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RESEARCH ARTICLE

Universal Expansion of Semigroup Varieties by Regular Involution

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1. Preliminaries

This paper is an "expansion" of, and makes a whole with, its predecessor [7]. For the reader's convenience, we recall here the basic definitions and put in a word on the motivation.

A category K is said to be

universal if every category C of algebras can be fully embedded into K,

monoid universal if every monoid M is isomorphic to the endomorphism monoid $End(A)$ for some object A of K.

We are concerned here with characterizations of universality for certain subvarieties of the variety $\mathbf{R}$ of *regular involution semigroups* (the universality of $\mathbf{R}$ was stated by Demlová and Koubek [3]), the latter being defined as semigroups with an additional unary operator $x \to x^+$, the *regular involution*, tied up to the semigroup multiplication by the identities

$$x^{++} = x, xx^+x = x.$$

The kind of subvarieties of $\mathbf{R}$ we are interested in are what we call "expansions" of semigroup varieties by regular involution. The aim of this paper is to characterize the universal expansions of varieties of semigroups by regular involution. The universal expansions of varieties of bands (i.e. idempotent semigroups) were described by Demlová and Koubek [4].

For a set E of semigroup equations (i.e. without the involution symbol in their terms), let $\mathbf{R}(E)$ denote the subvariety of $\mathbf{R}$ determined by E. For a semigroup variety (without involution) $\mathbf{V}$, let $Eq(\mathbf{V})$ denote the set of equations satisfied in $\mathbf{V}$ (i.e. the identities of $\mathbf{V}$).

Now, the *expansion of V by regular involution* (shortly: the *expansion of V*) is the subvariety

$$\mathbf{R}(\mathbf{V}) = \mathbf{R}(Eq(\mathbf{V}))$$

of $\mathbf{R}$.

There is no simple way of recovering $\mathbf{V}$ from $\mathbf{R}(\mathbf{V})$ – certainly not by just forgetting the involution. We cannot even expect that the subclass of $\mathbf{V}$ thus obtained would generate $\mathbf{V}$. Still it is felt that in a case – an especially interesting one – when $\mathbf{V}$ is not universal (the universal varieties of semigroups were described by Koubek and Sichler [8]) and $\mathbf{R}(\mathbf{V})$ is, it is the semigroup variety $\mathbf{V}$ which is a major agent responsible for the universality of $\mathbf{R}(\mathbf{V})$. As if $\mathbf{V}$ possessed a sort of "hidden" univerality "revealed" by the added regular involution, the role of the latter being rather an accessory one.

In [7] we have obtained several equivalent characterizations of the universal expansions of *band varieties*, summed up in the following

Theorem 1.1. *For a variety* **V** *of bands the following are equivalent:*
(1) **R(V)** *is universal;*
(2) **R(V)** *is monoid universal;*
(3) **V** *contains either all left normal bands or all right normal bands or all rectangular bands;*
(4) **V** *contains a semigroup* S *which is neither a semilattice nor a left-zero semigroup, nor a right-zero semigroup.* ∎

Note that condition (3) specifies the minimal band varieties whose expansion is universal:

LNB – the variety of left normal bands, i.e. semigroups satisfying the identities $x^2 = x$, $xyz = xzy$;

RNB – the variety of right normal bands, i.e. semigroups satisfying the identities $x^2 = x$, $zyx = yzx$;

RCB – the variety of rectangular bands, i.e. semigroups satisfying the identities $x^2 = x$, $x = xyx$;

Adding to this list, for every prime p, the varieties

$\mathbf{LAB}_p$ – the variety of left elementary p-groups, where p is a prime, i.e. semigroups satisfying the identities $xzy = xyz$, $x^{p+1} = x$, $x^p y^p = x^p$;

$\mathbf{RAB}_p$ – the variety of right elementary p-groups, where p is a prime, i.e. semigroups satisfying the identities $zyx = yzx$, $x^{p+1} = x$, $y^p x^p = x^p$.

We are able to establish in this paper the following more general (we recall that an algebra A is *rigid* if $End(A)$ is trivial).

Theorem 1.2. *For a variety* **V** *of semigroups the following are equivalent:*
(1) **R(V)** *is universal;*
(2) **R(V)** *is monoid universal;*
(3) *for every group* G *there exists a regular semigroup* $S \in \mathbf{R}(\mathbf{V})$ *with* $End(S) \simeq G$*;*
(4) *there exists a non-singleton rigid regular semigroup* $S \in \mathbf{R}(\mathbf{V})$*;*
(5) **V** *contains one of the varieties* **LNB**, **RNB**, **RCB**, $\mathbf{LAB}_p$, $\mathbf{RAB}_p$ *where* p *is a prime;*
(6) **V** *contains a semigroup* S *such that for every* $s \in S$ *there exists* $t \in S$ *with* $s = sts$ *and* S *is neither an inverse semigroup nor a left-zero semigroup nor a right-zero semigroup.*

2. Constructions for universality of $\mathbf{R}(\mathbf{LAB}_p)$

In this chapter we prove that for every prime p, the varieties $\mathbf{R}(\mathbf{LAB}_p)$ and $\mathbf{R}(\mathbf{RAB}_p)$ are universal. Proof will be given only for the variety $\mathbf{R}(\mathbf{LAB}_p)$, for $\mathbf{R}(\mathbf{RAB}_p)$ the proof is dual. Like Pigozzi and Sichler [9], we would rather work with the category $\mathbf{P}(\mathbf{LAB}_p)$ extending $\mathbf{R}(\mathbf{LAB}_p)$ by permitting partial involution. For this reason, define the category $\mathbf{P}(\mathbf{LAB}_p)$ of left elementary p-groups with partial involution:

objects are pairs (S, f) where $S \in \mathbf{LAB}_p$ is a left elementary p-group, $f : S \to S$ is a partial mapping such that for every $s \in S$, if $f(s)$ is defined then $sf(s)s = s$, $f(f(s))$ is also defined and $f(f(s)) = s$,

morphisms from (S, f) *into* (T, g) are all semigroup homomorphisms $\chi : S \to T$ satisfying $\chi(f(s)) = g(\chi(s))$ for every $s \in S$ whenever the left side is defined.

It is obvious that $\mathbf{P}(\mathbf{LAB}_p)$ is a category and for every object (S, f) if f is a total mapping, then $(S, f) \in \mathbf{R}(\mathbf{LAB}_p)$. For (S, f), $(T, g) \in \mathbf{R}(\mathbf{LAB}_p)$, homomorphisms from (S, f) to (T, g) and morphisms of $\mathbf{P}(\mathbf{LAB}_p)$ from (S, f) to (T, g) coincide, thus $\mathbf{R}(\mathbf{LAB}_p)$ is a full subcategory of $\mathbf{P}(\mathbf{LAB}_p)$.

Denote by $\mathbf{GRA}$ the category of all undirected graphs without loops and isolated elements such that every edge belongs to a triangle.

First we construct a full embedding Φ from the category $\mathbf{GRA}$ into $\mathbf{P}(\mathbf{LAB}_p)$. Then we describe a reflection $\Lambda : \mathbf{P}(\mathbf{LAB}_p) \to \mathbf{R}(\mathbf{LAB}_p)$ and we show that $\Lambda \circ \Phi$ is a full embedding from $\mathbf{GRA}$ into $\mathbf{R}(\mathbf{LAB}_p)$. Since $\mathbf{GRA}$ is universal, see [5] or [10], the universality of $\mathbf{R}(\mathbf{LAB}_p)$ will be proved.

Let p be a prime. For a set X denote by $F(X)$ a free semigroup over the set X in the variety $\mathbf{AB}_p$ of semigroups determined by identities $xy = yx$, $x^p y = y$.

By [1], every semigroup $S \in \mathbf{LAB}_p$ can be represented as $S = G \times L$ for an elementary p-group G and a left-zero semigroup L. Then (G, L) is called a *canonical decomposition* of S. Every morphism $\chi : G \times L \to G' \times L'$ in $\mathbf{LAB}_p$ splits into $\chi = \varphi \times \psi$ for $\varphi : G \to G'$ and $\psi : L \to L'$.

Assume that $(S, f) \in \mathbf{P}(\mathbf{LAB}_p)$ and (G, L) is a canonical decomposition of S, then for every $(t, u) \in S$, if $f(t, u)$ is defined, then $f(t, u) = (t^{p-1}, u')$ for some $u' \in L$. Indeed, if $f(t, u) = (t', u')$, then $(t, u) = (t, u)f(t, u)(t, u) = (t, u)(t', u')(t, u) = (tt't, u)$ and hence $t' = t^{p-1}$.

For a graph $\Gamma = (V, E) \in \mathbf{GRA}$ denote V_i, $i \in 3$ three disjoint copies of V, the element of V_i corresponding to $v \in V$ is denoted by v_i. Set $X_\Gamma = \cup\{V_i \mid i \in 3\}$. Let $A = \{a, b, c\}$ be a three-element left-zero semigroup. Put $S_\Gamma = F(X_\Gamma) \times A$ then $S_\Gamma \in \mathbf{LAB}_p$. Define $f_\Gamma : S_\Gamma \to S_\Gamma$ as the smallest partial mapping with the property that $f_\Gamma(f_\Gamma(s)) = s$ whenever $f_\Gamma(s)$ is defined and satisfies:

(a) $f_\Gamma(1, a) = (1, b)$
(b) $f_\Gamma(v_0, a) = ((v_0)^{p-1}, b)$ for every $v \in V$
(c) $f_\Gamma(v_1, b) = ((v_1)^{p-1}, c)$ for every $v \in V$
(d) $f_\Gamma(v_2, c) = ((v_2)^{p-1}, a)$ for every $v \in V$
(e) $f_\Gamma(v_0 v_1 v_2, b) = ((v_0 v_1 v_2)^{p-1}, b)$ for every $v \in V$
(f) $f_\Gamma(v_1 w_1, c) = ((v_1 w_1)^{p-1}, a)$ for every $\{v, w\} \in E$
(g) $f_\Gamma(u_1 v_1 w_1, c) = ((u_1 v_1 w_1)^{p-1}, a)$ for every triangle $\{u, v, w\}$ in (V, E).

Clearly, (S_Γ, f_Γ) is an object of $\mathbf{P}(\mathbf{LAB}_p)$. Define $\Phi\Gamma = (S_\Gamma, f_\Gamma)$. For a compatible mapping $g : \Gamma \to \Gamma'$, where $\Gamma = (V, E)$, $\Gamma' = (V', E')$ denote by φ_g the homomorphism from $F(X_\Gamma)$ into $F(X_{\Gamma'})$ satisfying $\varphi_g(v_i) = (g(v))_i$ for every $i \in 3$ and $v \in V$. Let $\Phi g = \varphi_g \times \iota_A$ where ι_A is the identity of A. We immediately obtain that Φg is a homomorphism from S_Γ into $S_{\Gamma'}$, and because $\Phi g \circ f_\Gamma = f_{\Gamma'} \circ \Phi g$, we conclude that Φg is a morphism of $\mathbf{P}(\mathbf{LAB}_p)$. Thus

Proposition 2.1. $\Phi : \mathbf{GRA} \to \mathbf{P}(\mathbf{LAB}_p)$ *is an embedding.* ■

We prove that Φ is full. Assume that $\Gamma = (V, E)$, $\Gamma' = (V', E')$ are graphs from $\mathbf{GRA}$. Assume that $\chi : \Phi\Gamma \to \Phi\Gamma'$ is a homomorphism. Then there exist homomorphisms $\varphi : F(X_\Gamma) \to F(X_{\Gamma'})$, $\psi : A \to A$ such that $\chi(x, d) = (\varphi(x), \psi(d))$ for every $(x, d) \in F(X_\Gamma) \times A$. Since φ is a homomorphism we have $\varphi(1) = 1$.

Lemma 2.2. $\psi(b) = b$, $\psi(a) = a$. *Moreover, for every* $v \in V$, $\varphi(v_0) \in V'_0 \cup \{1\}$ *and* $\varphi(v_0 v_1 v_2) \in \{v'_0 v'_1 v'_2 \mid v' \in V'\} \cup \{(v'_0 v'_1 v'_2)^{p-1} \mid v' \in V'\}$.

Proof. If $x \in F(X_\Gamma)$, $d \in A$ and $f_\Gamma(x,d) = (y,d)$, then $d = b$ and $x \in \{v_0v_1v_2 \mid v \in V\} \cup \{(v_0v_1v_2)^{p-1} \mid v \in V\}$, therefore $\psi(b) = b$ and for every $v \in V$, $\varphi(v_0v_1v_2) \in \{v'_0v'_1v'_2 \mid v' \in V'\} \cup \{(v'_0v'_1v'_2)^{p-1} \mid v' \in V'\}$. Further, $f_\Gamma(1,b)$ is defined and, because of $\varphi(1) = 1$, we obtain $\psi(a) = a$. For every $v \in V$, $f_\Gamma(v_0,a) = (v_0^{p-1},b)$ and since $f_{\Gamma'}(x,a) = (x^{p-1},b)$ for $x \in F(X_\Gamma)$ if and only if $x \in V'_0 \cup \{1\}$ we conclude that for every $v \in V$, $\varphi(v_0) \in V'_0 \cup \{1\}$. ∎

Lemma 2.3. $\psi(c) = c$.

Proof. First assume that $\psi(c) = a$. Then for $v \in V$, there exists $x \in F(X_{\Gamma'})$ with $\chi(v_2,c) = (x,a)$. By Lemma 2.2, $\chi(v_2^{p-1},a) = (y,a)$ for some $y \in F(X_{\Gamma'})$ and thus $f_{\Gamma'}(x,a) = (y,a)$ – a contradiction with the definition of $f_{\Gamma'}$. Thus $\psi(c) \neq a$. Assume that $\psi(c) = b$. Since $f_\Gamma(v_1,b) = (v_1^{p-1},c)$ for every $v \in V$, we obtain $\varphi(v_1)$, $\varphi(v_1^{p-1}) \in \{v'_0v'_1v'_2 \mid v' \in V'\} \cup \{(v'_0v'_1v'_2)^{p-1} \mid v' \in V'\}$. Further, $f_\Gamma(v_1w_1,c) = ((v_1w_1)^{p-1},a)$ for every $\{v,w\} \in E$. Hence $\varphi(v_1w_1) \in \{(v'_0)^{p-1} \mid v' \in V'\} \cup \{1\}$ by Lemma 2.2. If we combine both facts, we obtain for every $\{v,w\} \in E$ that $\{\varphi(v_1),\varphi(w_1)\} = \{z'_0z'_1z'_2,(z'_0z'_1z'_2)^{p-1}\}$ for some $z' \in V'$ and $\varphi(v_1w_1) = 1$. Since (V,E) contains a triangle, there exist three edges $\{v,w\}$, $\{u,v\}$, $\{u,w\} \in E$. If $\{\varphi(v_1),\varphi(w_1)\} = \{z'_0z'_1z'_2,(z'_0z'_1z'_2)^{p-1}\}$ for some $z' \in V'$, then also $\{\varphi(v_1),\varphi(u_1)\} = \{z'_0z'_1z'_2,(z'_0z'_1z'_2)^{p-1}\}$. Hence $\varphi(u_1) = \varphi(w_1)$ and thus $z'_0z'_1z'_2 = (z'_0z'_1z'_2)^{p-1}$, therefore $p = 2$. In this case $\varphi(u_1v_1w_1) = z'_0z'_1z'_2$ is a contradiction to $\varphi(u_1v_1w_1) \in \{(v'_0)^{p-1} \mid v' \in V'\} \cup \{1\}$ because $f_\Gamma(u_1v_1w_1,c) = ((u_1v_1w_1)^{p-1},a)$. Hence $\psi(c) \neq b$ and thus $\phi(c) = c$. ∎

Lemma 2.4. *For every $v \in V$ there exists $v' \in V'$ such that for every $i \in 3$, $\varphi(v_i) = v'_i$.*

Proof. Since $\chi \circ f_\Gamma(x) = f_{\Gamma'} \circ \chi(x)$ whenever $f_\Gamma(x)$ is defined, we conclude by Lemmas 2.2 and 2.3 and (c) that $\varphi(V_1) \subseteq V'_1$. This and (d), (f), (g) imply $\varphi(V_2) \subseteq V'_2$ and according to Lemma 2.2, we obtain $\varphi(V_0) \subseteq (V'_0)$. Taking the second half of Lemma 2.2, we obtain that $\varphi(v_0) = v'_0$ implies $\varphi(v_1) = v'_1$ and $\varphi(v_2) = v'_2$ for every $v \in V$. ∎

Define $g : V \to V'$ such that $\varphi(v_0) = (g(v))_0$.

Lemma 2.5. *g is a compatible mapping from (V,E) into (V',E').*

Proof. For $\{v,w\} \in E$, we have $f_{\Gamma'}(g(v)_1g(w)_1,c) = f_{\Gamma'}(\chi(v_1w_1,c)) = \chi(f_\Gamma(v_1w_1,c)) = \chi((v_1w_1)^{p-1},a) = ((g(v)_1g(w)_1)^{p-1},a)$ and by the definition of $f_{\Gamma'}$, we conclude that $\{g(v),g(w)\} \in E'$. ∎

Theorem 2.6. *$\Phi : \mathbf{GRA} \to \mathbf{P}(\mathbf{LAB}_p)$ is a full embedding. Thus $\mathbf{P}(\mathbf{LAB}_p)$ is universal.*

Proof. We prove that $\Phi g = \chi$. Since $\varphi(v_i) = (g(v))_i$ for every $v \in V$, $i \in 3$ and since $\chi = (\varphi \times \iota_A)$ where ι_A is the identity of A, we obtain $\Phi g = \chi$. Now Proposition 2.1 completes the proof. ∎

The second step contains a description of a free completion of left elementary p-groups with partial involution. A regular semigroup $(S,^+) \in \mathbf{R}(\mathbf{LAB}_p)$ is a *free completion* of a left elementary p-group (T,f) with a partial involution if there exists an injective morphism $\mu : (T,f) \to (S,^+)$ in $\mathbf{P}(\mathbf{LAB}_p)$ such that for every morphism $\sigma : (T,f) \to (S',^+)$ in $\mathbf{P}(\mathbf{LAB}_p)$ where $(S',^+) \in \mathbf{R}(\mathbf{LAB}_p)$, there exists a unique homomorphism $\chi : (S,^+) \to (S',^+)$ with $\sigma = \chi \circ \mu$.

Our aim is to prove that every left elementary p-group with partial involution has a free completion and to investigate properties of the free completion. Analogous results were obtained by Pigozzi and Sichler [9] for Steiner triples and Koubek [7] for rectangular bands with partial involution.

Lemma 2.7. *For every left elementary p-group (S,f) with partial involution, there exists a left elementary p-group (T,g) with partial involution such that the following hold:*

1. *if (G_S, L_S) and (G_T, L_T) are the canonical decompositions of S and T, then $G_S = G_T$ and $L_S \subseteq L_T$;*
2. *the inclusion $\iota : S \to T$ is a morphism from (S,f) to (T,g) in $\mathbf{P}(\mathbf{LAB}_p)$;*
3. *for every $s \in S$, $g(s)$ is defined;*
4. *for every morphism $\chi : (S,f) \to (S',^+)$ in $\mathbf{P}(\mathbf{LAB}_p)$ where $(S',^+) \in \mathbf{R}(\mathbf{LAB}_p)$ there exists exactly one morphism $\sigma : (T,g) \to (S',^+)$ in $\mathbf{P}(\mathbf{LAB}_p)$ with $\chi = \sigma \circ \iota$.*

Moreover, if $u_i \in G_T$, $i \in 3$, $z_j \in L_T$, $j \in 3$ with $g(u_i, z_i) = (u_i^{p-1}, z_{i+1})$ for $i \in 3$ (where the addition is taken modulo 3), then $z_j \in L_S$ for every $j \in 3$.

Proof. Set $B = \{(u,v) \mid u \in G_S, v \in L_S, f(u,v) \text{ is not defined}\}$. Without any loss of generality we can assume that $B \cap L_S = \emptyset$ and set $L_T = B \cup L_S$. Let $T = G_S \times L_T$. Define a partial mapping $g : T \to T$ such that for every $s \in S$, $g(s) = f(s)$ whenever $f(s)$ is defined. Further, if $(u,v) \in S$ and $f(u,v)$ is not defined, then $g(u,v) = (u^{p-1},(u,v))$, $g(u^{p-1},(u,v)) = (u,v)$. Denote by $\iota : S \to T$ the inclusion. Clearly, (1), (2), and (3) hold, and it suffices to prove (4), the rest is clear. Let $\chi : (S,f) \to (S',^+)$ be a morphism of $\mathbf{P}(\mathbf{LAB}_p)$ such that $(S',^+) \in \mathbf{R}(\mathbf{LAB}_p)$. Assume that $(G_{S'}, L_{S'})$ is a canonical decomposition of S', then there exist homomorphisms $\varphi : G_S \to G_{S'}$, $\psi : L_S \to L_{S'}$ such that $\chi = \varphi \times \psi$. Define $\psi_1 : L_T \to L_{S'}$ as follows: for $u \in L_S$, $\psi_1(u) = \psi(u)$, for $(u,v) \in B$, if $(\chi(u,v))^+ = (x,y)$, then $\psi_1(u,v) = y$. Set $\sigma = \varphi \circ \psi_1$. Clearly, $\chi = \sigma \circ \iota$ and $\sigma : T \to S'$ is a semigroup homomorphism. By a direct inspection we obtain for every $t \in T$ that $\sigma(g(t)) = \sigma(t)^+$ whenever $g(t)$ is defined. Clearly, σ is unique because (S,f) generates (T,g). ∎

Theorem 2.8. *Every left elementary p-group (S,f) with partial involution has a free completion $(T,^+)$. Moreover,*

1. *if (G_S, L_S) and (G_T, L_T) are the canonical decompositions of S and T, then $G_S = G_T$ and $L_S \subseteq L_T$;*
2. *if for $i \in 3$, $u_i \in G_T$, $z_i \in L_T$ with $(u_i, z_i)^+ = (u_i^{p-1}, z_{i+1})$, then $z_i \in L_S$ for every $i \in 3$.*

Proof. Set $(S_0, f_0) = (S,f)$ and for every positive integer n, let (S_n, f_n) be the left elementary p-group with partial involution given by Lemma 2.7 for the left elementary p-group (S_{n-1}, f_{n-1}) with partial involution. Set $T = \cup\{S_n \mid n \text{ is a natural number}\}$, and for $t \in T$, if $t \in S_n$, then $t^+ = f_{n+1}(t)$. By a direct inspection we obtain that $(T,^+) \in \mathbf{R}(\mathbf{LAB}_p)$ and from Lemma 2.7 we immediately obtain that $(T,^+)$ is a free completion of (S,f). The properties of (1) and (2) follow from (1) of Lemma 2.7 and the last statement of Lemma 2.7. ∎

For a left elementary p-group (S,f) with partial involution, denote by $\Lambda(S,f)$ a free completion of (S,f). For a morphism $\chi : (S,f) \to (T,g)$, by a universal property of a free extension, there exists a unique homomorphism $\Lambda\chi : \Lambda(S,f) \to \Lambda(T,g)$ such that $\iota \circ \chi = \Lambda\chi \circ \theta$ where $\iota : (T,g) \to \Lambda(T,g)$, $\theta : (S,f) \to \Lambda(S,f)$ are the inclusions. Then $\Lambda : \mathbf{P}(\mathbf{LAB}_p) \to \mathbf{R}(\mathbf{LAB}_p)$ is a reflection functor.

Theorem 2.9. $\Lambda \circ \Phi : \mathbf{GRA} \to \mathbf{R}(\mathbf{LAB}_p)$ *is a full embedding.*

Proof. From the definition of Λ, we immediately obtain that $\Lambda \circ \Phi$ is an embedding. To prove that $\Lambda \circ \Phi$ is full, assume that $\Gamma = (V,E)$, $\Gamma' = (V',E')$ are graphs from **GRA** and that $\chi : \Lambda \circ \Phi\Gamma \to \Lambda \circ \Phi\Gamma'$ is a homomorphism. If (G_Γ, L_Γ) and $(G_{\Gamma'}, L_{\Gamma'})$ are canonical decompositions of $\Lambda \circ \Phi\Gamma$ and $\Lambda \circ \Phi\Gamma'$, then $G_\Gamma = F(X_\Gamma)$, $G_{\Gamma'} = F(X_{\Gamma'})$, $A \subseteq L_\Gamma, L_{\Gamma'}$ and there exist homomorphisms $\varphi : F(X_\Gamma) \to F(X_{\Gamma'})$, $\psi : L_\Gamma \to L_{\Gamma'}$ with $\chi = \varphi \times \psi$. By Theorem 2.8 and by the definition of Φ we have for $w_i \in L_\Gamma$, $i \in 3$ that $\{w_i \mid i \in 3\} = A$ if and only if there exist $u_j \in F(X_\Gamma)$, $j \in 3$ with $(u_i, w_i)^+ = (u_i^{p-1}, w_{i+1})$, $i \in 3$. Since this property is perserved by homomorphisms, we obtain that $\psi(A) = A$ and thus $\chi(\Phi\Gamma) \subseteq \Phi\Gamma'$. According to Theorem 2.6, there exists a compatible mapping $g : \Gamma \to \Gamma'$ such that the domain-range restriction of χ to $\Phi\Gamma$ and $\Phi\Gamma'$ is equal to Φg. Since $\Lambda \circ \Phi\Gamma$ is generated by $\Phi\Gamma$, and χ and $\Lambda \circ \Phi g$ coincide on $\Phi\Gamma$, we obtain that $\chi = \Lambda \circ \Phi g$. Hence $\Lambda \circ \Phi$ is full. ∎

Corollary 2.10. *For every prime p, the varieties* $\mathbf{R}(\mathbf{LAB}_p)$ *and* $\mathbf{R}(\mathbf{RAB}_p)$ *are universal.*

Proof. Since **GRA** is universal, the proof follows from Theorem 2.9 and its dual. ∎

3. Main Result

We prove Theorem 1.2. First (1) $\Rightarrow$ (2), see [10]. The implication (2) $\Rightarrow$ (3) is clear. By Hedrlín-Sichler Theorem [6] or [10], we obtain (1) $\Rightarrow$ (4). We prove that (3) $\Rightarrow$ (6) and (4) $\Rightarrow$ (6). For this reason we recall the following easy lemma:

Lemma 3.1. *The following hold:*

(1) *If S is an inverse semigroup, then it has a unique regular involution $^+$ and there exists a constant endomorphism of $(S,^+)$.*

(2) *If S is either a left-zero semigroup or a right-zero semigroup, then for every regular semigroup $(S,^+)$ either a constant mapping is an endomorphism of $(S,^+)$ or there exists a non-idempotent endomorphism of $(S,^+)$ with $|Im(f)| = 2$.*

Proof. We prove (1). By [1], paragraph 2.3, there exists an idempotent $s \in S$. Since S is inverse, $s^+ = s$, thus a constant mapping $f : S \to S$ to s is an endomorphism of $(S,^+)$. We prove (2). Let S be a left-zero semigroup. If there exists $s \in S$ with $s^+ = s$, then a constant mapping $f : S \to S$ to s is an endomorphism of $(S,^+)$. Assume that for every $s \in S$, $s^+ \neq s$. Choose $s \in S$; then there exists a mapping $f : S \to S$ such that $Im(f) = \{s, s^+\}$, $f(s) \neq s$, and $f(t^+) = f(t)^+$ for every $t \in S$ - it suffices for every set $\{t, t^+\} \neq \{s, s^+\}$ to choose a representative u and to set $f(u) = s$, $f(u^+) = s^+$, and $f(s) = s^+$, $f(s^+) = s$. Then f is a non-idempotent endomorphism of $(S,^+)$. If S is a right-zero semigroup, then the proof is dual. ∎

Lemma 3.1 implies (3) $\Rightarrow$ (6) and (4) $\Rightarrow$ (6). We show (6) $\Rightarrow$ (5).

Lemma 3.2. *Let S be a semigroup which is neither an inverse semigroup nor a left-zero semigroup nor a right-zero semigroup and such that for every $s \in S$ there exists $t \in S$ with $s = sts$. Then the variety* $\mathbf{V}$ *generated by S contains either* **LNB** *or* **RNB** *or* **RCB** *or* $\mathbf{LAB}_p$ *or* $\mathbf{RAB}_p$ *for a prime p.*

Proof. If for every $s \in S$ there exists $t \in T$ with $s = sts$, then by [1] every left Green class and every right Green class contains an idempotent, see [1], paragraph 2.3. Since S is not an inverse semigroup, there exists either a left Green class of a right Green class containing two idempotents and hence $\mathbf{V}$ contains either a two-element left-zero semigroup or a two-element right-zero semigroup, again by [1], paragraph 2.3. Whence either every left-zero semigroup or every right-zero semigroup belongs to $\mathbf{V}$. Therefore, if $\mathbf{V}$ contains a two-element semilattice, then either $\mathbf{LNB} \subseteq \mathbf{V}$ or $\mathbf{RNB} \subseteq \mathbf{V}$. Since for every variety $\mathbf{W}$ of semigroups either $\mathbf{W}$ contains all commutative semigroups or every semigroup in $\mathbf{W}$ is periodical, and since $\mathbf{V}$ contains a two-element semilattice whenever S is not simple, we can assume that S is simple and periodical, thus S is completely simple. In this case, obviously, either $\mathbf{LAB}_p \subseteq \mathbf{V}$, or $\mathbf{RAB}_p \subseteq \mathbf{V}$ for some prime p, or $\mathbf{RCB} \subseteq \mathbf{V}$. ∎

The implication (6) $\Rightarrow$ (5) follows from Lemma 3.2. The implication (5) $\Rightarrow$ (1) follows from Theorem 1.1 and Corollary 2.10. The proof of Theorem 1.2 is complete. ∎

At the end we like to recall two open problems: A variety $\mathbf{V}$ is called finite-to-finite universal if there exists a full embedding from **GRA** into $\mathbf{V}$ taking finite graphs into finite algebras in $\mathbf{V}$. It is an open question whether varieties $\mathbf{R}(\mathbf{RCB})$, $\mathbf{R}(\mathbf{LAB}_p)$, and $\mathbf{R}(\mathbf{RAB}_p)$ are finite-to-finite universal. The second open problem is a characterization of all universal varieties of regular semigroups.

References

[1] Clifford, A.H. and G.B. Preston, "The Algebraic Theory of Semigroups", Amer. Math. Soc., Providence, R.I., 1961.

[2] Demlová, M. and V. Koubek, *Minimal congruences and coextensions in semigroups*, in "Semigroup Theory and Applications," Lecture Notes in Mathematics 1320, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo 1988, 24–83.

[3] Demlová, M. and V. Koubek, *Endomorphism monoids of bands*, Semigroup Forum **38** (1989), 305–329.

[4] Demlová, M. and V. Koubek, *Endomorphism monoids in small varieties of bands*, to appear in Acta Sci. Math. (Szeged) **54** (1992).

[5] Hedrlín, Z. and A. Pultr, *On full embeddings of categories of algebras*, Illinois J. Math. **10** (1966), 392–406.

[6] Hedrlín, Z. and J. Sichler, *Any boundable binding category contains a proper class of mutually disjoint copies of itself*, Algebra Universalis **1** (1971), 97–103.

[7] Koubek, V. *Representations in varieties of regular involution bands*, Comment. Math. Univ. Carolinae **31** (1990), 201–211.

[8] Koubek, V. and J. Sichler, *Universal varieties of semigroups*, J. Austral. Math. Soc. (Series A) **36** (1984), 143–152.

[9] Pigozzi, D. and J. Sichler, *Homomorphisms of partial and of complete Steiner triple systems and quasigroups*, in "Universal Algebra and Lattice Theory," Lecture Notes in Mathematics 1149, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985, 224–237.

[10] Pultr, A. and V. Trnková, "Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories," North Holland, Amsterdam, 1980.

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