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RESEARCH ARTICLE

SOME PROPERTIES OF LOCAL SUBSEMIGROUPS INHERITED BY LARGER SUBSEMIGROUPS

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Communicated by B.M. Schein

To Professor L.M. Gluskin, on his 60th birthday

1. INTRODUCTION AND SUMMARY

By the local subsemigroups of a semigroup S we mean the semigroups of the form eSe where e is an idempotent of S, as in [9]. Problem N3 in the DeKalb 1979 conference proceedings, posed by Nambooripad, is as follows: if S is a regular semigroup such that for each idempotent $e \in S$, the idempotents of eSe form a band, then do the idempotents of each eS (dually Se) also form a band? In other words, for regular semigroups is the property of having idempotents form a band inherited from the local subsemigroups by the (usually) larger subsemigroups eS and Se ?

Section 3 begins with an affirmative answer. The above question arose from the paper [10] by Meakin and Nambooripad, where there is a structure theorem for regular semigroups S that satisfy the condition that the idempotents of eS and of Se form bands, for each idempotent $e \in S$. The above affirmative answer shows that this class is precisely the class of regular semigroups with the local property of being orthodox (by a local property of a semigroup S we mean any property held by each of its local subsemigroups).

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It was natural to ask next, what other local properties are inherited by the subsemigroups eS and/or Se ? We were able to find a further nine such properties (Section 3); there are sure to be more.

In Section 4, we consider regular semigroups in which the local subsemigroups are E-solid (defined below). We show that the idempotent classes of the least congruence on such a semigroup giving a local inverse semigroup image are completely simple subsemigroups (by a local inverse semigroup we mean any semigroup in which the local subsemigroups are inverse semigroups). This congruence unifies three of its special cases occurring in the literature: the least inverse semigroup congruence on an orthodox semigroup; the least inverse semigroup congruence on a regular E-solid semigroup (the present author's appendix to [11]); and the least local inverse semigroup congruence on a regular semigroup S in which the idempotents of eS and Se form bands, for each idempotent $e \in S$ (Meakin and Nambooripad [10]), i.e., from the affirmative answer above, on any regular locally orthodox semigroup S. The author is happy to acknowledge that his consideration of this question about the above congruence on a regular locally E-solid semigroup arose from reading some seminar notes on regular semigroups by D.B. McAlister.

2. PRELIMINARIES

For any adjective, A say, describing a type of semigroup, we shall say that a semigroup S is a locally A semigroup, or locally A, if each local subsemigroup is an A semigroup (this follows McAlister [9]). We note that the word inverse is not an adjective but a noun and so we shall call any semigroup in which the local subsemigroups are inverse semigroups a local inverse semigroup; likewise, a local semilattice would be any semigroup in which the local subsemigroups are semilattices.

For any semigroup S, we denote by $E(S)$ the set of idempotents of S, and we sometimes shorten $E(S)$ to E, when no ambiguity will result. C. Eberhart introduced the following condition on $E(S)$ in a letter to the author around 1969 (see the acknowledgement in [6], page 6): for any $e,f,g \in E$ such that $e \mathcal{L} f \mathcal{R} g$, there exists $h \in E$ such that $e \mathcal{R} h \mathcal{L} g$. When this condition is satisfied, following Clifford [1], we shall say that E is *solid* and that S is E-*solid*

The following result was found independently by Fitz-Gerald (unpublished) and the author [6, Theorem 3].

RESULT 1. *For any regular semigroup* S, *the subsemigroup generated by the set of idempotents of* S *is a union of groups if and only if* S *is E-solid*.

For any semigroup S, we denote by $\mathrm{Reg}(S)$ the set of regular elements of S, i.e. $\mathrm{Reg}(S) = \{a \in S: axa = a \text{ for some } x \in S\}$. It is of interest to know when $\mathrm{Reg}(S)$ is a subsemigroup of S; two necessary and sufficient conditions on $E(S)$ can be gleaned from the literature.

RESULT 2. *For any semigroup* S *the following conditions are equivalent*:

(i) $\mathrm{Reg}(S)$ *is a subsemigroup of* S;

(ii) $\langle E(S) \rangle$ *is a regular subsemigroup of* S;

(iii) *for all* $e,f \in E = E(S)$, *the product* ef *is regular in* S (i.e. $E^2 \subseteq \mathrm{Reg}(S)$).

Proof. That (i) implies (ii) is due to Fitz-Gerald ([3] or [8, Chapter II, Exercise 15]), clearly (ii) implies (iii), and that (iii) implies (i) follows from [2, Theorem 2.4].

For any equivalence relation ρ on any set X, and for any $Y \subseteq X$, we shall denote $\rho \cap (Y \times Y)$, the restriction of ρ to Y, by $\rho|Y$.

We use wherever possible, and usually without comment, the notations and conventions of Clifford and Preston [2] and Howie [8].

3. SOME LOCAL PROPERTIES

The affirmative answer to Nambooripad's question follows from part (i) of Theorem 1. The next three parts of Theorem 1 and Theorem 5 give the further nine local properties inherited by each eS and/or Se.

THEOREM 1. *Let e be any idempotent of any semigroup S.*

(i) *If $E(eSe)$ is a band, then $E(eS)$ is a band.*

(ii) *If eSe is E-solid, then eS is E-solid.*

(iii) *If eSe has at most one idempotent per $\mathcal{L}$-class, then eS has at most one idempotent per $\mathcal{L}$-class.*

(iv) *If $\mathrm{Reg}(eSe)$ is a subsemigroup (e.g. if S is regular) then $\mathrm{Reg}(eS)$ is a subsemigroup; equivalently, if $\langle E(eSe)\rangle$ is regular then $\langle E(eS)\rangle$ is regular.*

Proof. (i) Take any $f,g \in E(eS)$. Then easy checking shows that $fe, ge \in E(eSe)$ and so $fege$ is idempotent, giving that

$$(fg)(fg) = fgfgg = f(eg)(ef)(eg)(eg) \quad (\text{since } f,g \in eS)$$
$$= (fege)(fege)g = (fege)g = fgg = fg,$$

i.e. fg is an idempotent. Thus $E(eS)$ is a band as required.

(ii) Take any idempotents $f,g,h \in eS$ such that $h \mathcal{L} f \mathcal{R} g$. We show first that the idempotents he,fe,ge of eSe satisfy $he \mathcal{L} fe \mathcal{R} ge$. Now $(fe)f = f(ef) = ff = f$ so $fe \mathcal{R} f$ and likewise $ge \mathcal{R} g$, so $fe \mathcal{R} ge$; and since $\mathcal{L}$ is a right congruence we have $he \mathcal{L} fe$ in S and hence also in eSe.

Thus, since eSe is E-solid, there exists an idempotent $k \in eSe$ such that $he \mathcal{R} k \mathcal{L} ge$. Routine calculations now show that kg is an idempotent of eS such that, from Green's Lemma, $h \mathcal{R} kg \mathcal{L} g$, whence eS is E-solid as required.

(iii) Take idempotents f,g in eS such that $f \mathcal{L} h$. From above, fe and he are idempotents of eSe such that $f \mathcal{R} fe \mathcal{L} he$. From Green's Lemma and $(fe)f = f$ we have $(he)f = h$. However $fe = he$ since eSe has at most one idempotent per $\mathcal{L}$-class so $f = h$, giving us that eS has at most one idempotent per $\mathcal{L}$-class, as required.

We note that routine arguments show that

$$\mathcal{L}(eS) = \mathcal{L}(S)|eS$$

and

$$\mathcal{L}(eSe) = \mathcal{L}(eS)|eSe = \mathcal{L}(S)|eSe.$$

(iv) We assume then that Reg(eSe) is a subsemigroup and we take any elements $x,y \in \mathrm{Reg}(eS)$ and any $x' \in V(x) \cap eS$ and $y' \in V(y) \cap eS$. Then $x'e \in V(xe) \cap eSe$, $y'e \in V(ye) \cap eSe$ and so $xe, ye \in \mathrm{Reg}(eSe)$ giving that $xeye = xye \in \mathrm{Reg}(eSe)$. Take any $(xye)' \in V(xye) \cap (eSe)$. Then from $y' \in eS$ we have

$$xy(xye)'xy = xy(e(xye)')xyy'y = (xye)(xye)'xyey'y$$

$$= xyey'y = xyy'y = xy$$

and easily now we see that $(xye)' \in V(xy)$, whence xy is regular in eS and so Reg(eS) is a subsemigroup, as required.

Note that if S is a regular semigroup then each local subsemigroup eSe is also regular, so by part (iv) each Reg(eS) is a subsemigroup.

The reverse implications of (i) to (iv) are obvious or easily proved; the reverse implication of (iv) also follows from Lemma 4(i) below.

LEMMA 2. *Let e be any idempotent of any semigroup S and let ϕ be the morphism of eS onto eSe given by $x\phi = xe$ for each $x \in eS$. The restriction map $\psi = \phi|\mathrm{Reg}(eS)$ maps Reg(eS) onto Reg(eSe) and satisfies the following conditions:*

(i) *ψ is $\mathcal{R}$-class preserving (i.e. $x\psi \,\mathcal{R}\, x$ in eS for each $x \in \mathrm{Reg}(eS)$), whence $\psi \circ \psi^{-1} \subseteq \mathcal{R}(eS)$;*

(ii) *ψ maps each regular $\mathcal{L}$-class of eS one-to-one and onto a regular $\mathcal{L}$-class of eSe;*

(iii) *for each idempotent* $f \in eSe$, $f\psi^{-1}$ *is a right zero subsemigroup; in particular, when* Reg(eSe) *is a subsemigroup, the onto morphism* ψ: Reg(eS) $\to$ Reg(eSe) *is idempotent-determined.*

Proof. Routine checking shows that ϕ is a morphism, so ϕ maps Reg(eS) into Reg(eSe); since (Reg(eSe))ϕ = Reg(eSe) we have that ϕ maps Reg(eS) onto Reg(eSe).

Take any elements $x, x' \in eS$ such that $x' \in V(x)$. Then $(xe)x'x = x(ex')x = xx'x = x$ so $xe\,\mathcal{R}\,x$ in eS, i.e. $x\psi\,\mathcal{R}\,x$ in eS, as required for (i). Statement (ii) is now seen to be part of Green's Lemma. Also, if $x\psi$ is an idempotent then x is an idempotent since

$$x^2 = xxx'x = x(ex)(ex')x = (xe)(xe)x'x = (xe)x'x = x(ex')x = x.$$

Statement (iii) now follows.

LEMMA 3. *Let* e *be any idempotent in any regular semigroup* S.

(i) *Each idempotent separating congruence on* eSe *extends to an idempotent separating congruence on* S *and to a unique idempotent separating congruence on* SeS.

(ii) *If* $\mathcal{H}$(eSe) (=$\mathcal{H}$(S)|eSe) *is a congruence on* eSe *then* $\mathcal{H}$(SeS) (=$\mathcal{H}$(S)|SeS) *is a congruence on* SeS.

Proof. (i) Take any idempotent separating congruence ρ on eSe. By Hall and Jones [7, Proposition 4.5], ρ extends to a congruence on S, for example ρ^*, the congruence on S generated by ρ (i.e. $\rho^*|eSe = \rho$). Denoting by $\mu(T)$ the maximum idempotent separating congruence on any regular semigroup T, we have $\rho \subseteq \mu(eSe)$ and from the author [6, Corollary 6] we have $\mu(eSe) \subseteq \mu(S)$; from $\rho \subseteq \mu(S)$ we have that $\rho^* \subseteq \mu(S)$, i.e. that ρ^* is idempotent separating (as required). Further, $\rho^*|SeS$ is therefore also idempotent separating and of course also extends ρ. It remains to show that $\rho^*|SeS$ is the only idempotent separating extension of ρ to SeS (from which it follows that it is the congruence on SeS generated by ρ).

For any $\mathcal{H}$-class, H say, of SeS, by Clifford and Preston [2, Section 8.4, Exercise 3], there exists an element (in fact an idempotent) $f \in eSe$ such that H and f are in the same $\mathcal{D}$-class, D say, of SeS (equivalently, of S). Now by Green's Lemma and its dual there exist elements $x,y,u,v \in (SeS)^1$ (in fact in SeS since SeS is regular) such that the maps

$$\theta_{x,y}: H_f \to H, \quad : s \mapsto xsy, \qquad \theta_{u,v}: H \to H_f, \quad : t \mapsto utv,$$

are mutually inverse bijections between H_f and H. Clearly then, as is well-known, for any idempotent separating congruence σ on S, $\sigma|D$ is determined uniquely by $\sigma|H_f$ (in fact for any $\mathcal{H}$-class $H_f \subseteq D$) since, for all $t_1,t_2 \in H$, we have $(t_1,t_2) \in \sigma$ if and only if $(ut_1v,ut_2v) \in \sigma$ (of course $\sigma \subseteq \mathcal{H}(S)$, by [8, Proposition II.4.8]).

Since $f \in eSe$ we have that $H_f \subseteq eSe$ and that $\rho|H_f(=\rho^*|H_f)$ determines $\rho^*|H$ uniquely and hence $\rho^*|SeS$ ($\subseteq \mathcal{H}(SeS)$) is the unique idempotent separating congruence on SeS extending ρ.

(ii) From the proof above of statement (i), statement (ii) easily follows.

LEMMA 4. *Let e be any idempotent in any semigroup S.*

(i) $\mathrm{Reg}(eSe) = (eSe) \cap \mathrm{Reg}(eS) = (\mathrm{Reg}(eS))e$.

(ii) $\langle E(eSe) \rangle = (eSe) \cap \langle E(eS) \rangle = \langle E(eS) \rangle e$.

Proof. (i) It is clear that

$$\mathrm{Reg}(eSe) \subseteq (eSe) \cap \mathrm{Reg}(eS) \subseteq (\mathrm{Reg}(eS))e$$

and from Lemma 2, $(\mathrm{Reg}(eS))e = (\mathrm{Reg}(eS))\psi = \mathrm{Reg}(eSe)$; this proves part (i).

(ii) It is clear that

$$\langle E(eSe) \rangle \subseteq (eSe) \cap \langle E(eS) \rangle \subseteq \langle E(eS) \rangle e$$

and the morphism $\phi: eS \to eSe$ of Lemma 2 of course satisfies $\langle E(eS) \rangle \phi \subseteq \langle E(eSe) \rangle$, i.e. $\langle E(eS) \rangle e \subseteq \langle E(eSe) \rangle$; this proves part (ii).

THEOREM 5. *Let e be any idempotent of any semigroup S.*

(i) *If* $\langle E(eSe) \rangle$ *is a union of groups then* $\langle E(eS) \rangle$ *is a* union of groups.

(ii) If $\langle E(eSe)\rangle$ is a band of groups then $\langle E(eS)\rangle$ is a band of groups.

(iii) If $Reg(eSe)$ is an idempotent-generated subsemigroup then $Reg(eS)$ is an idempotent-generated subsemigroup.

(iv) If the subgroups of eSe are trivial then the subgroups of eS are trivial.

(v) If $Reg(eSe)$ is a subsemigroup on which $\mathcal{H}$ is a congruence then $Reg(eS)$ is a subsemigroup on which $\mathcal{H}$ is a congruence.

(vi) If $Reg(eSe)$ is a union of groups (not necessarily a subsemigroup) then $Reg(eS)$ is a union of groups.

Proof. (i) This follows from Theorem 1(i) and (iv) and Result 1.

(ii) Now $\langle E(eS)\rangle = T$ say, is a union of groups from part (i). Clearly $e \in T$ and $eT = T$; from Lemma 4(ii) we have $Te = \langle E(eSe)\rangle$. From $\mathcal{H}$ being a congruence on $eTe = \langle E(eSe)\rangle$ we have from Lemma 3(ii) that $\mathcal{H}$ is a congruence on $TeT = T$, so $T = \langle E(eS)\rangle$ is a band of groups, as required.

(iii) Take any element $a \in Reg(eS)$ and any idempotent $f\,\mathcal{L}\,a$ in eS. Then $ae\,\mathcal{L}\,fe$ in eSe (Lemma 2(ii)). From $f\,\mathcal{R}\,fe = (fe)^2$ we have $(fe)f = f$ whence $(ae)f = a$ from Green's Lemma. From ae being a product of idempotents in eSe we have $a = (ae)f$ is a product of idempotents in eS.

(iv) This follows from Lemma 2(i) and (ii).

(v) From Theorem 1(iv), $Reg(eS)$ is a subsemigroup. We put $T = Reg(eS)$ and note that again $eT = T$, $eTe = Te = Reg(eSe)$ by Lemma 4(i), and $TeT = T$. The required result now follows from Lemma 3(ii).

(vi) This follows from Lemma 2(ii) and (iii).

Again, the reverse implications of (i) to (vi) are obvious or easily proved.

4. REGULAR LOCALLY E-SOLID SEMIGROUPS

Let S be any regular semigroup all of whose local subsemigroups eSe $(e \in E(S))$ are E-solid. In this section we consider the least congruence ρ on S such that S/ρ is a local inverse semigroup. We show that for each $e \in E = E(S)$, the congruence class $e\rho$ is a completely simple subsemigroup, and we determine the (normal) partition of E induced by ρ (equivalently, we determine $\rho|E$).

It is easy to see that ρ exists, and that ρ is in fact the congruence on S generated by the relation

$$\rho_0 = \{(f,g) \in E \times E \text{: for some } e \in E,\ f\,\mathcal{D}\,g \text{ in } \langle E(eSe)\rangle\};$$

for example, this can be proved as follows from Lallement's Lemma. For the moment we denote ρ_0^*, the congruence on S generated by ρ_0, by σ. By Lallement's Lemma, any local subsemigroup of S/σ is of the form $(e\sigma)(S/\sigma)(e\sigma) = (eSe)\sigma^{\natural} \cong eSe/(\sigma|eSe)$ for some $e \in E$. Now $\langle E(eSe)\rangle$ is a union of groups by Result 1 so $\mathcal{D} = \mathcal{D}(\langle E(eSe)\rangle)$ is a congruence on $\langle E(eSe)\rangle$ and $\langle E(eSe)\rangle/\mathcal{D}$ is a semilattice. Since $\mathcal{D} \subseteq \rho_0 \subseteq \sigma|eSe$, we have from Lallement's Lemma again that $\langle E((eSe)\sigma^{\natural})\rangle$ is a homomorphic image of the semilattice $\langle E(eSe)\rangle/\mathcal{D}$, and so is itself a semilattice. Hence $(eSe)\sigma^{\natural}$ is an inverse semigroup and so S/σ is a local inverse semigroup. It is clear that ρ_0^* is the least such congruence, so ρ exists and $\rho = \rho_0^*$.

We denote by ρ_1 the compatible closure of ρ_0; that is, we define

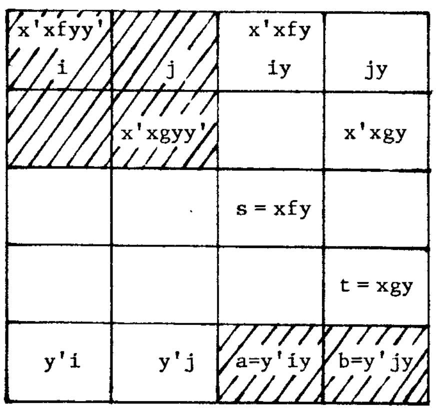
$$\rho_1 = \{(xfy, xgy) \in S \times S\text{: } x,y \in S^1, (f,g) \in \rho_0\}.$$

Then of course $\rho = \rho_1^t$, the transitive closure of ρ_1 (since ρ_1 is reflexive and compatible).

LEMMA 6. <u>For any pair</u> $(s,t) \in \rho_1$ <u>there exist idempotents</u> $a,b \in S$ <u>such that</u> $s\,\mathcal{L}\,a\,\mathcal{R}\,b\,\mathcal{L}\,t$.

Proof. There exist $x,y \in S^1$, $e,f,g \in E(S)$ such that $s = xfy$, $t = xgy$ and $f\,\mathcal{D}\,g$ in $\langle E(eSe)\rangle$. Since $f,g \leqslant e$ we have $s = xefey$, $t = xegey$ so we can assume without loss of generality that $x = xe$,

$y = ey$. Take any $x' \in V(x)$, $y' \in V(y)$; then $ex' \in V(x)$ and $y'e \in V(y)$ so we can assume without loss that $x' = ex'$, $y' = y'e$. Thus we have $x'x, yy' \in E(eSe)$ and since $\langle E(eSe)\rangle$ is a union of groups (Result 1), we have $x'xfyy' \,\mathcal{D}\, x'xgyy'$ in $\langle E(eSe)\rangle$ and of course $x'xfyy' \,\mathcal{R}\, x'xfy \,\mathcal{L}\, xfy$ and likewise $x'xgyy' \,\mathcal{R}\, x'xgy \,\mathcal{L}\, xgy$ in S. The egg-box diagram below will help explain the proof; shaded boxes indicate some of the group $\mathcal{H}$-classes of S. Let i,j denote the idempotents satisfying $i \,\mathcal{H}\, x'xfyy' \,\mathcal{R}\, j \,\mathcal{L}\, x'xgyy'$. From Green's Lemma we have $i \,\mathcal{R}\, iy \,\mathcal{L}\, xfy$ and likewise $j \,\mathcal{R}\, jy \,\mathcal{L}\, xgy$. From $i = iyy'$ and $j = jyy'$ we see that $y'i \in V(iy)$, $y'j \in V(jy)$ and $i \,\mathcal{L}\, y'i \,\mathcal{R}\, y'j \,\mathcal{L}\, j$ and then that the idempotents $a = y'iy$, $b = y'jy$ satisfy the required condition, namely that $s \,\mathcal{L}\, a \,\mathcal{R}\, b \,\mathcal{L}\, t$.



Define a relation γ on S, containing $\mathcal{L}$, by

$$\gamma = \cup\{L_a \times L_b : a \,\mathcal{R}\, b \text{ and } a,b \in E(eSe) \text{ for some } e \in E(S)\}$$

and define $\mathcal{L}^{\#}$ to be the transitive closure of γ. (Note that $\mathcal{L} \subseteq \mathcal{L}^{\#} \subseteq \mathcal{D}$.)

In the notation of the proof of Lemma 6 we have that the idempotents $a = y'iy$, $b = y'jy$ are in $y'ySy'y$ and are $\mathcal{R}$-related in S, so $(s,t) \in L_a \times L_b \subseteq \gamma$, which gives us that $\rho_1 \subseteq \gamma$ and so $\rho \subseteq \gamma^t = \mathcal{L}^{\#}$. Conversely, given any a,b as in the definition of γ we see that $(a,b) \in \rho_0 \subseteq \rho$ so for any $(x,y) \in \gamma$ we have that x and y map into the same $\mathcal{L}$-class of S/ρ, and thus likewise for any $(x,y) \in \gamma^t = \mathcal{L}^{\#}$.

Dually, we define δ and $\mathcal{R}^{\#}$ by

$$\delta = \cup\{R_a \times R_b : a\,\mathcal{L}\,b \text{ and } a,b \in E(eSe) \text{ for some } e \in E(S)\},$$

and $\mathcal{R}^{\#} = \delta^t$, the transitive closure of δ; and we have that $\rho \subseteq \mathcal{R}^{\#}$. Putting $\mathcal{H}^{\#} = \mathcal{L}^{\#} \cap \mathcal{R}^{\#}$ we have also that $\rho \subseteq \mathcal{H}^{\#}$ and $\mathcal{H} \subseteq \mathcal{H}^{\#}$. This does not concern us here, but $\mathcal{L}^{\#}/\rho$, $\mathcal{R}^{\#}/\rho$ and $\mathcal{H}^{\#}/\rho$ are Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ respectively on S/ρ; since $\rho \subseteq \mathcal{D}$ we have that $\mathcal{D}/\rho$ and $\mathcal{J}/\rho$ are $\mathcal{D}$ and $\mathcal{J}$ on S/ρ, by the author [5, Theorems 10 and 13].

THEOREM 7. (i) *For any $(x,y) \in \mathcal{R}^{\#}$, the pattern of group $\mathcal{H}$-classes in R_x is identical to the pattern of group $\mathcal{H}$-classes in R_y, in the sense that for each $\mathcal{L}$-class L of S, $L \cap R_x$ is a subgroup if and only if $L \cap R_y$ is a subgroup.*

(ii) *For each $e \in E = E(S)$, $e\mathcal{H}^{\#}$ is a completely simple subsemigroup of S, whence $e\rho$ is also a completely simple subsemigroup of S.*

(iii) $\rho|E = \mathcal{H}^{\#}|E$.

Proof. (i) First we take any $(x,y) \in \delta$; then there exist $e \in E(S)$ and $a,b \in E(eSe)$ such that $x\,\mathcal{R}\,a\,\mathcal{L}\,b\,\mathcal{R}\,y$. Note that $R_x, R_y \subseteq eS$. Now Theorem 1(ii) comes to our assistance, for eS is thereby E-solid, and since $a^2 = a\,\mathcal{L}\,b = b^2$ we have the required conclusion for any $(x,y) \in \delta$ (namely that the pattern of group $\mathcal{H}$-classes in R_x is identical to that of R_y) and thus also for any $(x,y) \in \delta^t = \mathcal{R}^{\#}$.

(ii) Take any element $s \in e\mathcal{H}^{\#}$. From $e\,\mathcal{R}^{\#}\,s$ and part (i) above we have that $L_e \cap R_s$ contains an idempotent f say, and then from $e\,\mathcal{L}^{\#}\,s$ and the dual of part (i) above we have that H_s contains an idempotent, g say, and H_s is a group. From $\mathcal{H} \subseteq \mathcal{H}^{\#}$ we have that $H_s \subseteq e\mathcal{H}^{\#}$ and so $e\mathcal{H}^{\#}$ is a union of groups. From $e\,\mathcal{L}\,f\,\mathcal{R}\,s$ we have $e\,\mathcal{L}^{\#}\,f\,\mathcal{R}^{\#}\,s\,\mathcal{R}^{\#}\,e$ so $f \in e\mathcal{H}^{\#}$; and dually there is an idempotent $h \in e\mathcal{H}^{\#}$ satisfying $e\,\mathcal{R}\,h\,\mathcal{L}\,s$. Thus, regarding H_e and H_g as arbitrary $\mathcal{H}$-classes in $e\mathcal{H}^{\#}$, we see that the group $\mathcal{H}$-classes of S which constitute $e\mathcal{H}^{\#}$ form a rectangular array in the egg-box picture of the containing $\mathcal{D}$-class of S;

i.e. $eH^{\#}$ is a completely simple subsemigroup of S, as required. Of course $\rho|(eH^{\#})$ is a congruence on $eH^{\#}$ so $(e\rho)\cap(eH^{\#}) = e\rho$ is also a completely simple semigroup, since any idempotent congruence class in a completely simple semigroup is itself completely simple.

(iii) We have already that $\rho \subseteq H^{\#}$ so we now show that $H^{\#}|E \subseteq \rho|E$. Take any pair $(e,f) \in H^{\#}|E$. As shown before, from $(e,f) \in L^{\#}$ we have that e and f map into the same L-class of S/ρ, and dually, into the same R-class and hence the same H-class of S/ρ; thus $e\rho = f\rho$, i.e. $(e,f) \in \rho$ and so $H^{\#}|E \subseteq \rho|E$ as required.

REMARK. Of course $\rho_0 \subseteq \rho|E = H^{\#}|E$ so ρ is the least congruence on S corresponding to its normal partition of E.

The following corollary follows also from [10, Theorem 1.5] and Theorem 1(i) (since part (i) is easily proved from part (ii)), but for completeness we give a proof of it from Theorem 7.

COROLLARY 8. *Let S be any regular locally orthodox semigroup. For each idempotent $h \in S$*

(i) $\rho|hSh = Y(hSh)$, *the least inverse semigroup congruence on the orthodox semigroup hSh,*

and

(ii) $h\rho$ *is a rectangular band.*

Proof. Take any $s,t \in hSh$ such that $s\ \rho_1\ t$ and take the elements e,f,g,x,y as in the proof of Lemma 6. Since $s = hsh = hxefeyh$ and $t = hxegeyh$ we can assume without loss that $x = hxe$, $y = eyh$ and then that $x' = ex'h \in V(x)$, $y' = hy'e \in V(y)$. Thus the idempotents $a = y'iy$ and $b = y'jy$ are elements of hSh and dually the idempotents $c = xjx'$, $d = xkx'$ (where $k = x'xgyy' \in E(eSe)$) are in hSh and $s\,R\,c\,L\,d\,R\,t$. Now from $a\ R\ b$ we have $(a,b) \in Y(hSh) \doteq Y$ say, and likewise $(c,d) \in Y$. Routine checking shows that $s = cta$ and so we have $s = cta\,Y\,dtb = t$, giving that $\rho_1|hSh \subseteq Y$.

Take any $x,y \in hSh$ such that $(x,y) \in \rho = \rho_1^t$. Then

$$x = s_1\rho_1 s_2\rho_1 \ldots \rho_1 s_n = y$$

for some $s_2, s_3, \ldots, s_{n-1} \in S$, and so

$$x = hxh = hs_1h \ \rho_1 \ hs_2h \ \rho_1 \ldots \rho_1 \ hs_nh = hyh = y.$$

Thus

$$x = hs_1h \,\mathcal{Y}\, hs_2h \,\mathcal{Y} \ldots \mathcal{Y}\, hs_nh = y$$

giving that $(x,y) \in \mathcal{Y}$. Thus $\rho|hSh \subseteq \mathcal{Y}$ and the reverse containment is obvious since the image of hSh in S/ρ is inverse.

(ii) Take any element $s \in h\rho$, a completely simple semigroup by Theorem 7(ii). Thus $s \,\mathcal{H}\, e \,\rho\, h$ for some $e \in E(S)$. Then $s \in eSe$ and $s \,\rho\, e$, so from part (i), $s \in (e\rho) \cap (eSe) = e\mathcal{Y}(eSe) = \{e\}$, whence $s = e$. Thus $h\rho$ is a rectangular band, as required.

There are obvious converses to Theorem 7 and Corollary 8, namely the following.

THEOREM 9. *Let S be any regular semigroup.*

(i) *If there is a congruence* ρ *on S such that* S/ρ *is a local inverse semigroup and* $e\rho$ *is a union of groups for each* $e \in E(S)$, *then S is locally E-solid.*

(ii) *If there is a congruence* ρ *on S such that* S/ρ *is a local inverse semigroup and* $e\rho$ *is a band for each* $e \in E(S)$, *then S is locally orthodox.*

Proof. (i) Suppose such a congruence ρ exists and take any $e \in E(S)$ and any $f,g,h \in E(eSe)$ such that $f \,\mathcal{L}\, g \,\mathcal{R}\, h$. To show that eSe is E-solid we must show there exists $k \in E(eSe)$ such that $f \,\mathcal{R}\, k \,\mathcal{L}\, h$. Since $(eSe)\rho^{\natural}$ is a local subsemigroup of S/ρ it is an inverse semigroup, so from $f \,\mathcal{L}\, g \,\mathcal{R}\, h$ we obtain $f\rho = g\rho = h\rho$, i.e. $f \,\rho\, g \,\rho\, h$. Thus $gh \in f\rho$, a union of groups, so there is an idempotent $k \in f\rho$ such that $gh \,\mathcal{H}\, k$ in $f\rho$ and hence also in S. Now $g \,\mathcal{R}\, gh \,\mathcal{L}\, h$ from [2, Theorem 2.17] so of course $g \,\mathcal{R}\, k \,\mathcal{L}\, h$ in S. From $g,h \leqslant e$ we have $k \leqslant e$, i.e. $k \in E(eSe)$, and so $g \,\mathcal{R}\, k \,\mathcal{L}\, h$ in eSe, i.e. eSe is E-solid, as required.

(ii) Given that such a congruence ρ exists, let us take any $e \in E(S)$. By part (i), we have that $\langle E(eSe) \rangle$ is a union of groups.

Take any $f, g \in E(eSe)$ such that $f \mathcal{D} g$ in $\langle E(eSe) \rangle$; we show that fg is idempotent. Now $\langle E(eSe) \rangle \rho^{\natural}$ is a semilattice so $(f,g) \in \mathcal{D}(\langle E(eSe) \rangle) \subseteq \rho$; thus $(f, fg) \in \rho$ and so fg is idempotent since $f\rho$ is a band. It follows that each $\mathcal{D}$-class of $\langle E(eSe) \rangle$, besides being a completely simple semigroup, is also orthodox, whence, from $\mathcal{D} = \mathcal{J}$ on $\langle E(eSe) \rangle$, we have that each principal factor of $\langle E(eSe) \rangle$ is orthodox, which gives us that $\langle E(eSe) \rangle$ is orthodox by the author [4, Corollary 1] (or [8, Chapter VI, Exercise 2]), i.e. $E(eSe)$ is a band, giving that S is locally orthodox, as required.

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ADDENDUM

The author has recently learned that the word "inverse" is in fact an adjective as well as a noun. The phrase "locally inverse semigroup" now seems to the author to be preferable to "local inverse semigroup".

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