}$ is a subset of a $z(\mathcal{P})$-class if and only if $B \cup C$ is a subset of a $\mathcal{P}$-class. To see that z is well-defined assume $\mathcal{P} \in \mathcal{E}_{\rangle}^{(\nabla)} \cap \mathcal{E}_{|}^{(\nabla)}$, and let F be a $\mathcal{P}$-class such that $F = \cup\,\{G \mid G \in \mathcal{E}_i\} = \cup\,\{H \mid H \in \mathcal{E}_j\}$. Since F is a union of fewer than r classes of $\mathcal{E}_i$ or $\mathcal{E}_j$ and μ is r union-preserving, we have that $\cup\,\{\tilde{G} \mid G \in \mathcal{E}_i\} = \cup\,\{\tilde{H} : H \in \mathcal{E}_j\}$, as required.

Define a mapping τ on S as follows. For $f \in S^\#$, let $\tau(f) = \mu(f)$. For $f \in S$ with $R(f) \in \mathcal{M}_\alpha$ and $\pi(f) \in \mathcal{E}_i^{(r)}$, let $\pi(\tau(f)) = z(\pi(f))$, and $\tau(f)(x) = h_\alpha f h_\alpha^{-1}(x)$ if $x \in W$, while for an $[x] \in \mathcal{B}(z(\mathcal{E}_i))$, $\tau(f)(x) = h_\alpha f y_i^{-1}(D)$, where $D \subseteq [x]$, $D \in \mathcal{E}_i$.

To see that $\tau(f)$ is a mapping assume that $[x] \in \mathcal{B}(z(\mathcal{E}_i))$ and there exists $u \in W$, $u \in C \in z(\mathcal{E}_i)$ such that x, u are in the same class of $\pi(\tau(f))$. Then $\tau(f)(x) = h_\alpha f y_i^{-1}(x)$, $\tau(f)(u) = h_\alpha f h_\alpha^{-1}(u)$, and since by the definition of z, $f h_\alpha^{-1}(C) = f y_i^{-1}(x)$ we have that $h_\alpha f y_i^{-1}(x) = h_\alpha f h_\alpha^{-1}(u)$, as required. The proof that τ is a homomorphism is analogous to that of Proposition 3 (see [2, §4]). ∎

We now turn to the description of the Green's relations on S. Just as the maximal Croisot-Teissier subsemigroup $S^\# = \{f \in S \mid \pi(ft) = \pi(t) \text{ for all } t \in S\}$ of S played a crucial role in the description of the automorphisms of S, so the maximal regular subsemigroup of S aids in the description of the Green's relations on S. Let $E(S)$ be the set of all idempotents of S, and define

$$N = \{f \in S \mid R(ft) = R(f) \text{ for some } t \in S\}\ .$$

Then N is a subsemigroup of S containing $E(S)$. Moreover N contains all the regular transformations in S, for if f is regular then $fgf = f$, for some $g \in S$, and $R(f(gf)) = R(f)$. We show in Proposition 20 that N is the maximal regular subsemigroup of S.

Proposition 18. *For distinct $f, g \in S$, $f\ \mathcal{R}\ g$ iff $f, g \in N$ with $R(f) = R(g)$.*

Proof. Assume $f\ \mathcal{R}\ g$, then $fs = g$ and $gt = f$, for some $s, t \in S$. Therefore $R(f) = R(g)$, and so $R(f) = R(g) = R(fs) = R(gt)$, hence $f, g \in N$. Conversely, assume $f, g \in N$ with $R(f) = R(g)$. Then there exist $A, B \in C_q$ such that A and B

are transversals of $\pi(f)$ and $\pi(g)$ respectively. Let $h : B \to A$ be a bijection such that $h(b) = \{f^{-1}g(b)\} \cap A$, for each $b \in B$. Define $s \in S$ such that $R(s) = A$, $\pi(s) = \pi(g)$, and for each $b \in B$, a transversal of $\pi(s)$, $s(b) = h(b)$. Then $fs = g$, and a transformation $t \in S$ such that $gt = f$ may be constructed similarly. ■

Proposition 19. *For distinct $f, g \in S$, $f \mathcal{D} g$ iff either $\pi(f) = \pi(g)$, or $f, g \in N$.*

Proof. Assume $f \mathcal{D} g$, so that $f \mathcal{L} s$ and $s \mathcal{R} g$, for some $s \in S$. Then $\pi(f) = \pi(s)$ (Lemma 7) and if $s \neq g$, then $s, g \in N$ so that $f \in N$ also. Conversely, if $f, g \in N$, choose $s \in S$ with $R(s) = R(f)$ and $\pi(s) = \pi(g)$. Then $s \in N$ and $f \mathcal{L} s \mathcal{R} g$. ■

Since N consists of precisely those elements f of S whose partition $\pi(f)$ has a total transversal amongst $\mathcal{C}_q$ sets, N is a $\mathcal{D}$-class of S. Moreover N is a $\mathcal{D}$-class of S containing the set of idempotents of S, so that every element of N is regular. This, in conjunction with the earlier observation that N contains all the regular elements of S, proves the next result.

Proposition 20. *N is the maximal regular subsemigroup of S.* ■

Proposition 21. *S is simple.*

Proof. Since N is a $\mathcal{D}$-class of S (see the remark after Proposition 19) and $\mathcal{D} \subseteq \mathcal{J}$, it suffices to show that for any $f \in S$ there exists $g \in N$ such that $f \mathcal{J} g$. A proof similar to that of Lemma 9 yields that for an $f \in S$ there exists $\mathcal{P} \in (\pi(f))^{(r)}$ such that for $g \in S$ with $\mathcal{P} = \pi(g)$, we have that $g \in N$. Now let i be such that $\pi(f) \in \mathcal{E}_i^{(r)}$, and choose $\mathcal{Q} \in \mathcal{E}_i^{(r)}$ such that for $A, B \in \mathcal{E}_i$, A and B are in the same class of $\mathcal{Q}$ if and only if both $A \cap R(f)$ and $B \cap R(g)$ are non-empty, and $f^{-1}(A)$ and $f^{-1}(B)$ are in the same class of $\mathcal{P}$. Then for $s \in S$ with $\pi(s) = \mathcal{Q}$ we have that $\pi(sf) = \mathcal{P}$, and so $sf \in N$. Let $sf = g$. We show that there exist $u, v \in S$ such that $f = ugv$. Let v be such that $R(v)$ is a transversal of $\pi(g)$ and $\pi(v) = \pi(f)$. Then $R(gv) = R(g)$ and $\pi(gv) = \pi(f)$. Choose a bijection w from $R(g)$ onto $R(f)$ such that $w(gv(x)) = f(x)$ for all $x \in X$. Let $u \in N$ be such that $R(g)$ is a transversal of $\pi(u)$, and for each $y \in R(g)$, $u(y) = w(y)$. Then $f = ugv$, as required. ■

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