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RESEARCH ARTICLE

A Hille-Yosida Theorem for weakly continuous semigroups

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Abstract

We introduce a new class of weakly continuous semigroups and give a characterization of their infinitesimal generators, generalizing the classical Hille-Yosida Theorem for strongly continuous semigroups. The results are illustrated by the example of transition semigroups corresponding to the solutions of certain stochastic differential equations.

1. Introduction

Let X be a Banach space and let $A : D(A) \subset X \longrightarrow X$ be a closed linear operator. As well known, the Hille - Yosida Theorem gives the necessary and sufficient conditions in order that A generates a strongly continuous semigroup e^{tA} on X. Such conditions are:

1. there exist $M, \omega \in \mathbb{R}$ such that, if $\lambda > \omega$,

$$R(\lambda, A) = (\lambda - A)^{-1} \in \mathcal{L}(X) \ \text{ and } \ \|R^k(\lambda, A)\| \leq \frac{M}{(\lambda - \omega)^k} \ \ \forall\, k \in \mathbb{N},$$

2. $D(A)$ is dense in X.

The aim of this paper is to examine the case in which $D(A)$ is not dense. This problem has been considered in several papers (see [2] and [3]) dealing with the non homogeneous initial value problem

$$\begin{cases} u'(t) = A(t)u(t) + f(t) \\ \\ u(0) = x, \end{cases} \tag{1}$$

in the autonomous [2] and non autonomous [3] case. Some results on existence of solutions for (1) have been proved under suitable hypotheses, for any X-valued continuous function f and any $x \in \overline{D(A)}$.

It is easy to see that if hypothesis 1 is satisfied but $D(A)$ is not dense, then $Y = \overline{D(A)}$ is an e^{tA}-invariant closed subspace of X, for every $t \geq 0$, $e^{tA}_{|Y}$ is strongly continuous on Y and the part of A on Y is its infinitesimal generator. Our purpose is to construct a semigroup of bounded linear operators e^{tA} associated to certain operators A satisfying 1 but not 2, such that

1. e^{tA} is continuous in a weaker sense, to be specified later.

2. $R(\lambda, A)$ is the Laplace transform of e^{tA}, in a suitable sense.

Jefferies has dealt with this problem in [4] and in [5], introducing a class of weakly integrable semigroups on locally convex spaces, and giving a characterization of their infinitesimal generators. But the approach followed in his articles, where he uses weak integration and weaker topologies for the underlying spaces, seems not to be suited for the study of transition semigroups corresponding to the solutions of certain stochastic differential equations (see [1]).

To fill this gap, we introduce a class of semigroups of bounded linear operators on $UC_b(H)$ (the Banach space of all uniformly continuous and bounded functions from the separable Hilbert space H to $\mathbb{R}$) which is well adapted to the concrete cases studied in [1]. Infinitesimal generators are defined, as in Jefferies, in terms of resolvents instead of the usual definition by differentiation; this approach allows us to prove some properties of differentiability for weakly continuous semigroups. We also give a characterization of such generators and conclude giving the example of a weakly continuous semigroup which arises in studying Kolmogorov equations in Hilbert spaces.

2. Weakly continuous semigroups

Troughout this paper H will denote a separable Hilbert space endowed with the scalar product $< \cdot, \cdot >$ and the norm $\|\cdot\|$. We will use $UC_b(H)$ to indicate the Banach space of all uniformly continuous and bounded functions from H to $\mathbb{R}$, where the usual norm is given

$$\|\varphi\|_\infty = \sup_{x \in H} |\varphi(x)|, \quad \forall\, \varphi \in UC_b(H).$$

$\mathcal{L}(UC_b(H))$ will denote the space of all bounded linear operators on $UC_b(H)$.

Definition 2.1. A semigroup of bounded linear operators $\{\, S(t) \mid t \in [0, +\infty)\,\}$ defined on $UC_b(H)$ is said to be *weakly continuous (of negative type)* if there exist $M, \omega > 0$ such that

1. The family of functions in $UC_b(H)$

$$\{\, S(t)\varphi \,:\, t \in [0, +\infty)\,\}$$

is equi-uniformly continuous, for every $\varphi \in UC_b(H)$.

2. For every $\varphi \in UC_b(H)$ and for every compact set $K \subset H$, it holds

$$\lim_{t \to 0} \sup_{x \in K} |S(t)\varphi(x) - \varphi(x)| = 0. \tag{2}$$

3. For every $\varphi \in UC_b(H)$ and for every sequence $\{\varphi_j\} \subset UC_b(H)$ such that

$$\begin{cases} \sup_{j \in \mathbb{N}} \|\varphi_j\|_\infty < +\infty \\ \\ \lim_{j \to +\infty} \sup_{x \in K} |\varphi_j(x) - \varphi(x)| = 0, \quad \forall\, K \subset H \text{ compact set,} \end{cases} \tag{3}$$

it holds

$$\lim_{j \to +\infty} \sup_{x \in K} |S(t)\varphi_j(x) - S(t)\varphi(x)| = 0, \tag{4}$$

for every compact set $K \subset H$. Furthermore the limit is uniform in t.

4. We have

$$\|S(t)\|_{\mathcal{L}(UC_b(H))} \leq Me^{-\omega t}, \quad \forall\, t \geq 0. \tag{5}$$

We denote by $\mathcal{G}_w(M,\omega)$ the set of all weakly continuous semigroups $S(\cdot)$ satisfying (1)-(4) above.

3. Infinitesimal generator

Proposition 3.1. *Let $S(\cdot)$ be a weakly continuous semigroup in $\mathcal{G}_w(M,\omega)$. Define*

$$F(\lambda)\varphi(x) = \int_0^{+\infty} e^{-\lambda t} S(t)\varphi(x)\, dt, \quad \varphi \in UC_b(H), \quad x \in H. \tag{6}$$

Then we have that $F(\lambda) \in \mathcal{L}(UC_b(H))$, $\forall\, \lambda > -\omega$.

Proof. From weak continuity of $S(\cdot)$ it follows that the function

$$[0,+\infty) \longrightarrow \mathbb{R}, \quad t \longrightarrow e^{-\lambda t} S(t)\varphi(x)$$

is continuous $\forall\, \varphi \in UC_b(H)$ and $\forall\, x \in H$; moreover for every $x \in H$ we have

$$|\, S(t)\varphi(x) \,| \leq Me^{-\omega t}\|\varphi\|_\infty, \quad \varphi \in UC_b(H).$$

Then the above definition of $F(\lambda)$ is meaningful.

If $\lambda > -\omega$ it follows

$$|\, F(\lambda)\varphi(x) \,| \leq \int_0^{+\infty} |\, e^{-\lambda t} \,||\, S(t)\varphi(x) \,|\, dt \leq M\|\varphi\|_\infty \int_0^{+\infty} e^{-(\lambda+\omega)t}\, dt = \frac{M}{\lambda+\omega}\|\varphi\|_\infty.$$

Let us fix $\epsilon > 0$. Since $\lambda + \omega > 0$ we can find $T > 0$ such that

$$\left(\frac{2M\|\varphi\|_\infty}{\lambda+\omega}\right) e^{-(\lambda+\omega)T} < \frac{\epsilon}{2}.$$

Now, we recall that the family of functions in $UC_b(H)$

$$\{\, S(t)\varphi \,:\, t \in [0,+\infty) \,\}$$

is equi-uniformly continuous $\forall\, \varphi \in UC_b(H)$. Then there exists $\delta_\epsilon > 0$ such that

$$\|y\| \leq \delta_\epsilon \Rightarrow$$

$$|S(t)\varphi(x+y) - S(t)\varphi(x)| \leq \frac{\epsilon}{2}\left(\frac{\lambda}{1-e^{-\lambda T}}\right), \quad \forall\, t \in [0,+\infty), \quad \forall\, x \in H.$$

Therefore, if $\|y\| \leq \delta_\epsilon$, we have

$$|F(\lambda)\varphi(x+y) - F(\lambda)\varphi(x)| \leq$$

$$\int_0^T e^{-\lambda t}|S(t)\varphi(x+y) - S(t)\varphi(x)|\, dt + \int_T^{+\infty} e^{-\lambda t}|S(t)\varphi(x+y) - S(t)\varphi(x)|\, dt \leq$$

$$\frac{\epsilon}{2}\left(\frac{\lambda}{1-e^{-\lambda T}}\right)\int_0^T e^{-\lambda t}\, dt + 2M\|\varphi\|_\infty \int_T^{+\infty} e^{-(\lambda+\omega)t}\, dt \leq \epsilon.$$

Then $F(\lambda) \in \mathcal{L}(UC_b(H))$, $\forall\, \lambda > -\omega$. ■

We now want to introduce the notion of infinitesimal generator for weakly continuous semigroups. To this purpose let us first show that $F(\lambda)$ is the resolvent of some closed operator.

Proposition 3.2. *Assume that $S(\cdot)$ is a weakly continuous semigroup belonging to $\mathcal{G}_w(M,\omega)$. Then there exists a unique closed linear operator $G_S : D(G_S) \subseteq UC_b(H) \longrightarrow UC_b(H)$ such that $\forall\, \lambda > -\omega$ it holds*

$$R(\lambda, G_S)\varphi(x) = F(\lambda)\varphi(x), \quad \forall\, \varphi \in UC_b(H),\ \forall\, x \in H. \tag{7}$$

Proof. By using the Fubini- Tonelli theorem, it is easy to check that for every $\lambda, \mu > -\omega$

$$F(\lambda) - F(\mu) = (\mu - \lambda)F(\lambda)F(\mu), \tag{8}$$

so that F is a pseudo resolvent on $UC_b(H)$. Then, by a well known result (see for instance [6], Th VIII 4.1), in order to prove the proposition we have only to show that the kernel of $F(\lambda)$ is equal to $\{0\}$, for every $\lambda > -\omega$. From the uniqueness of Laplace transforms it follows that for every measurable and exponentially bounded function f for which there exists set of values λ containing a limit point such that

$$\int_0^{+\infty} e^{-\lambda t} f(t)\, dt = 0,$$

it holds

$$f(t) = 0 \ \text{ a.s.} \tag{9}$$

Now, from the resolvent equation (8) we obtain

$$F(\lambda)\varphi(x) = 0 \text{ for some } \lambda > -\omega \Rightarrow F(\lambda)\varphi(x) = 0 \text{ for all } \lambda > -\omega$$

and then, as the function $t \longrightarrow S(t)\varphi(x)$ is continuous $\forall\, x \in H$, we have

$$F(\lambda)\varphi(x) = 0 \Rightarrow S(t)\varphi(x) = 0, \quad \forall\, t \in [0, +\infty). \tag{10}$$

In particular $\varphi(x) = S(0)\varphi(x) = 0$. ∎

Definition 3.3. The *infinitesimal generator* of the weakly continuous semigroup $S(\cdot)$ in $\mathcal{G}_w(M,\omega)$ is the unique closed linear operator $G_S : D(G_S) \subseteq UC_b(H) \longrightarrow UC_b(H)$ such that $\forall\, \lambda > -\omega$

$$R(\lambda, G_S)\varphi(x) = F(\lambda)\varphi(x) = \int_0^{+\infty} e^{-\lambda t} S(t)\varphi(x)\, dt, \quad \forall\, \varphi \in UC_b(H),\ \forall\, x \in H. \tag{11}$$

Remark 3.4. We could have defined the infinitesimal generator of the weakly continuous semigroup $S(\cdot)$ as for strongly continuous semigroups, i.e as the linear operator $A : D(A) \subseteq UC_b(H) \longrightarrow UC_b(H)$ such that

$$\begin{cases} D(A) = \left\{ \varphi \in UC_b(H) : \begin{array}{l} \exists \lim_{h\to 0^+} \Delta_h\varphi(x) \ \ \forall\, x \in H \\ \lim_{h \to 0^+} \Delta_h \varphi(\cdot) \in UC_b(H) \end{array} \right\} \\[2ex] A\varphi(x) = \lim_{h\to 0^+} \Delta_h\varphi(x), \end{cases} \tag{12}$$

where Δ_h is the incremental ratio

$$\Delta_h = \frac{S(h)-I}{h} \quad h > 0.$$

If $S(\cdot)$ is strongly continuous, then the two definitions above coincide; otherwise we have only the obvious inclusion

$$G_S \subset A. \tag{13}$$

■

We now remark that from (11) it easily follows that

$$\|R^k(\lambda, G_S)\| \leq \frac{M}{(\lambda+\omega)^k}, \quad k \in \mathbb{N}. \tag{14}$$

Then if $D(G_S)$ is dense in $UC_b(H)$ we have that G_S fulfills the hypotheses of the Hille Yosida Theorem, so that $S(\cdot)$ is strongly continuous. Thus, if $S(\cdot)$ is not strongly continuous, $D(G_S)$ must not be dense in $UC_b(H)$. However the following weaker result holds.

Proposition 3.5. *Let $S(\cdot)$ be a weakly continuous semigroup in $\mathcal{G}_w(M,\omega)$. Then for every function φ in $UC_b(H)$ and for every $k \in \mathbb{N}$, there exists a sequence $\{\varphi_n^k\} \subset D(G_S^k)$ such that*

$$\sup_{n\in\mathbb{N}} \|\varphi_n^k\|_\infty < +\infty \quad \forall\, k \in \mathbb{N} \tag{15}$$

and

$$\lim_{n\to+\infty} \sup_{x\in K} |\varphi_n^k(x) - \varphi(x)| = 0, \quad \forall\, K \subset H \text{ compact.} \tag{16}$$

Proof. For every $k \in \mathbb{N}$ and for every $n \in \mathbb{N}$, we put

$$\varphi_n^k(x) = n^k R^k(n, G_S)\varphi(x). \tag{17}$$

Clearly $\{\varphi_n^k\} \subset D(G_S^k)$, $\forall\, k \in \mathbb{N}$; moreover, from (14) it follows that

$$\|\varphi_n^k\|_\infty \leq \left(\frac{n}{n+\omega}\right)^k M\|\varphi\|_\infty \leq M\|\varphi\|_\infty, \quad \forall\, n \in \mathbb{N}.$$

We now show that $\forall\, k \in \mathbb{N}$ and for every compact set $K \subset H$ compact it holds

$$\lim_{n\to+\infty} \sup_{x\in K} |n^k R^k(n, G_S)\varphi(x) - \varphi(x)| = 0.$$

Indeed, since

$$\lim_{t\to 0} \sup_{x\in K} |S(t)\varphi(x) - \varphi(x)| = 0, \quad \forall\, K \subset H \text{ compact},$$

then for any fixed compact set $K \subset H$ and any fixed $\epsilon > 0$, there exists $\delta > 0$ such that

$$|t| \leq \delta \Rightarrow \sup_{x\in K} |S(t)\varphi(x) - \varphi(x)| \leq \epsilon.$$

Therefore for every $n \in \mathbb{N}$ we have that

$$\sup_{x\in K} |n^k R^k(n, G_S)\varphi(x) - \varphi(x)| \leq$$

$$\sup_{x\in K} \frac{n^k}{(k-1)!} \int_0^{+\infty} t^{k-1} e^{-nt} |S(t)\varphi(x) - \varphi(x)| \, dt \leq$$

$$\epsilon \frac{n^k}{(k-1)!} \int_0^{\delta} t^{k-1} e^{-nt} \, dt + (1+M)\|\varphi\|_\infty \frac{n^k}{(k-1)!} \int_\delta^{+\infty} t^{k-1} e^{-nt} \, dt \leq$$

$$\epsilon + (1+M)\|\varphi\|_\infty \left[\frac{n^{k-1}}{(k-1)!} \delta^{k-1} e^{-n\delta} + \ldots + n\delta e^{-n\delta} + e^{-n\delta} \right],$$

and our claim follows as $n \to +\infty$. ■

Remark 3.6. If $S(\cdot) \in \mathcal{G}_w(M, \omega)$ is a weakly continuous semigroup on $UC_b(H)$ and if $G_S : D(G_S) \subset UC_b(H) \longrightarrow UC_b(H)$ is its infinitesimal generator, then $Y = \overline{D(G_S)}$ is a $S(\cdot)$- invariant closed subspace of $UC_b(H)$, $\{S(t)_{|_Y} \mid t \geq 0\}$ is a strongly continuous semigroup in Y and $G_{S_{|_Y}}$ is its infinitesimal generator. ■

4. Differentiability of weakly continuous semigroups

If $S(\cdot)$ is a strongly continuous semigroup and A is its infinitesimal generator, $\forall\, x \in D(A)$ we have

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax \quad \forall\, t \geq 0.$$

For weakly continuous semigroups this is not true. Nevertheless we will state a similar result, namely that $\forall\, \varphi \in D(G_S)$ and $\forall\, x \in H$ the function

$$[0, +\infty) \longrightarrow \mathbb{R}, \quad t \longrightarrow S(t)\varphi(x)$$

is differentiable. First we want to describe some properties of the infinitesimal generator of $S(\cdot)$.

Lemma 4.1. *Let $S(\cdot) \in \mathcal{G}_w(M, \omega)$ be a weakly continuous semigroup and let G_S be its infinitesimal generator. Then the following statements hold*

1. *$\forall\, t \geq 0$ and $\lambda > -\omega$ we have*

$$S(t)R(\lambda, G_S) = R(\lambda, G_S)S(t). \tag{18}$$

2. *$\forall\, t \geq 0$ we have*

$$S(t)D(G_S) \subset D(G_S) \tag{19}$$

and $\forall\, \varphi \in D(G_S)$

$$G_S S(t)\varphi = S(t)G_S\varphi. \tag{20}$$

Proof. In order to prove (18) it is enough to verify that $\forall\, T > 0$ and $\forall\, \lambda > -\omega$

$$S(t)\int_0^T e^{-\lambda s}S(s)\varphi(x)\,ds = \int_0^T e^{-\lambda s}S(t)S(s)\varphi(x)\,ds, \quad \forall\, x \in H, \quad \forall\, \varphi \in UC_b(H). \tag{21}$$

Indeed, assume (21). Let us fix $\varphi \in UC_b(H)$ and put $\psi_n(x) = \int_0^n e^{-\lambda s}S(s)\varphi(x)\,ds$, for all $n \in \mathbb{N}$ and $x \in H$. Then we have $\|\psi_n\|_\infty \leq M\|\varphi\|_\infty/\lambda + \omega$, $\forall\, n \in \mathbb{N}$ and

$$\lim_{n\to+\infty}\sup_{x\in H}\left|\psi_n(x) - \int_0^{+\infty} e^{-\lambda s}S(s)\varphi(x)\,ds\right| = 0.$$

Finally, from the weak continuity of $S(\cdot)$ and from (21), it follows that $\forall\, x \in H$, $\forall\, \varphi \in UC_b(H)$ and $\forall\, t \geq 0$

$$S(t)R(\lambda, G_S)\varphi(x) = \lim_{n\to+\infty} S(t)\psi_n(x) =$$

$$\lim_{n\to+\infty} S(t)\int_0^n e^{-\lambda s}S(s)\varphi(x)\,ds = \lim_{n\to+\infty}\int_0^n e^{-\lambda s}S(t)S(s)\varphi(x)\,ds =$$

$$\int_0^{+\infty} e^{-\lambda s}S(s)S(t)\varphi(x)\,ds = R(\lambda, G_S)S(t)\varphi(x).$$

Let us now prove (21). Consider partitions $0 = s_0^h < s_1^h < \ldots < s_{j_h}^h = T$ of the interval $[0,T]$, with size $\delta_h = \max_{j\in\{1,\ldots,j_h\}}(s_j^h - s_{j-1}^h)$, $\forall\, h \in \mathbb{N}$, and assume that $\delta_h \to 0$, as $h \to +\infty$. For every $h \in \mathbb{N}$ we put

$$I_h(x) = \sum_{j=1}^{j_h}(s_j^h - s_{j-1}^h)e^{-\lambda s_j^h}S(s_j^h)\varphi(x), \quad x \in H.$$

It is immediate to verify that $\sup_{h\in\mathbb{N}}\|I_h\|_\infty < +\infty$. Moreover for any compact set $K \subset H$ it holds

$$\lim_{h\to+\infty}\sup_{x\in K}|I_h(x) - \int_0^T e^{-\lambda s}S(s)\varphi(x)\,ds| = 0. \tag{22}$$

In fact, for every $\epsilon > 0$ we can choose $\delta_\epsilon > 0$ such that

$$|s - t| \leq \delta_\epsilon \Rightarrow \sup_{x\in K}|e^{-\lambda s}S(s)\varphi(x) - e^{-\lambda t}S(t)\varphi(x)| \leq \epsilon.$$

Therefore, since $\lim_{h\to+\infty}\delta_h = 0$, there exists $\bar{h}$ such that $\delta_h \leq \delta_\epsilon$, $\forall\, h \geq \bar{h}$ and then we have

$$\sup_{x\in K}|I_h(x) - \int_0^T e^{-\lambda s}S(s)\varphi(x)\,ds| \leq$$

$$\sum_{j=1}^{j_h}\sup_{x\in K}\int_{s_{j-1}^h}^{s_j^h}|e^{-\lambda s_j^h}S(s_j^h)\varphi(x) - e^{-\lambda s}S(s)\varphi(x)|\,ds \leq \epsilon T, \quad \forall\, h \geq \bar{h}.$$

Our claim follows from (22) and the weak continuity of $S(\cdot)$.

We now prove 2). For every $\lambda > -\omega$ we have

$$D(G_S) = R(\lambda, G_S)(UC_b(H)),$$

then from (18) it follows

$$S(t)D(G_S) \subset D(G_S). \tag{23}$$

Moreover if $\varphi \in D(G_S)$, there exist $\psi \in UC_b(H)$ and $\lambda > -\omega$ such that

$$R(\lambda, G_S)\psi = \varphi,$$

therefore from (18) we have

$$\begin{aligned} G_S S(t)\varphi = G_S S(t) R(\lambda, G_S)\psi = G_S R(\lambda, G_S) S(t)\psi = \\ \lambda R(\lambda, G_S) S(t)\psi - S(t)\psi = S(t)\left[\lambda R(\lambda, G_S) - I\right]\psi = \\ S(t) G_S R(\lambda, G_S)\psi = S(t) G_S \varphi. \end{aligned}$$

■

Proposition 4.2. *Let $S(\cdot)$ be a weakly continuous semigroup. Then $\forall\, \varphi \in D(G_S)$ we have*

$$S(t)\varphi(x) = \varphi(x) + \int_0^t S(s) G_S \varphi(x)\, ds. \tag{24}$$

In particular the function $S(\cdot)\varphi(x)$ is differentiable and

$$\frac{d}{dt} S(t)\varphi(x) = S(t) G_S \varphi(x) = G_S S(t)\varphi(x). \tag{25}$$

Proof. Let us fix $\varphi \in D(G_S)$, $x \in H$ and $\lambda > -\omega$. Then we have

$$\begin{aligned} \varphi(x) = R(\lambda, G_S)(\lambda I - G_S)\varphi(x) = \int_0^{+\infty} e^{-\lambda t} S(t)(\lambda I - G_S)\varphi(x)\, dt = \\ \lambda \int_0^{+\infty} e^{-\lambda t} S(t)\varphi(x)\, dt - \int_0^{+\infty} e^{-\lambda t} S(t) G_S \varphi(x)\, dt. \end{aligned} \tag{26}$$

We now remark that if $-\omega < \mu < \lambda$ we have

$$\begin{aligned} |e^{-\lambda t} \int_0^t S(s) G_S \varphi(x)\, ds| \le e^{-\lambda t} \int_0^t |S(s) G_S \varphi(x)|\, ds \le \\ e^{-(\lambda-\mu)t} \int_0^t e^{-\mu s} |S(s) G_S \varphi(x)| ds \le e^{-(\lambda-\mu)t} \int_0^{+\infty} e^{-\mu s} |S(s) G_S \varphi(x)|\, ds, \end{aligned}$$

and then

$$\lim_{t \to +\infty} e^{-\lambda t} \int_0^t S(s) G_S \varphi(x)\, ds = 0. \tag{27}$$

Integrating by parts and taking into account (26), we get

$$\int_0^{+\infty} e^{-\lambda t} S(t)\varphi(x)\, dt - \int_0^{+\infty} e^{-\lambda t} \int_0^t S(s) G_S \varphi(x)\, ds\, dt = \int_0^{+\infty} e^{-\lambda t} \varphi(x)\, dt,$$

so that from the uniqueness of Laplace transforms and the continuity of the application $t \longrightarrow S(t)\varphi(x)$ it follows

$$S(t)\varphi(x) = \varphi(x) + \int_0^t S(s) G_S \varphi(x)\, ds. \tag{28}$$

■

5. Generation theorem

Our aim is here to generalize the Hille Yosida Theorem to weakly continuous semigroups on $UC_b(H)$ giving a characterization of their infinitesimal generator.

If X is a Banach space and $A : D(A) \subseteq X \longrightarrow X$ is a linear operator such that $\rho(A) \supset]0,+\infty)$ the *Yosida approximation* A_n, $n \in \mathbb{N}$, is the bounded linear operator on X defined by

$$A_n x = nAR(n,A)x = n^2 R(n,A)x - nx, \quad x \in X. \tag{29}$$

Theorem 5.1. *Let $A : D(A) \subset UC_b(H) \longrightarrow UC_b(H)$ be a linear operator. Then A is the infinitesimal generator of a weakly continuous semigroup $S(\cdot)$ in $\mathcal{G}_w(M,\omega)$ if and only if the following statements hold*

1. *A is closed.*
2. *$\rho(A) \supset \{\, \lambda > -\omega \,\}$ and $\|R^k(\lambda,A)\|_{\mathcal{L}(UC_b(H))} \leq \frac{M}{(\lambda+\omega)^k}$, $\forall\, \lambda > -\omega$, $\forall\, k \in \mathbb{N}$.*
3. *For every $\varphi \in UC_b(H)$ and for every compact set $K \subset H$*

$$\lim_{n\to+\infty} \sup_{x\in K} |n^k R^k(n,A)\varphi(x) - \varphi(x)| = 0, \quad \forall\, k \in \mathbb{N}. \tag{30}$$

4. *For every $\varphi \in UC_b(H)$ and for every sequence $\{\varphi_j\} \subset UC_b(H)$ satisfying properties (3) it holds*

$$\lim_{j\to+\infty} \sup_{x\in K} |n^k R^k(n,A)\varphi_j(x) - n^k R^k(n,A)\varphi(x)| = 0, \tag{31}$$

   for every compact set $K \subset H$, and this limit is uniform in n and k.

5. *For every $\varphi \in UC_b(H)$ the family of functions in $UC_b(H)$*

$$\left\{ n^k R^k(n,A)\varphi \; : \; k,n \in \mathbb{N} \right\} \tag{32}$$

   is equi-uniformly continuous.

Proof. **Necessity** - The first statement follows from the definition of infinitesimal generator . The second and the third one were proved in Proposition 3.5.

Let us check the fourth. Fix $\varphi \in UC_b(H)$ and let $\{\varphi_j\} \subset UC_b(H)$ satisfy properties (3). Since $S(\cdot)$ is weakly continuous, for every compact set $K \subset H$ and for every $\epsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that

$$\sup_{x\in K} |S(t)\varphi_j(x) - S(t)\varphi(x)| \leq \epsilon, \quad \forall\, j \geq j_0 \text{ and } \forall\, t \in [0,+\infty).$$

Then for every $j \geq j_0$ and for every $x \in K$ we have

$$\sup_{x\in K} |n^k R^k(n,A)\varphi_j(x) - n^k R^k(n,A)\varphi(x)| \leq$$

$$\sup_{x\in K} \frac{n^k}{(k-1)!} \int_0^{+\infty} t^{k-1} e^{-nt} |S(t)\varphi_j(x) - S(t)\varphi(x)|\, dt \leq$$

$$\epsilon \frac{n^k}{(k-1)!} \int_0^{+\infty} t^{k-1} e^{-nt}\, dt = \epsilon.$$

In order to prove the fifth statement, we use again the weak continuity of $S(\cdot)$ and we get that $\forall \epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$\|y\| \leq \delta_\epsilon \Rightarrow |S(t)\varphi(x+y) - S(t)\varphi(x)| \leq \epsilon \quad \forall x \in H, \quad \forall t \in [0, +\infty).$$

Then we proceed as above.

Sufficiency We split up the proof into several steps.

Step 1. As for the classical Hille-Yosida Theorem we can prove that for every $\varphi \in D(A^2)$

$$\lim_{n \to +\infty} A_n \varphi = A\varphi \quad \text{in } UC_b(H),$$

and the following inequality holds

$$\|e^{tA_n}\| \leq M e^{\frac{-n\omega t}{n+\omega}}, \quad \forall n \in \mathbb{N}, \quad \forall t \in [0, +\infty). \tag{33}$$

Step 2. For every $\varphi \in D(A)$ we have

$$\|e^{tA_n}\varphi - e^{tA_m}\varphi\|_\infty \leq M^2 t e^{-\frac{\omega}{1+\omega}t} \|A_n\varphi - A_m\varphi\|_\infty\,. \tag{34}$$

In particular $\forall \varphi \in D(A^2)$

$$\left\{e^{tA_n}\varphi\right\}_n$$

is a Cauchy sequence in $UC_b(H)$, uniformly in $t \in [0, +\infty)$.

The proof is the same as for the classical Hille-Yosida Theorem.

Step 3. $\forall \varphi \in UC_b(H)$ and $\forall \{\varphi_j\} \subset UC_b(H)$ satisfying properties (3), it holds

$$\lim_{j \to +\infty} \sup_{x \in K} |e^{tA_n}\varphi_j(x) - e^{tA_n}\varphi(x)| = 0, \tag{35}$$

for any compact set $K \subset H$, uniformly in $t \in [0, +\infty)$ and $n \in \mathbb{N}$.

Indeed, from assumption 4, we have that $\forall \epsilon > 0$ and for every compact set $K \subset H$ there exists $j_0 \in \mathbb{N}$ such that $\forall n, k \in \mathbb{N}$

$$\sup_{x \in K} |n^k R^k(n, A)\varphi_j(x) - n^k R^k(n, A)\varphi(x)| \leq \epsilon, \quad \forall j \geq j_0.$$

Then $\forall j \geq j_0$ we have

$$\sup_{x \in K} |e^{tA_n}\varphi_j(x) - e^{tA_n}\varphi(x)| \leq$$

$$e^{-nt} \sum_{k=1}^{+\infty} \frac{n^k t^k}{k!} \sup_{x \in K} |n^k R^k(n, A)\varphi_j(x) - n^k R^k(n, A)\varphi(x)| \leq \epsilon e^{-nt} \sum_{k=1}^{+\infty} \frac{n^k t^k}{k!} = \epsilon.$$

Step 4. For every $\varphi \in UC_b(H)$ and for every compact set $K \subset H$, the sequence $\left\{e^{tA_n}\varphi\right\}_n$ is a Cauchy sequence in $C(K)$, uniformly in $t \in [0, +\infty)$, namely, $\forall \epsilon > 0$ there exists $\bar{n}$ such that

$$\sup_{x \in K} |e^{tA_n}\varphi(x) - e^{tA_m}\varphi(x)| \leq \epsilon, \quad \forall n, m \geq \bar{n}, \quad \forall t \geq 0.$$

We set for every $j \in \mathbb{N}$

$$\varphi_j = j^2 R^2(j, A)\varphi.$$

Clearly $\{\varphi_j\} \subset D(A^2)$. Moreover $\{\varphi_j\}$ satisfies properties (3). Indeed we have that $\sup_{j\in\mathbb{N}} \|\varphi_j\|_\infty < +\infty$ and from assumption 3

$$\lim_{j\to+\infty} \sup_{x\in K} |\varphi_j(x) - \varphi(x)| = 0.$$

Then, since

$$|e^{tA_n}\varphi(x) - e^{tA_m}\varphi(x)| \leq$$
$$|e^{tA_n}\varphi(x) - e^{tA_n}\varphi_j(x)| + |e^{tA_n}\varphi_j(x) - e^{tA_m}\varphi_j(x)| + |e^{tA_m}\varphi_j(x) - e^{tA_m}\varphi(x)|,$$

the conclusion easily follows from the second and the third step.

From step 4 we can define $\forall\, \varphi \in UC_b(H)$ and $\forall\, x \in H$

$$S(t)\varphi(x) = \lim_{n\to+\infty} e^{tA_n}\varphi(x). \tag{36}$$

Clearly $\forall\, \varphi \in UC_b(H)$ we have

$$\|S(t)\varphi\|_\infty \leq M e^{-\omega t}\|\varphi\|_\infty \quad \forall\, t \in [0,+\infty).$$

Indeed

$$|S(t)\varphi(x)| = \lim_{n\to+\infty} |e^{tA_n}\varphi(x)| \leq \limsup_{n\to+\infty} \|e^{tA_n}\|_{\mathcal{L}(UC_b(H))}\|\varphi\|_\infty \leq$$

$$\limsup_{n\to+\infty} M e^{\frac{-n\omega t}{n+\omega}}\|\varphi\|_\infty = M e^{-\omega t}\|\varphi\|_\infty, \quad \forall\, x \in H.$$

Now we will verify that $S(\cdot)$ is a weakly continuous semigroup belonging to $\mathcal{G}_w(M,\omega)$ and that A is its infinitesimal generator.

Step 5. For all $\varphi \in UC_b(H)$ the family of functions in $UC_b(H)$

$$\left\{ e^{tA_n}\varphi \;:\; n \in \mathbb{N}, t \in [0,+\infty) \right\} \tag{37}$$

is equi-uniformly continuous.

From assumption 5, for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $\forall\, x \in H$ we have

$$\|y\| \leq \delta_\epsilon \Rightarrow$$
$$|n^k R^k(n,A)\varphi(x+y) - n^k R^k(n,A)\varphi(x)| \leq \epsilon, \quad \forall\, n,k \in \mathbb{N}.$$

Therefore, we proceed as in the third step and we get that for every $n \in \mathbb{N}$ and for every $t \in [0,+\infty)$

$$\|y\| \leq \delta_\epsilon \Rightarrow |e^{tA_n}\varphi(x+y) - e^{tA_n}\varphi(x)| \leq \epsilon \quad \forall\, x \in H.$$

Step 6. For every $\varphi \in UC_b(H)$ the family of functions

$$\{S(t)\varphi \;:\; t \in [0,+\infty)\}$$

is equi-uniformly continuous.

From the precedent step the family of functions

$$\left\{ e^{tA_n}\varphi \ : \ n \in \mathbb{N}, t \geq 0 \right\}$$

is equi-uniformly continuous, so that for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $\forall\, x \in H$

$$\|y\| \leq \delta_\epsilon \Rightarrow |e^{tA_n}\varphi(x+y) - e^{tA_n}\varphi(x)| < \epsilon, \quad \forall\, t \geq 0, \quad \forall\, n \in \mathbb{N}.$$

Then, since

$$|S(t)\varphi(x+y) - S(t)\varphi(x)| \leq |S(t)\varphi(x+y) - e^{tA_n}\varphi(x+y)| +$$
$$|e^{tA_n}\varphi(x+y) - e^{tA_n}\varphi(x)| + |e^{tA_n}\varphi(x) - S(t)\varphi(x)|,$$

and it follows that $\forall\, x \in H \ \ \forall\, t \geq 0$

$$\|y\| \leq \delta_\epsilon \Rightarrow$$
$$|S(t)\varphi(x+y) - S(t)\varphi(x)| \leq \epsilon + |S(t)\varphi(x+y) - e^{tA_n}\varphi(x+y)| +$$
$$|S(t)\varphi(x) - e^{tA_n}\varphi(x)|$$

and letting $n \to +\infty$ we prove our claim.

Step 7. For every $\varphi \in UC_b(H)$ and for every compact set $K \subset H$

$$\lim_{t \to 0} \sup_{x \in K} |S(t)\varphi(x) - \varphi(x)| = 0.$$

Once $\epsilon > 0$ is fixed, there exists $\bar{n}$ such that

$$\sup_{x \in K} |S(t)\varphi(x) - e^{tA_{\bar{n}}}\varphi(x)| \leq \epsilon, \quad \forall\, t \geq 0.$$

Then

$$\sup_{x \in K} |S(t)\varphi(x) - \varphi(x)| \leq \sup_{x \in K} |S(t)\varphi(x) - e^{tA_{\bar{n}}}\varphi(x)| +$$
$$\sup_{x \in K} |e^{tA_{\bar{n}}}\varphi(x) - \varphi(x)| \leq \epsilon + \|e^{tA_{\bar{n}}}\varphi - \varphi\|_\infty, \quad \forall\, t \geq 0$$

and letting $t \to 0^+$ our claim follows.

Step 8. For every $\varphi \in UC_b(H)$ and for every sequence $\{\varphi_j\} \subset UC_b(H)$ satisfying (3), it holds

$$\lim_{j \to +\infty} \sup_{x \in K} |S(t)\varphi_j(x) - S(t)\varphi(x)| = 0, \tag{38}$$

for every compact set $K \subset H$, uniformly in t.

We have that $\forall\, x \in H$

$$|S(t)\varphi_j(x) - S(t)\varphi(x)| \leq$$
$$|S(t)\varphi_j(x) - e^{tA_n}\varphi_j(x)| + |e^{tA_n}\varphi_j(x) - e^{tA_n}\varphi(x)| + |e^{tA_n}\varphi(x) - S(t)\varphi(x)|$$

and using the third and the fourth step we can say that (38) holds.

Step 9. The semigroup law holds.

Let us recall that $\forall \varphi \in UC_b(H)$ there exists a sequence $\{\varphi_j\} \subset D(A^2)$ satisfying (3), therefore

$$
\begin{gathered}
|e^{(t+s)A_n}\varphi(x) - S(t)S(s)\varphi(x)| \leq \\
|e^{(t+s)A_n}\varphi(x) - e^{(t+s)A_n}\varphi_j(x)| + |e^{tA_n}e^{sA_n}\varphi_j(x) - e^{tA_n}S(s)\varphi_j(x)| + \\
|e^{tA_n}S(s)\varphi_j(x) - e^{tA_n}S(s)\varphi(x)| + |e^{tA_n}S(s)\varphi(x) - S(t)S(s)\varphi(x)| = \\
L_1(n,j)(x) + L_2(n,j)(x) + L_3(n,j)(x) + L_4(n,j)(x).
\end{gathered}
$$

It is easy to check that $S(s)\varphi$ and the sequence $\{S(s)\varphi_j\}$ satisfy (3), $\forall\, s \geq 0$, then from the third step, we have that

$$\lim_{j \to +\infty} L_1(n,j)(x) = \lim_{j \to +\infty} L_3(n,j)(x) = 0$$

uniformly in n. Moreover

$$L_2(n,j)(x) \leq \|e^{tA_n}\|_{\mathcal{L}(UC_b(H))}\|e^{sA_n}\varphi_j - S(s)\varphi_j\|_\infty \leq M\|e^{sA_n}\varphi_j - S(s)\varphi_j\|_\infty,$$

and from the second step it follows that

$$\lim_{n \to +\infty} L_2(n,j)(x) = 0, \quad \forall\, j \in \mathbb{N},\ \forall\, x \in H.$$

Choosing $j_0 \in \mathbb{N}$ such that

$$L_1(n,j_0)(x) + L_3(n,j_0)(x) \leq \epsilon, \quad \forall\, n \in \mathbb{N},$$

we have

$$
\begin{gathered}
|e^{(t+s)A_n}\varphi(x) - S(t)S(s)\varphi(x)| \leq \\
\epsilon + L_2(n,j_0)(x) + |e^{tA_n}S(s)\varphi(x) - S(t)S(s)\varphi(x)|,
\end{gathered}
$$

and taking the limit as $n \to +\infty$, $\forall\, x \in H$ and $\forall\, \varphi \in UC_b(H)$ we get

$$S(t)S(s)\varphi(x) = \lim_{n \to +\infty} e^{(t+s)A_n}\varphi(x) = S(t+s)\varphi(x), \quad \forall\, x \in H. \tag{39}$$

Step 10. The operator A is the infinitesimal generator of the semigroup $S(\cdot)$, namely for every $\varphi \in UC_b(H)$ we have

$$\int_0^{+\infty} e^{-\lambda t} S(t)\varphi(x)\, dt = R(\lambda, A)\varphi(x), \quad \forall\, x \in H. \tag{40}$$

Let $\varphi \in D(A^2)$. Then for every compact set $K \subset H$ it holds

$$\lim_{n \to +\infty} \sup_{x \in K} |e^{tA_n}A_n\varphi(x) - S(t)A\varphi(x)| = 0. \tag{41}$$

Indeed we have

$$
\begin{gathered}
\sup_{x \in K} |e^{tA_n}A_n\varphi(x) - S(t)A\varphi(x)| \leq \\
\sup_{x \in K} |e^{tA_n}A_n\varphi(x) - e^{tA_n}A\varphi(x)| + \sup_{x \in K} |e^{tA_n}A\varphi(x) - S(t)A\varphi(x)| \leq \\
M\|A_n\varphi - A\varphi\|_\infty + \sup_{x \in K} |e^{tA_n}A\varphi(x) - S(t)A\varphi(x)|,
\end{gathered}
$$

and (41) follows as $n \to +\infty$. Since $\forall\varphi \in UC_b(H)$ and $\forall x \in H$ we have

$$\frac{d}{dt}(e^{tA_n}\varphi(x)) = e^{tA_n}A_n\varphi(x), \quad \forall n \in \mathbb{N},$$

it follows that $\forall n \in \mathbb{N}$ and $\forall \lambda > -\omega$

$$\lambda \int_0^{+\infty} e^{-\lambda t}e^{tA_n}\varphi(x)\,dt = \varphi(x) + \int_0^{+\infty} e^{-\lambda t}e^{tA_n}A_n\varphi(x)\,dt. \tag{42}$$

We now remark that $\forall\varphi \in D(A)$

$$|e^{tA_n}\varphi(x)| \leq M\|\varphi\|_\infty, \quad \forall n \in \mathbb{N}$$

and

$$|e^{tA_n}A_n\varphi(x)| = |e^{tA_n}nR(n,A)A\varphi(x)| \leq M^2\frac{n}{n+\omega}\|A\varphi\|_\infty \leq M^2\|A\varphi\|_\infty,$$

then, from the Dominated Convergence Theorem, taking the limit in (42), as $n \to +\infty$, we get

$$\lambda \int_0^{+\infty} e^{-\lambda t}S(t)\varphi(x)\,dt = \varphi(x) + \int_0^{+\infty} e^{-\lambda t}S(t)A\varphi(x)\,dt, \quad \forall\varphi \in D(A^2) \tag{43}$$

Now, let $\varphi \in D(A)$. Since

$$nR(n,A)\varphi \in D(A^2),$$

from (43) it follows that

$$\begin{gathered}\lambda \int_0^{+\infty} e^{-\lambda t}S(t)nR(n,A)\varphi(x)\,dt = \\ nR(n,A)\varphi(x) + \int_0^{+\infty} e^{-\lambda t}S(t)nR(n,A)A\varphi(x)\,dt.\end{gathered} \tag{44}$$

Moreover

$$\begin{cases} \sup\limits_{n\in\mathbb{N}} \|nR(n,A)\varphi\|_\infty < +\infty \\ \\ \lim\limits_{n\to+\infty} \sup\limits_{x\in K} |nR(n,A)\varphi(x) - \varphi(x)| = 0, \quad \forall K \subset H \text{ compact,} \end{cases}$$

therefore, because of the weak continuity of $S(\cdot)$ we can say that

$$\lim_{n\to+\infty} S(t)nR(n,A)\varphi(x) = S(t)\varphi(x), \quad \forall x \in H.$$

Taking the limit in (44), as $n \to +\infty$, by Dominated Convergence Theorem we have

$$\lambda \int_0^{+\infty} e^{-\lambda t}S(t)\varphi(x)\,dt = \varphi(x) + \int_0^{+\infty} e^{-\lambda t}S(t)A\varphi(x)\,dt, \tag{45}$$

and then, using notations of proposition 6, $\forall\varphi \in D(A)$ it holds

$$F(\lambda)(\lambda - A)\varphi(x) = \varphi(x). \tag{46}$$

Finally, let us remark that $\forall\varphi \in UC_b(H)$, $\forall x \in H$

$$R(\lambda, A)\varphi(x) = (\lambda - A)^{-1}\varphi(x) = F(\lambda)\varphi(x). \tag{47}$$

The proof is thus complete. ∎

6. An example: Transition Semigroups

Let us consider the following stochastic equation

$$\begin{cases} dX(t) = AX(t)dt + dW(t) \\ X(0) = x, \end{cases} \tag{48}$$

where A is the infinitesimal generator of a C_0 - semigroup $S(\cdot)$ on a separable Hilbert space H, W is an H-valued Q-Wiener process (defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$), Q being a self-adjoint and positive bounded linear operator on H and $x \in H$.

For all $t \geq 0$ we define the bounded linear operator Q_t

$$Q_t x = \int_0^t S(s) Q S^*(s) x \, ds \quad \forall\, x \in H. \tag{49}$$

It is well known (see for instance [1]) that if

$$\mathrm{Tr}\, \mathrm{Q_t} < +\infty, \quad \forall \mathrm{t} > 0, \tag{50}$$

then the mild solution of (48) is given by

$$X(t,x) = S(t)x + \int_0^t S(t-s) dW(s) \tag{51}$$

and $X(t,x)$ is the Gaussian random variable $\mathcal{N}(S(t)x, Q_t)$, with mean $S(t)x$ and covariance operator Q_t, for all $t \geq 0$ and $x \in H$.

The aim of this section is to describe, under the hypothesis (50), the transition semigroup $P(t)$, $t \geq 0$, associated with the stochastic equation (48) and defined $\forall\, \varphi \in UC_b(H)$ and $\forall\, x \in H$ by

$$P(t)\varphi(x) = \mathrm{E}[\varphi(X(t,x))] = \begin{cases} \int_H \varphi(y) \mathcal{N}(S(t)x, Q_t) & t > 0 \\ \\ \varphi(x) & t = 0. \end{cases} \tag{52}$$

It is easy to check that for arbitrary $\varphi \in UC_b(H)$ the function $u : [0, +\infty) \times H \longrightarrow \mathbb{R}$ defined by

$$u(t,x) = P(t)\varphi(x)$$

is continuous.

However, the semigroup $P(t)$ is not strongly continuous on $UC_b(H)$ in general, as the following example shows.

Example 6.1. Let $H = \mathbb{R}$ and let $Ax = -\frac{1}{2}x$ and $Qx = x$, $\forall\, x \in \mathbb{R}$. If we set $q(t) = 1 - e^{-t}$, for every $t > 0$ we have $Q_t x = q(t)x$ and

$$P(t)\varphi(x) = \begin{cases} \int_{\mathbb{R}} \varphi(y) \mathcal{N}(e^{-t/2}x, q(t))\, dy & t > 0 \\ \\ \varphi(x) & t = 0, \end{cases} \tag{53}$$

for every $\varphi \in UC_b(\mathbb{R})$. Moreover it is easy to check that the function

$$u : [0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}, \quad (t,x) \longrightarrow P(t)\varphi(x)$$

is the unique solution in $UC_b(\mathbb{R})$ of the following Cauchy problem

$$\begin{cases} u_t(t,x) = \frac{1}{2}u_{xx}(t,x) - \frac{1}{2}xu_x(t,x) & t>0, \quad x \in \mathbb{R} \\ u(0,x) = \varphi(x) & x \in \mathbb{R}. \end{cases}$$

Now let us introduce for every $\varphi \in UC_b(\mathbb{R})$ the function

$$\psi : [0,+\infty) \times \mathbb{R} \longrightarrow \mathbb{R}, \quad (t,x) \longrightarrow T(t)\varphi(x),$$

where $T(\cdot)$ is the strongly continuous semigroup on $UC_b(\mathbb{R})$ defined by

$$T(t)\varphi(x) = \begin{cases} \int_{\mathbb{R}} \varphi(y)\,\mathcal{N}(x,t)\,dy & t>0 \\ \varphi(x) & t=0. \end{cases}$$

Since the function ψ is the unique solution in $UC_b(\mathbb{R})$ of the Cauchy problem

$$\begin{cases} \psi_t(t,x) = \frac{1}{2}\psi_{xx}(t,x) & t>0, \quad x \in \mathbb{R} \\ \psi(0,x) = \varphi(x) & x \in \mathbb{R}, \end{cases}$$

it is easy to show that $\forall\,(t,x) \in [0,+\infty) \times \mathbb{R}$ we have

$$u(t,x) = \psi(q(t), e^{-t/2}x) = T(q(t))\varphi(e^{-t/2}x).$$

If we set $\varphi(x) = \sin x$, $t_n = 2\log\left(\frac{2n+1}{2n-1}\right)$ and $x_n = \pi\left(\frac{1}{1-e^{-t_n/2}}\right) = \frac{\pi}{2}(2n+1)$, $\forall\, n \in \mathbb{N}$, we have that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} |u(t_n,x) - \sin x| \geq |u(t_n,x_n) - \sin x_n| \geq \\
&\left|T(q(t_n))\sin\left((2n-1)\frac{\pi}{2}\right) - T(q(t_n))\sin\left((2n+1)\frac{\pi}{2}\right)\right| - \\
&\left|T(q(t_n))\sin\left((2n+1)\frac{\pi}{2}\right) - \sin\left((2n+1)\frac{\pi}{2}\right)\right| \geq \\
&\left|T(q(t_n))\sin\left((2n-1)\frac{\pi}{2}\right) - T(q(t_n))\sin\left((2n+1)\frac{\pi}{2}\right)\right| - \sup_{x\in\mathbb{R}} |T(q(t_n))\sin x - \sin x| = \\
&\left|\int_{\mathbb{R}} \left(\sin\left(y + (2n-1)\frac{\pi}{2}\right) - \sin\left(y + (2n+1)\frac{\pi}{2}\right)\right) \mathcal{N}(0,q(t_n))\,dy\right| - \\
&\sup_{x\in\mathbb{R}} |T(q(t_n))\sin x - \sin x| = \\
&\left|\left(\sin(2n-1)\frac{\pi}{2} - \sin(2n+1)\frac{\pi}{2}\right)\int_{\mathbb{R}} \cos y\,\mathcal{N}(0,q(t_n))\,dy + \right. \\
&\left.\left(\cos(2n-1)\frac{\pi}{2} - \cos(2n+1)\frac{\pi}{2}\right)\int_{\mathbb{R}} \sin y\,\mathcal{N}(0,q(t_n))\,dy\right| - \\
&\sup_{x\in\mathbb{R}} |T(q(t_n))\sin x - \sin x| = \\
&2\,|T(q(t_n))\cos(0)| - \sup_{x\in\mathbb{R}} |T(q(t_n))\sin x - \sin x|.
\end{aligned}$$

Therefore, since the semigroup $T(\cdot)$ is strongly continuous and $\lim_{n\to+\infty} q(t_n) = 0$, it holds

$$\liminf_{n\to+\infty} \sup_{x\in\mathbb{R}} |u(t_n,x) - \sin x| \geq 2,$$

so that the semigroup $P(\cdot)$ is not strongly continuous. ∎

Proposition 6.2. *Let $S(\cdot)$ be a C_0-semigroup of negative type on H, namely assume that there exist $L, \alpha \geq 0$ such that $\|S(t)\|_{\mathcal{L}(UC_b(H))} \leq Le^{-\alpha t}$, $t \geq 0$. Then, if (50) holds, the semigroup $P(\cdot)$ on $UC_b(H)$ defined by (52) satisfies properties 1), 2) and 3) of definition 2.1.*

Thus the semigroup $\{ e^{-\omega t} P(t) \mid t \geq 0 \}$ is a weakly continuous semigroup in $\mathcal{G}_w(1, \omega)$, for every $\omega \geq 0$.

Proof. **Step 1.** If $\varphi \in UC_b(H)$, then for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$\|y\| \leq \delta_\epsilon \Rightarrow |\varphi(x+y) - \varphi(x)| < \epsilon, \quad \forall\, x \in H.$$

Moreover, from strong continuity of the semigroup $S(\cdot)$, it follows that for every compact set $K \subset H$

$$\lim_{t \to 0} \sup_{x \in K} \|S(t)x - x\| = 0,$$

so that there exists $\bar{t} > 0$ such that

$$\|y\| \leq \delta_\epsilon / 2 \Rightarrow \sup_{x \in K} \|y + S(t)x - x\| \leq \delta_\epsilon, \quad \forall\, t \leq \bar{t}.$$

Therefore for every $t \leq \bar{t}$ we have

$$\sup_{x \in K} |P(t)\varphi(x) - \varphi(x)| \leq$$

$$\sup_{x \in K} \int_{\{\|y\| \leq \delta_\epsilon/2\}} |\varphi(y + S(t)x) - \varphi(x)| \, \mathcal{N}(0, Q_t) dy +$$

$$2\|\varphi\|_\infty \int_{\{\|y\| > \delta_\epsilon/2\}} \mathcal{N}(0, Q_t) dy \leq \epsilon + \frac{8\|\varphi\|_\infty}{\delta_\epsilon^2} \operatorname{Tr} Q_t,$$

and the assertion follows as $t \to 0^+$.

Step 2. Since $S(t)$ is of negative type, we have

$$\|S(t)z\| \leq L\|z\|, \quad \forall\, z \in H, \quad \forall\, t \geq 0.$$

Then, if $\varphi \in UC_b(H)$, $\forall\, \epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $\forall\, t \in [0, +\infty)$

$$\|z\| \leq \delta_\epsilon / L \Rightarrow |P(t)\varphi(x+z) - P(t)\varphi(x)| \leq$$

$$\int_H |\varphi(y + S(t)x + S(t)z) - \varphi(y + S(t)x)| \, \mathcal{N}(0, Q_t) dy \leq \epsilon, \quad \forall\, x \in H.$$

Step 3. We state first the following

Lemma 6.3. *Assume that $S(\cdot)$ is of negative type. Then the family of probability measures*

$$\{ \mathcal{N}(S(t)x, Q_t) \, : \, t \in [0, +\infty), \quad x \in K \} \tag{54}$$

is tight, for every compact set $K \subset H$.

Proof. Since $\lim_{t\to+\infty}\|S(t)\|_{\mathcal{L}(UC_b(H))}=0$, we have that (see [1])

$$\sup_{t\geq 0}\operatorname{Tr} Q_t<+\infty. \tag{55}$$

Therefore the Gaussian measure $\mathcal{N}(0,Q_\infty)$, where

$$Q_\infty x=\int_0^{+\infty}S(t)QS^*(t)x\,dt,\quad x\in H$$

is well defined. Now let $K\subset H$ be a compact set. We recall that the function

$$[0,+\infty)\times H\longrightarrow\mathbb{R},\quad (t,x)\longrightarrow P(t)\varphi(x)$$

is continuous, $\forall\,\varphi\in UC_b(H)$, then from compactness of K it suffices to prove that for every sequence $\{t_n\}\subset[0,+\infty)$ and $\{x_n\}\subset K$ such that

$$\begin{cases} t_n\uparrow+\infty & \\ & \text{as } n\to+\infty\\ x_n\to x\in H, & \end{cases}$$

we have

$$\mathcal{N}(S(t_n)x_n,Q_{t_n})\rightharpoonup\mathcal{N}(0,Q_\infty)\quad\text{as } n\to+\infty. \tag{56}$$

For every $n\in\mathbb{N}$, let us consider the stochastic differential equation

$$\begin{cases} dX(t)=AX(t)dt+dW(t)\\ X(-t_n)=x_n. \end{cases} \tag{57}$$

and denote by $X(t,-t_n,x_n)$ its mild solution

$$X(t,-t_n,x_n)=S(t+t_n)x_n+\int_{-t_n}^t S(t-r)\,dW(r).$$

We remark that $X(t_n,0,x_n)$ and $X(0,-t_n,x_n)$ have the same distribution, for every $n\in\mathbb{N}$ (see [1]), then, if we show that $\{X(0,-t_n,x_n)\}_n$ is a Cauchy sequence in $L^2(\Omega,\mathcal{F},\mathbf{P})$, (56) follows immediately. For $n,p\in\mathbb{N},p>0$ we have

$$\mathbf{E}\left(\|X(0,-t_{n+p},x_{n+p})-X(0,-t_n,x_n)\|^2\right)\leq$$

$$2\|S(t_{n+p})x_{n+p}-S(t_n)x_n\|^2+2\mathbf{E}\left(\int_{-t_{n+p}}^{-t_n}S(-r)\,dW(r)\right)^2=$$

$$2\|S(t_{n+p})x_{n+p}-S(t_n)x_n\|^2+\operatorname{Tr}\int_{t_n}^{t_{n+p}}S(r)QS^*(r)\,dr.$$

Our claim follows recalling that (55) holds and that, as K is bounded, we have

$$\lim_{n\to+\infty}S(t_n)x_n=0.$$ ■

We now conclude the proof of the proposition.

Let $K\subset H$ be a compact set and $\epsilon>0$. Then there exists a compact set $K_\epsilon\subset H$ such that

$$\mathcal{N}(S(t)x,Q_t)(K_\epsilon)\geq 1-\epsilon,\quad\forall\,t\in[0,+\infty),\quad\forall\,x\in K.$$

Fix a function $\varphi \in UC_b(H)$ and a sequence $\{\varphi_j\} \subseteq UC_b(H)$ satisfying properties (3) and choose $j_0 \in \mathbb{N}$ such that

$$\sup_{y \in K_\epsilon} |\varphi_j(y) - \varphi(y)| \leq \epsilon, \quad \forall j \geq j_0.$$

It follows

$$\sup_{x \in K} |P(t)\varphi_j(x) - P(t)\varphi(x)| \leq$$

$$\sup_{x \in K} \int_{K_\epsilon} |\varphi_j(y) - \varphi(y)| \mathcal{N}(S(t)x, Q_t)dy + \sup_{x \in K} \int_{K_\epsilon^c} |\varphi_j(y) - \varphi(y)| \mathcal{N}(S(t)x, Q_t)dy \leq$$

$$\epsilon + (\|\varphi_j\|_\infty + \|\varphi\|_\infty)\epsilon. \quad \blacksquare$$

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