BOOK REVIEW

Techniques of Semigroup Theory
by Peter M. Higgins
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Boris M. Schein

Communicated by Gerard J. Lallement

The first book ever on the theory of semigroups was the monograph [1] of A. K. Suschkewitsch[1] published in 1937. The book contained virtually everything known on semigroups (and many things on quasigroups, loops, and other algebraic systems) at that time. In fact, it covered more than was known, for there were incisive remarks, guesses and outlines of possible research and now, with our perfect hindsight, we appreciate their almost uncanny precision. Many remarks in that book were developed into full-blown branches of theories of semigroups, quasigroups and loops decades later.

Yet when I think of the first monograph in the general theory of semigroups, I think of the book of Lyapin [2] that was published in 1960 (the first edition of its English translation appeared in 1963). Indeed, it is difficult to consider the book of Suschkewitsch as the first in the theory of semigroups, if for no other reason than because it was published too early—before the theory of semigroups came into existence. Moreover, the book was almost unavailable, even in the USSR[2] In contrast, by the end of the fifties, when Lyapin's book appeared, the theory of semigroups had emerged on the algebraic horizon, and the second generation of semigroup theorists, the students of the pioneers, was already active. Even now, more than a third of a century after the event, I remember the thrill produced by the book: we were coming of age! It was almost encyclopedical. It contained everything in the algebraic theory of semigroups, and the list of references included everything published in the world by that time, with very few exceptions. The book was very influential. In particular, it helped coalesce the semigroup community in its country, established terminology, outlined the main directions of research existing in the fifties and did many other useful things.

In 1961 volume I of Clifford and Preston's book [3] appeared, with volume II following in 1967. It (especially volume II) contained a wealth of new results that had become known after Lyapin's book was written. Also, because of geopolitical realities, the book of Clifford and Preston had a much wider circulation than the book of Lyapin and was even more influential.

After that, books in the general algebraic theory of semigroups appeared more or less regularly and the seventies brought a record crop. I do not touch

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[1] Here I follow a German transliteration of the name used by Suschkewitsch in some of his articles; other transliterations were used occasionally. Likewise, Lyapin's name used to be transliterated as "Ljapin" by the American Mathematical Society.

[2] The edition was kept in Kharkov, a city that, during the WW II, changed hands four times, each time with considerable hostilities and fires. Only those few copies of the book that were kept elsewhere survived destruction.

upon books in topological semigroups, semigroups of operators, and on books devoted to special topics in the algebraic theory of semigroups, although there is no shortage of these. I mention here only the books that belong to the general theory. The next book was the monograph [4] of Tamura, that appeared in 1972. In 1973 the book [5] of Petrich was published. Its second volume [6] was written but, for various reasons, appeared only in 1977 under a different title. 1976 saw two new books: Yamada [7] and Howie [8]. In 1977, besides the book by Petrich [6], there appeared a volume [9] by Creangă and Simovici and, in 1979, the book [10] of Lallement was published.

Then there came a lull and we have had to wait 13 years for a new monograph. Not that the research in semigroup theory has slown down — there is more than a score of books dedicated to specific aspects of the algebraic theory of semigroups. Suffice it to say that only in the area of inverse semigroups there appeared a volume almost 800 pages long. Many other books contained important and nontrivial contributions to semigroups. Also, more and more collections of semigroup papers have started to appear as separate books. After the first collection [11] was published in Russia in 1965, they proliferated, becoming more and more common. Usually (but not always) they were proceedings of semigroup meetings, both national and international.[3] The first international conference on semigroups was organized by Štefan Schwarz in 1968 in Smolenice, Czechoslovakia (now Slovakia), it was influential in many ways and, in particular, finalized previous tentative contacts between the semigroup theorists from different countries concerning the first international journal dedicated to semigroups. After that, meetings in semigroups gathered more frequently (in fact, so frequently, that there were overlaps, or, in just one year, you might be tempted to take part in conferences organized in all four — and more! — corners of the world). Proceedings of many of these conferences were published in various places.

Anyway, finally there is a new monograph devoted to the general algebraic theory of semigroups by Peter Higgins. What are its distinctive features?

Now, when the world literature in semigroups boasts tens of thousands of titles, it is impossible to survey all of them in one book, as was done in the book of Lyapin. The author has to make a judicious choice. Here is a brief list of contents of the book.

It begins with a foreword written by G. B. Preston and consists of six chapters. Chapter 1 "Fundamentals" is the longest. It contains the main definitions and facts. For example, the definitions of semigroups and monoids, homomorphisms, rectangular bands, semilattices, representations by transformations, the full transformation semigroups, one- and two-sided ideals, monogenic semigroups and their structure, group-bound, regular, orthodox, completely regular and inverse semigroups (and representations of inverse semigroups) as well as semigroups of partial transformations and binary relations, congruences, standard "isomorphism" theorems and very much more are introduced in 11 pages of the "Introduction". Already this shows that this is not a book for a beginner. Chapter 1 is also dedicated to Green's relations, descending and ascending chain conditions, the structure of completely 0-simple and completely regular semigroups, various properties of regularity (including orders and quasi orders, biordered sets and corresponding results of Nambooripad and and others. A dis-

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[3] For example, the first semigroup conference in the then USSR was organized in 1966 in Kääriku, Estonia by Eugen Gabovich and Jaak Hion. For diplomatic reasons it was called a "symposium in general algebra". However, in 1969, a conference in semigroups, organized by L. N. Shevrin and others in Sverdlovsk (now Yekaterinburg), ran under its own name.

tinctive feature of this book is a substantial use of graph-theoretical approaches and various semigroup diagrams, so the corresponding concepts are introduced in the first chapter too.

The contents of Chapter 2 are clear from its title "Free inverse semigroups and the theorems of McAlister." Of the three existing descriptions of elements of free inverse semigroups the author considers Munn's "birooted trees," deducing Scheiblich's original description as a corollary. That leads him naturally to P-semigroups and "P-theorems" of McAlister, although the proofs given in the book are different from the original long proofs of these theorems.

Chapter 3 is entitled "Biordered sets" and considers an axiomatic system for biordered sets, problems of reconstructing some semibands (that is, the semigroups generated by their idempotents) from their biordered sets, and biordered sets of bands, completely regular semibands, and completely regular semigroups.

Chapter 4 "Zigzags and their applications" contains the zigzag theorem (the author suggests a new and short proof of it) as well as a variety of applications concerning epimorphisms and semigroup amalgams. Among them there are characterizations of various classes of right inner, flat and saturated semigroups.

Chapter 5 is called "Semigroup diagrams and word problems." It shows how diagrammatic methods can be used to solve special semigroup word problems. For example, there is a proof of the existence of an algorithm solving the word problem for finitely presented semigroups the defining relations of which satisfy a version of a small overlap condition. Group diagrams of combinatorial group theory are considered as well.

Finally, Chapter 6 "Combinatorial aspects of transformation semigroups" is the second longest in the book. It contains a variety of materials. Section 6.1 is devoted to probabilistic features of the full transformation semigroup on a finite set and the semigroup of all order-preserving mappings of a finite chain. Typical results describe how quickly random products of transformations tend to collapse into lower $\mathcal{D}$-classes, or what is the number of components of a randomly chosen transformation, etc. Section 6.2 considers solutions of certain equations in finite full transformation semigroups (e.g., which transformation have square roots and how to find these roots, which transformations belong to inverse subsemigroups of the full transformation semigroup, etc.) The last Section 6.3 is devoted to products of idempotents in a finite full transformation semigroup.

In a recent talk Karl Hofmann mused over "ghetto existence of the semigroup theory." I would like to add that you are not driven to ghetto by your own choice: usually it is your external "protectors" who see to that. However, although it may sound paradoxical, the "semigroup ghetto" is created to a large extent by the semigroup theorists. Every new field of mathematics had to overcome a good deal of prejudice, and semigroup theory is no exception. Prejudices about semigroups do tend to disappear, yet that process has been taking many years, and I feel that the semigroup community has been doing too little to hasten the process. That harms both the theory of semigroups and mathematics at large. By the twenties and thirties, after the theory of groups (or at least the theory of finite groups) overcame prejudices and became an integral part of mathematics, there sprung numerous imitations generalizing groups (see [1]). Almost all of them disappeared. Sometimes the theory of semigroups is identified with those "generalizations for generalization's sake." Yet our subject is not merely a "generalization," and we did not disappear, because semigroups strike vital concepts (e.g., composition of mappings and concatenation of strings of symbols). Neither groups, nor rings, nor other algebraic systems are a fully adequate vehicle for these fundamental concepts of mathematics. The original

motivation for the theory of semigroups as well as all of its pioneers, except Suschkewitsch, came from other parts of mathematics, a fact that we, their students, tend to forget. Not only did we not disappear, but, quite the contrary, we overcame the danger of premature death in early childhood, grew, proved our viability, and enriched mathematics, no matter how well or poorly we were funded and supported. We even have imitators. We have seen a remarkable use of semigroup techniques, approaches and methods that have permitted elegant and short proofs of results, sometimes classical, from other fields of mathematics. Yet more often than not specialists in semigroups did not show much interest in applications of semigroup methods to other fields nor in problems emerging in other fields that led to natural questions in the theory of semigroups.

This is why I specifically want to applaud chapter 6 of the book, which applies — quite successfully — some of the semigroup techniques developed earlier to produce very satisfactory solutions to a few selected problems that can interest specialists in other fields (particularly in mathematical genetics).[4] Of the semigroup books listed in the references, only [10] gives a good deal of attention to applications of semigroups to other fields.

While discussing various results, the author gives references and also mentions some of the papers not used in the book, which are on the same or similar subjects. Therefore, the list of references is fairly long, it contains more than 300 books and papers. The book ends with a glossary of notation and an index. There are not many misprints, a list of them can be obtained from the author (his e-mail address is peteh@essex.ac.uk).

It is clear from the short description of chapters that the book is not intended as an introduction to the subject for a novice. Its goal is different. It presents some of the recently developed ideas and techniques that have not been available in a book form (or were not available in a single volume). Of course, the choice reflects personal predilections of the author, but that is true of any book, except probably [1] and [2]. Besides, at the present stage of the theory of semigroups, it is impossible to include *everything* in one book.

There are 170 exercises placed in the end of the sections. They contain new results not included in the main body. Also, they can help the reader to understand the material. Therefore, the book may be used as a text for sufficiently advanced graduate students specializing in the semigroup theory.

Having said all that, I approach the crucial questions. Is another book on the algebraic theory of semigroups really needed at this time? Does the book of Higgins say something sufficiently new compared with the extant books? In my opinion, both questions are to be answered positively.

The book is written with clarity, the proofs and arguments are usually presented in a brief and elegant way, and even familiar things are often viewed

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[4] The author attributes the work on the square roots in finite transformation semigroups to semigroup theorists (1982–88). However, the solution—both for the finite and infinite cases — appeared in [12] in 1950. Apparently, the author was not aware that this problem (for arbitrary and not only square roots of functions) was first raised by Charles Babbage, who considered a special case of it in 1815. There exist many papers on this topic for various classes of functions, and Jacques Hadamard and Karl Menger are only two of the researchers who were interested in the problem (a real name-dropper might have mentioned Niels Henrik Abel whose work was instrumental for finding the n-th roots of certain functions (those which possess the Abel function)). In general functional equations with compositions of unknown functions arise from numerous problems coming from geometry, algebra, or functional analysis. This is a rich source of problems where semigroup techniques may be helpful.

from a fresh perspective. Although the pace is fast, this enviably crisp and clear exposition is a pleasure to read.

The book has few overlaps with other semigroup books. It is relatively short and concise, and yet it contains much interesting material, quite a few of them at the forefront of the contemporary research. It will be very useful for semigroup theorists and graduate students as well as for specialists in other fields who may use it as a reference. Altogether, this is a most welcome addition to the available monographs on semigroups.

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Department of Mathematical Sciences
University of Arkansas, SCEN-307
Fayetteville, Arkansas 72701-1202, USA
e-mail: bschein@uafsysb.uark.edu