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RESEARCH ARTICLE

ENDOMORPHISM MONOIDS OF BANDS

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Introduction.

Endomorphisms of any algebra A form a monoid End(A) under their composition. In this paper we investigate the endomorphism monoids of bands, and the question of when End(A) determines $A \in \underset{\sim}{V}$ within a variety $\underset{\sim}{V}$ of bands. An early result on determination by endomorphisms - not of algebras but partially ordered sets - is due to L.M. Gluskin [7]. It says that two partially ordered sets S and T whose endomorphism semigroups are isomorphic must themselves be either isomorphic or anti-isomorphic. Analogous results hold for some small varieties of algebras, for example, for a variety of distributive lattices, see B.M. Schein [19], R. McKenzie and C. Tsinakis [14], and by P. Ribenboim [18]; for distributive p-algebras see [1], for Boolean rings K.D. Magill [13], for semilattices [19], and for Brouwerian semilattices [10] and [21]. In all of these varieties there exist at most two non-isomorphic algebras with isomorphic endomorphism monoids. B.M. Schein [20] proved that there exist at most four non-isomorphic normal bands with ismorphic endomorphism monoids. Our aim is to generalize the results of B.M. Schein [19,20] to larger varieties of bands.

A band is an algebra with one binary operation which is associative and idempotent. All varieties of bands are described by A.P. Birjukov in [3], see also Ch. Fennemore [5], or J.A. Gerhard [6]. We show that the variety of bands defined by the identity $xyx = xy$ (or $xyx = yx$) contains at most two non-isomorphic bands with isomorphic endomorphism monoids (see Theorem 2.2 and Corollary 2.3) and the variety of bands defined by the identity $xyxz = xyz$ (or $zxyx = zyx$) - therefore it contains the variety of normal bands - possesses at most four non-isomorphic bands with isomorphic endomorphism monoids (see Theorem 2.9 and Corollary 2.10).

An extreme counterpart to this property is the notion of universality. A concrete category K is called *universal* if a full subcategory of K is isomorphic to the category of all graphs and compatible mappings. Universal categories possess a very rich structure; for instance, for every monoid M and every cardinal α there exist α objects in K whose endomorphism monoids are isomorphic to M and which are mutually rigid, see A. Pultr and V. Trnková [17].

The universality was intensively studied in varieties of semigroups. Z. Hedrlín and J. Lambek [9] proved that the variety of all semigroups is universal and, subsequently, all universal semigroup varieties were characterized in [12]. We continue this study by investigating universal varieties of algebras adding new operations to semigroups. The algebras $(S,\cdot,{}^+)$ obtained from a semigroup $(S,\cdot)$ by adding a unary operation ${}^+$ satisfying the identities

$$x^{++} = x,\ x^+\cdot x\cdot x^+ = x^+, \text{ and } x\cdot x^+\cdot x = x$$

will be called here *semigroups with a regular involution*, or simply *regular involution semigroups*. (Note that we do not require the identity $(xy)^+ = y^+x^+$.) If in addition, is idempotent then $(X,\cdot,{}^+)$ is a *regular involution band*. We prove that regular involution bands are universal (see

Theorem 3.5).

Distributive lattices with added nullary operations were investigated in [2] and [11], where the universality of the distributive lattices with three constants was demonstrated. This provides an interesting counterpart to the fact that at most two non-isomorphic distributive lattices possess isomorphic endomorphism monoids. As an analogy to this result we prove that the bands with three added nullary operations are universal (Theorem 3.10).

We also investigate the category Band(S) of bands with a given structural quotient semilattice S, whose morphisms are homomorphisms of bands which induce the identity mapping on their common structural quotient semilattice, and show that there exists a finite semilattice S such that Band(S) is universal (Theorem 3.11).

An auxiliary result concerns varieties of unary algebras with n operations φ_i; $i = 0,1,\dots,n-1$ satisfying all identities of the form $\varphi_i \circ \alpha \circ \varphi_i = \varphi_i$, $i = 0,1,\dots,n-1$ where α is an arbitrary word over the alphabet $\{\varphi_0,\varphi_1,\dots,\varphi_{n-1}\}$; such a variety is universal if and only if $n \geq 3$ (Theorem 4.7).

Several interesting questions remain unsolved:

1) Algebras in a variety $\underset{\sim}{V}$ are α-determined by endomorphism monoids where α is a cardinal if for every monoid M there exist less than α pairwise nonisomorphic algebras A in $\underset{\sim}{V}$ with $\mathrm{End}(A) \cong M$. Is it possible to prove that bands in a larger variety of bands are α-determined by endomorphism monoids for some cardinal α?

2) Characterize minimal universal varieties of regular involution bands (or bands with three nullary operations).

3) Characterize the semilattices S for which

Band(S) is universal. Clearly, if the semilattice S from Theorem 3.11 is a subsemilattice of a given semilattice T, then Band(T) is universal. On the other hand, if we generalize results of B.M. Schein [20] we obtain that Band(S) is not universal when S is a finite chain.

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1. Preliminaries

This section recalls basic facts about the structure of bands and about endomorphism monoids of bands. Classical semigroup notions used in this paper can be found in the monographs [4] or [15].

THEOREM 1.1: *The decomposition of a band B into D-classes is a congruence on B and the quotient semigroup of B by this congruence is a semilattice.*

Proof: See, for instance, [15].■

The semilattice S determined by the decomposition of a band B into D-classes is called the *structural semilattice* of B. We shall always assume that the structural semilattice is a meet semilattice. For an element $b \in B$ we will denote the image of b in the canonical homomorphism of band B onto S by D_b. For an element $s \in S$ denote by B_s the D-class of B corresponding to s. We assume that an abstract monoid End(B) of a band B is given. Then we obviously obtain

LEMMA 1.2: *Every constant mapping of B into itself is an endomorphism of B. An element $c \in \mathrm{End}(B)$ is a* *constant homomorphism if and only if it is a left zero of* End(B). ■

In what follows, an element $b \in B$ will be identified with a constant endomorphism with image $\{b\}$. By Lemma 1.2, we can thus recognize the underlying set from End(B), as the set of all left zeros of End(B). Furthermore, End(B) is determined uniquely as a transformation monoid up to a strong isomorphism - see the following proposition.(See also [20].)

PROPOSITION 1.3: *For every* $f \in \mathrm{End}(B)$ *we have* $\mathrm{Im}\, f = \{c;\ c$ *is a left zero of* $\mathrm{End}(B)$ *and* $f \circ d = c$ *for some* $d \in \mathrm{End}(X)\}$. *Moreover, for* $f \in \mathrm{End}(B)$ *we have* $f(d) = c$ *if and only if* $f \circ d = c$.

Proof is clear.■

Therefore in the following for simplicity of notions we shall assume that if B_1 and B_2 are bands with isomorphic endomorphism monoids then $\mathrm{End}(B_1) = \mathrm{End}(B_2)$ and hence the underlying sets of both bands are the same.

B.M. Schein proved

THEOREM 1.4[20]: *If* B_1, B_2 *are two bands with* $\mathrm{End}(B_1) = \mathrm{End}(B_2)$ *then the decompositions into D-classes of both bands are the same and structural semilattices of both bands are either isomorphic or anti-isomorphic (that is dual) chains.*■

LEMMA 1.5[20]: *Let* B *be a band with a two-element subset* $A \subseteq B$. *Then* A *is a subsemilattice of* B *if and only if* A *is not a subset of any D-class and there exists* $f \in \mathrm{End}(B)$ *with* $\mathrm{Im}\, f = A$.■

By means of Theorem 1.4 and Lemma 1.5 B.M. Schein proved

THEOREM 1.6: <u>There exist at most four normal bands with isomorphic endomorphism monoids.</u>■

In the following section we generalize Schein's theorem.

2. The left distributive bands.

The aim of this section is to investigate endomorphism monoids of bands from the variety $\underset{\sim}{V}$ of all left distributive bands. We recall that a band B is called *left distributive* if B satisfies the identity xyxz = xyz. We recall that this variety is the join of the variety of normal bands and the variety of semilattices of left zero-semigroups, see [5]. In the following we assume that a band B from the variety $\underset{\sim}{V}$ is given. Then we have

LEMMA 2.1: <u>For a mapping</u> $f:B\to B$ <u>and for an element</u> $x\in B$ <u>the following are equivalent:</u>

<u>a)</u> <u>for every</u> $y\in B$ <u>we have</u> $f(y) = xyx$;

<u>b)</u> f <u>is an idempotent endomorphism of</u> B <u>and for</u> $u\in B$ <u>we have</u> $u\in \mathrm{Im}\, f$ <u>if and only if the set</u> $\{u,x\}$ <u>is a subsemilattice of</u> B <u>with</u> $u\leq x$.

Proof: Assume that a) holds. Then

$$f^2(y) = xxyxx = xyx = f(y)$$

for every $y\in B$ and therefore f is idempotent. Since $f(yz) = xyzx = xyxzx = xyxxzx = f(y)f(z)$ for every $y,z\in B$ we conclude that f is an endomorphism of B. Assume that $u\in \mathrm{Im}\, f$. Then $u = xux$ implies $u = ux = xu$, hence $\{u,x\}$ is a subsemilattice of B with $u\leq x$. Thus b) holds.

Assume that b) holds. Then $u\in \mathrm{Im}\, f$ if and only if $u = xux$ and this is equivalent with there exists $y\in B$

with $u = xyx$. Thus $\mathrm{Im}\, f = \{xyx;\ y \in B\}$. Hence for every $y \in B$ we have $f(xyx) = xyx$. Since $x \in \mathrm{Im}\, f$ we have $f(x) = x$ and thus for every $y \in B$ we conclude

$$f(y) = xf(y)x = f(x)f(y)f(x) = f(xyx) = xyx$$

and a) is proved.∎

THEOREM 2.2: *For every monoid M there exist at most two non-isomorphic semilattices B of left zero semigroups with* $\mathrm{End}(B) \cong M$. *If two non-isomorphic semilattices of left zero-semigroups have isomorphic endomorphism monoids then their structural semilattices are anti-isomorphic chains. Thus algebras in the variety of all semilattices of left zero-semigroups are 3-determined by endomorphism monoids.*

Proof: Let $\mathrm{End}(B)$ be an endomorphism monoid of a semilattice of left zero-semigroups B. Then for every $x,y \in B$ we have $xyx = xy$. If we apply Theorem 1.4 we determine the decomposition of B into D-classes and the structural semilattice S except when S is a chain. If S is a chain then S is determined up to an anti-isomorphism, and we proceed with each semilattice separately. Assume that S is a structural semilattice of B then by Lemma 1.5, we obtain for every $x \in B$ the set $\{u \in B;\ \{x,u\}$ is a subsemilattice of B with $x \geq u\} = \{xyx;\ y \in B\}$. Lemma 2.1 completes the proof because for every $x,y \in B$ we determine xy. If a structural semilattice of B is not determined uniquely and we find a band for both semilattices then these two semilattices of left zero-semigroups can have isomorphic endomorphism monoids. (If for the semilattice S and some $x \in B$ there exists no idempotent endomorphism f satisfying the condition b) from Lemma 2.1 or if there are at least two such idempotent endomorphisms then no band $B \in \underset{\sim}{V}$ with the structural semilattice S and $\mathrm{End}(B)$ exists.)∎

COROLLARY 2.3: Algebras in the variety of all semilattices of right zero-semigroups are 3-determined by endomorphism monoids. Thus for every monoid M there exist at most two non-isomorphic semilattices B of right zero-semigroups with End(B) ≅ M. If two non-isomorphic semilattices of right zero-semigroups have isomorphic endomorphism monoids then their structural semilattices are anti-isomorphic chains.

Proof: If $(B,\cdot)$ is a semilattice of right zero-semigroups then $(B,\circledast)$ where $x\circledast y = y\cdot x$ is a semilattice of left zero-semigroups such that $\mathrm{End}(B,\cdot)$ and $\mathrm{End}(B,\circledast)$ are isomorphic. We apply Theorem 2.2.∎

We recall several well-known facts about the variety $\underset{\sim}{V}$ of left distributive bands. Every semigroup $B\in\underset{\sim}{V}$ is a subsemigroup of a product of a semilattice of left zero-semigroups and a right zero-semigroup. Thus for $x,y\in B$ from the same L-class of B (i.e. if $xy = x$, and $yx = y$) and for every $z\in B$ with $D_x \geq D_z$ we have $zx = zy$. Hence for every $u,v\in B$ we have $xuvx = xuvy = xuxvy = xuxyvy$ because $D_x \geq D_{xuv}$. This fact is often used in the following without a reference. In the sequel we assume that End(B) is given. If we know a structural semilattice then by Lemma 2.1 we can determine the product xyx for every $x,y\in B$ this fact is also used without any reference.

LEMMA 2.4: Let A be a left-zero subsemigroup of B and let f be an idempotent endomorphism of B with $\mathrm{Im}\, f = \{axa;\ a\in A,\ x\in B\}$. Then $T_a = \{x\in B;\ f(x) = axa\}$ is a right ideal for every $a\in A$ and the following hold:

a) $\bigcup\{T_a;\ a\in A\} = B$,

b) if $y\in T_a\cap T_b$ for $a,b\in A$ then $aya = byb$.

Proof: Assume that f is an idempotent endomorphism of B with $\mathrm{Im}\, f = \{axa;\ a\in A,\ x\in B\}$. For every $a\in A$

define $T_a = \{y \in B;\ f(y) = aya\}$. First we show that a) holds. Choose $y \in B$. From the properties of Im f it follows that there exists $a \in A$ with $f(y) = af(y)a$. Hence $f(y) = af(y)a = f(a)f(y)f(a) = f(aya) = aya$ because $a = aaa \in \mathrm{Im}\, f$ and f is idempotent. Thus $y \in T_a$ and a) is proved. We prove that T_a is a right ideal. Let $y \in T_a$, $z \in B$, then $f(yz) = f(y)f(z) = ayabzb = ayza$ for some $b \in A$ and hence $yz \in T_a$. Since B is a band we have $a \in T_a$ for every $a \in A$. Further for every $y \in T_a \cap T_b$ where $a,b \in A$ we have $aya = f(y) = byb$ whence b) holds.■

LEMMA 2.5: *Let A be a left zero-subsemigroup of B and let $\{T_a;\ a \in A\}$ be a family of right ideals satisfying a) and b) from Lemma 2.4 and such that $a \in T_a$ for every $a \in A$. Then a mapping f defined $f(y) = aya$ for $a \in A$, $y \in T_a$ is an idempotent endomorphism of B.*

Proof: Assume that we have a family $\{T_a;\ a \in A\}$ of right ideals satisfying a), b), and such that $a \in T_a$ for every $a \in A$. By Condition b) the mapping f is a correctly defined partial mapping. By a) f is a total mapping. Since $a \in T_a$ for every $a \in A$ we have $f(a) = a$ for every $a \in A$ and since T_a is a right ideal we have for every $y \in T_a$ that $aya \in T_a$. Since $aayaa = aya$ we conclude that f is idempotent. We show that f is an endomorphism. Let $y \in T_a$, $z \in T_b$ for $a,b \in A$. Since T_a is a right ideal we obtain that $yz \in T_a$ and hence $f(yz) = ayza = ayabzb =$
$= f(y)f(z)$.■

COROLLARY 2.6: *Let A be a left zero-subsemigroup of B such that for every distinct $a,b \in A$ there exists no $x \in B$ with $axa = bxb$. If there exists an idempotent endomorphism f of B with $\mathrm{Im}\, f = \{axa;\ a \in A,\ x \in B\}$ then for every idempotent mapping $g: A \to A$ there exists an idempotent endomorphism $h: B \to B$ with $h \restriction A = g$ and $\mathrm{Im}\, h = \{g(a)xg(a);\ a \in A,\ x \in B\}$.*

Proof: By Lemma 2.4 $\{T_a = \{x\in B;\ f(x) = axa\};\ a\in A\}$ is a family of right ideals satisfying a) and b). We show that $T_a \cap T_b = 0$ if a,b are distinct elements of A. Indeed, if $y\in T_a \cap T_b$ then $aya = f(y) = byb$ and by the assumption $a = b$. Therefore $T_a \cap T_b = 0$. For every $a\in \mathrm{Im}\ g$ define $S_a = \cup\{T_b;\ b\in A,\ g(b) = a\}$. Then $\{S_b;\ b\in \mathrm{Im}\ g\}$ is a family of right ideals with $b\in S_b$ for every $b\in \mathrm{Im}\ g$ because $f(a) = a$ for every $a\in A$. By a) and b) we conclude that $\{S_b;\ b\in \mathrm{Im}\ g\}$ also satisfies a) and b) (to prove b) we use that $T_a \cap T_b = 0$ if a,b are distinct elements of B). If we apply Lemma 2.5 we obtain the required statement.∎

In the following we define a pseudo D-partition; this notion describes the basic properties of D-classes. First we give auxiliary notions. If $\underline{P}$ is a decomposition of a set A, then for $a\in A$ denote by $\underline{P}(a)$ the set of $\underline{P}$ containing a. An idempotent endomorphism f of B is called *A-determined*, where A is a subset of B, if $\mathrm{Im}\ f = \{axa;\ a\in A,\ x\in B\}$. Let D be a D-class of B. A pair $\underline{D} = (\underline{E},\underline{F})$ of two decompositions of D is called a *pseudo D-decomposition* if the following conditions hold:

1) for every $F\in \underline{F}$, $E\in \underline{E}$ we have $|F\cap E| = 1$;

2) for every $A\subseteq F\in \underline{F}$ there exists an A-determined endomorphism of B;

3) for every endomorphism f of B we have that $x,y\in \mathrm{Im}\ f$ implies that $\underline{E}(x)\cap \underline{F}(y), \underline{E}(y)\cap \underline{F}(x) \subseteq \mathrm{Im}\ f$;

4) for every A-determined endomorphism f and C-determined endomorphism g, where $A\subseteq F$, $C\subseteq E$ for some $F\in \underline{F}, E\in \underline{E}$ with $A\cap C \neq 0$ there exists an $A\circledast C$-determined endomorphism h where $A\circledast C = \{x\in D;\ \exists a\in A,\ \exists c\in C$ with $\{x\} = \underline{E}(a)\cap \underline{F}(c)\}$;

5) if f is an A-determined endomorphism for some $A\subseteq F\in \underline{F}$ then for every $a\in F$ we have $\underline{E}(a)\subseteq f^{-1}(f(a))$;

6) for $a,b\in D$ we have $\underline{E}(a) = \underline{E}(b)$ whenever $axa = byb$ for some $x,y\in B$;

7) if $a,b \in E \in \underline{E}$ with $\{axa;\ x\in B\}\cap\{bxb;\ x\in B\} = 0$ then for every $E' \in \underline{E}$ we have $\{cxc;\ x\in B\}\cap\{dxd;\ x\in B\} = 0$ where $\{c\} = E'\cap \underline{F}(a)$, $\{d\} = E'\cap \underline{F}(b)$.

A pair of decompositions $\underline{D} = (\underline{E},\underline{F})$ on a D-class D of B is called *trivial* if the decomposition $\underline{E}$ is universal (i.e. $\underline{E}$ has exactly one class) and the decomposition $\underline{F}$ is identical (i.e. every class of $\underline{F}$ is a singleton). A pair of decompositions $\underline{D} = (\underline{L},\underline{R})$ on a D-class D of B is called an *LR-decomposition* if $\underline{L}$ is a decomposition of D into L-classes and $\underline{R}$ is a decomposition of D into R-classes.

PROPOSITION 2.7: Let D be a D-class of B. Then the LR-decomposition is a pseudo D-partition.

Proof: It is well-known that Condition 1) holds. We prove Condition 2). Let $A \subseteq R \in \underline{R}$. Since B satisfies the identity $xyxz = xyz$ we obtain that B is a subsemigroup of $B_1 \times B_2$ where B_1 is a semilattice of left zero-semigroups and B_2 is a right zero-semigroup. Denote by π_i the restriction of the i-th projection on B, $i\in\{1,2\}$. Choose an idempotent mapping $g:B_2 \to B_2$ such that $\mathrm{Im}\, g = \pi_2(A)$ and choose $a = (a_1,a_2)\in R$. Define a mapping $f:B\to B$ as follows:

for $x = (x_1,x_2)\in B$ set $f(x) = (a_1x_1, g(x_2))$.

We prove that f is an idempotent endomorphism of B. For $x = (x_1,x_2)$, $y = (y_1,y_2)\in B$ we have $f(x)f(y) = (a_1x_1,g(x_2))(a_1y_1,g(y_2)) = (a_1x_1a_1y_1,g(x_2)g(y_2)) = (a_1x_1y_1,g(x_2y_2)) = f(x_1y_1,x_2y_2) = f(xy)$ because g is an endomorphism of B_2. Further $ff(x) = f(a_1x_1,g(x_2)) = (a_1a_1x_1,g(g(x_2))) = (a_1x_1,g(x_2)) = f(x)$. Thus f is an idempotent endomorphism. Since for every $b = (b_1,b_2)\in A$ we have $b_1 = a_1$ and for every $x = (x_1,x_2)\in B$ we have $bxb = (b_1x_1b_1,b_2x_2b_2) = (a_1x_1,b_2)$ we conclude that $f(bxb) = bxb$ because $g(b_2) = b_2$. On the other hand, for $x = (x_1,x_2)\in B$, if $f(x) = (a_1x_1,g(x_2))$ then for $b =$

$= (a_1, g(x_2)) \in A$ we obtain $f(x) = bxb$. Whence f is A-determined and 2) is proved.

Since for every $x, y \in D$ we have that every subsemigroup C of B containing x, y contains also $\underline{L}(x) \cap \underline{R}(y)$ and $\underline{L}(y) \cap \underline{R}(x)$ and since for every endomorphism f of B we have that $\mathrm{Im}\, f \cap D$ is a subsemigroup of B we conclude 3). To prove 4) assume that B is a subsemigroup of $B_1 \times B_2$ where B_1 is a semilattice of left zero-semigroups, B_2 is a right zero-semigroup. Let f be an A-determined endomorphism, g be a C-determined endomorphism where $A \subseteq R \in \underline{R}$, $C \subseteq L \in \underline{L}$ with $A \cap C \neq 0$. Denote by π_i, $i = 1,2$ the restriction of the i-th projection from B to B_i. Since $C \subseteq L \in \underline{L}$ we conclude that $\pi_2 \circ g$ is a constant mapping. Let f_1 be an idempotent mapping from B_2 into itself with $\mathrm{Im}\, f_1 = \pi_2(A)$, then $f_1 \circ \pi_2$ is a homomorphism from B into B_2. By the properties of the product we obtain that there exists an endomorphism $h: B \to B$ with $\pi_1 \circ h = \pi_1 \circ f$, $\pi_2 \circ h = f_1 \circ \pi_2$. By a routine calculation we obtain that h is $A \oplus C$-determined and 4) is proved. To obtain 5) it suffices to notice that every endomorphism of B preserves the decomposition into L-classes. If we use that B is a subsemigroup of the product of a semilattice of left zero-semigroups with a right zero-semigroup we obtain Conditions 6) and 7) by a direct inspection. Thus the LR-decomposition is a pseudo D-partition.∎

PROPOSITION 2.8: Let $D \subseteq B$ be a D-class. Then the trivial pair of decompositions is a pseudo D-partition. Moreover, there exist at most three distinct pseudo D-partitions of D. If there exist three different pseudo D-partitions then the LR-decomposition is not trivial.

Proof: By a direct inspection a trivial pair of decompositions is a pseudo D-partition. By Proposition 2.7 the LR-decomposition is a pseudo D-partition. Let $\underline{D} = (\underline{L}, \underline{R})$ be the LR-decomposition. Assume that $\underline{D}' =$

= ($\underline{L}'$,$\underline{R}'$) is a non-trivial pseudo D-decomposition of D different from $\underline{D}$. By Conditions 2) and 3), for every $A \subseteq R \in \underline{R}$ and for every $d \in D$ if there exists an $A \cup \{d\}$-determined endomorphism of B then either A is a singleton or $d \in R$. Hence for every $x \in D$ either $\underline{R}'(x) = \underline{R}(x)$ or $\underline{R}'(x) \subseteq L(x)$. By Condition 5) in the former case we conclude that $\underline{D} = \underline{D}'$ which is a contradiction, and therefore we assume that for every $x \in D$ we have $\underline{R}'(x) \subseteq \underline{L}(x)$. Assume that $\underline{D}'' = (\underline{L}'',\underline{R}'')$ is a pseudo D-partition different from $\underline{D}$ and that $\underline{R}',\underline{R}''$ are non-trivial. Choose $x \in D$, by Condition 1) $|\underline{R}'(x)|,|\underline{R}''(x)|>1$ and if $\underline{R}' \neq \underline{R}''$ then by Conditions 2) and 3) we have $\underline{R}'(x) \subseteq \underline{L}''(x)$ and $\underline{R}''(x) \subseteq \underline{L}'(x)$. Choose $y \in \underline{R}'(x)$, $z \in \underline{R}''(x)$ with $y,z \neq x$. By Condition 2) there exist an $\{x,y\}$-determined endomorphism f and an $\{x,z\}$-determined endomorphism g of B. Let $\{u\} = \underline{L}'(y) \cap \underline{R}'(z)$. Then by Condition 4) there exists an $\{x,y,z,u\}$-determined endomorphism h of B. By Condition 6) $(\{yvy;\ v \in B\} \cup \{uvu;\ v \in B\}) \cap (\{xvx;\ v \in B\} \cup \{zvz;\ v \in B\}) = 0$ and $\{xvx;\ v \in B\} \cap \{zvz;\ v \in B\} = 0$ and by Condition 7) we obtain also $\{yvy;\ v \in B\} \cap \{uvu;\ v \in B\} = 0$. Therefore we can apply Corollary 2.6 and we conclude that there exists an $\{x,y,z\}$-determined endomorphism of B, this contradicts Conditions 2) and 3). Therefore $\underline{R}' = \underline{R}''$ and whence $\underline{D}' = \underline{D}''$. The proof is complete.■

<u>THEOREM</u> 2.9: <u>Algebras in the variety</u> $\underset{\sim}{V}$ <u>of all left distributive bands are 5-determined by endomorphism monoids, thus for every monoid</u> M <u>there exist at most four non-isomorphic left distributive bands</u> B <u>with</u> End(B) ≅ M.

Proof: Let End(B) be given. By Theorem 1.4 there exist at most two non-isomorphic semilattices which can be structural semilattices of B. Furthermore we know the decomposition of B into D-classes. For a structural semilatice by Lemma 2.1 for every $x \in B$ we determine an endomorphism f_x such that $f_x(y) = xyx$ for every

$y \in B$. Whence for a given D-class we can find all pseudo D-partitions and by Proposition 2.8 there exist at most two decompositions to L- and R-classes. Let D_1, D_2 be two D-classes of B with $D_2 \subseteq BD_1B$. If x,y are distinct elements of D_1 which belong to the same R-class, then for every $u \in D_2$ we have that xux, yuy are distinct elements of D_2 belonging to the same R-class. Hence the decomposition of D_1 into L- and R-classes determine the decomposition of D_2 into L- and R-classes. Since for every two D-classes D_1, D_2 of B the set $BD_1B \cap BD_2B$ contains a D-class we conclude that for every semilattice S which can be the structural semilattice of B there exist at most two decompositions to L- and R-classes. To finish the proof it suffices to show that f_x and the decompositions to the L- and R-classes determine the left translation of x. Let $y \in B$, then $\{xy\} = \underline{L}(xyx) \cap \underline{R}(yxy)$, thus xy is uniquely determined. Since for every structural semilattice and every decomposition to L- and R- classes we determine the multiplication uniquely, the proof is complete.∎

COROLLARY 2.10: _Algebras in the variety $\underset{\sim}{V}$ of all right distributive bands are 5-determined by endomorphism monoids, thus for every monoid M there exist at most four non-isomorphic right distributive bands B with $\mathrm{End}(B) \cong M$._

Proof: If $(B,\oplus)$ is a right distributive band then the opposite band $(B,\cdot)$, where $a \cdot b = b \oplus a$, is left distributive and both bands have the same endomorphism monoids. Theorem 2.9 completes the proof.∎

REMARK: Let $(S,\wedge)$ be the semilattice isomorphic to the natural numbers with the operation meet, let $(S,\vee)$ be the opposite semilattice. Let A be a non-singleton set and let $(A,*)$ be the left zero-semigroup and $(B,\circ)$ be the right zero-semigroup. Then the following semigroups $(S,\wedge)\times(A,*)$, $(S,\wedge)\times(A,\circ)$, $(S,\vee)\times(A,*)$, $(S,\vee)\times(A,\circ)$ are non-isomorphic normal bands with the same endomorphism monoids. Therefore Theorem 2.9 and Corollary 2.10 cannot be improved.∎

3. Universal constructions

First we show that the variety of regular involution bands is universal. To this end we use the following theorem (proved in the Appendix).

THEOREM 3.1: *There exists a universal full subcategory K of the category A(1,1) of unary algebras with two operations such that for every pair (A,φ_A,ψ_A), (B,φ_B,ψ_B) of objects of K there exists no mapping $f:A\to B$ with $\varphi_B\circ f = f\circ\psi_A$ and $\psi_B\circ f = f\circ\varphi_A$.*∎

Choose two elements x,y that are not elements of any algebra in K. For every object (A,φ,ψ) from K define $\Phi(A) = (\{x,y\}\cup A\times\{0,1\},\cdot,{}^+)$ with the binary operation defined by:

$y\cdot x = x\cdot x = x$, $x\cdot y = y\cdot y = y$,
$(a,i)\cdot(b,j) = (a,i)\cdot x = (a,i)\cdot y = (a,i)$ for every $a,b\in A$, $i,j\in\{0,1\}$,
$x\cdot(a,0) = y\cdot(a,0) = (a,0)$, $x\cdot(a,1) = (\varphi(a),0)$,
$y\cdot(a,1) = (\psi(a),0)$ for every $a\in A$,

and the unary operation $^+$ defined by:

$x^+ = y$, $y^+ = x$, and $(a,i)^+ = (a,1-i)$ for every $a\in A$, $i\in\{0,1\}$.

LEMMA 3.2: *$\Phi(A,\varphi,\psi)$ is a regular involution band.*

Proof: The operation $\cdot$ is clearly idempotent, and $A\times\{0,1\}$ is the set of left zeros of $\Phi(A,\varphi,\psi)$. Further, $\{x,y\}$ is a right zero-subsemigroup of $\Phi(A,\varphi,\psi)$. From

$\{x,y\}\cdot(A\times\{0,1\}) = A\times\{0\}$ and $x\cdot(a,0) = y\cdot(a,0) = (a,0)$ for every $a\in A$ we easily verify that $\cdot$ is associative. It is easy to verify that $^+$ satisfies the required identities. Thus $\Phi(A,\varphi,\psi)$ is a regular involution band.∎

For every homomorphism $f:(A,\varphi,\psi)\to(B,\varphi,\psi)$ in K define a mapping Φf from $\Phi(A,\varphi,\psi)$ into $\Phi(B,\varphi,\psi)$ by

$\Phi f(x) = x$, $\Phi f(y) = y$, and $\Phi f(a,i) = (f(a),i)$ for every $a\in A$, $i\in\{0,1\}$.

LEMMA 3.3: *Φf is a homomorphism from $\Phi(A,\varphi,\psi)$ into $\Phi(B,\varphi,\psi)$ for every homomorphism $f:(A,\varphi,\psi)\to(B,\varphi,\psi)$. Furthermore, Φ is an embedding functor from K into the category of regular involution bands.*

Proof: Clearly, Φf commutes with the operation $^+$. From the definition of Φf and $\Phi(A,\varphi,\psi)$ it follows that it suffices to verify that $\Phi f(x)\cdot\Phi f(a,1) = \Phi f(x\cdot(a,1))$ and $\Phi f(y)\cdot\Phi f(a,1) = \Phi f(y\cdot(a,1))$ for every $a\in A$. This is equivalent with $f(\varphi(a)) = \varphi(f(a))$ and $f(\psi(a)) = \psi(f(a))$; both equations hold because f is a homomorphism. The rest is obvious.∎

Next we prove that Φ is full. Assume that (A,φ,ψ), (B,φ,ψ) are objects of K, and $g:\Phi(A,\varphi,\psi)\to\Phi(B,\varphi,\psi)$ is a homomorphism. First we show that $g(A\times\{0,1\})\subseteq B\times\{0,1\}$. If there exists an $a\in A$ such that $g(a,i)\in\{x,y\}$ for some $i\in\{0,1\}$ then $g(a,1-i)\in\{x,y\}$; also $g(a,i) = x$ if and only if $g(a,1-i) = y$ because g commutes with $^+$. If $g(a,i) = x$, then $x = g(a,i) = g((a,i)\cdot(a,1-i)) = g(a,i)\cdot g(a,1-i) = x\cdot y = y$ – this is a contradiction. If $g(a,i) = y$ then a contradiction is obtained by exchanging (a,i) for $(a,1-i)$. Thus $g(A\times\{0,1\})\subseteq B\times\{0,1\}$. If $g(x) = (a,i)$ then $g(y) = (a,1-i)$, and the contradictory $(a,i) = g(x) = g(y\cdot x) = g(y)\cdot g(x) = (a,1-i)\cdot(a,i) = (a,1-i)$ follows. Hence $g(x)\in\{x,y\}$. From $x^+ = y$ and

$y^+ = x$ we have that $g(\{x,y\}) = \{x,y\}$. Since $(a,i)^+ = (a,1-i)$ for every $a \in A$ (or $a \in B$) and $i \in \{0,1\}$, we conclude that there exists a mapping $h: A \to B$ with $g(\{(a,0),(a,1)\}) = \{(h(a),0),(h(a),1)\}$. From $\{x,y\} \cdot (A \times \{0,1\}) \subseteq A \times \{0\}$ it follows that $g(a,0) = (h(a),0)$ and thus $g(a,i) = (h(a),i)$ for every $a \in A$, $i \in \{0,1\}$. For every $a \in A$ we have $(h(\varphi(a)),0) = g(\varphi(a),0) = g(x \cdot (a,1)) = g(x) \cdot g(a,1) = g(x) \cdot (h(a),1)$ and, analogously, $(h(\psi(a)),0) = g(y) \cdot (h(a),1)$.

If $g(x) = y$ then $g(y) = x$, and

$$(h(\varphi(a)),0) = y \cdot (h(a),1) = (\psi(h(a)),0),$$
$$(h(\psi(a)),0) = x \cdot (h(a),1) = (\varphi(h(a)),0).$$

In this case $h \circ \varphi = \psi \circ h$ and $h \circ \psi = \varphi \circ h$, but by the defining property of K such an h does not exist. Thus $g(x) = x$, $g(y) = y$ and $(h(\varphi(a)),0) = x \cdot (h(a),1) = (\varphi(h(a)),0)$ and $(h(\psi(a)),0) = y \cdot (h(a),1) = (\psi(h(a)),0)$. Hence $h \circ \varphi = \varphi \circ h$ and $\psi \circ h = h \circ \psi$, so that h is a homomorphism and $\Phi h = g$. We obtain

PROPOSITION 3.4: *Φ is a full embedding of K into the variety of regular involution bands.*■

THEOREM 3.5: *The variety of regular involution bands is universal.*

Proof: Combine Theorem 3.1 and Proposition 3.4.■

Next we aim to show that the variety of bands with three added nullary operations is universal. For this purpose we use the following theorem (proved in the Appendix). Recall that $3 = \{0,1,2\}$.

THEOREM 3.6: *The variety $\underset{\sim}{V}$ of unary algebras with three operations φ_i, $i \in 3$ fulfilling identities $\varphi_i \circ \alpha \circ \varphi_i = \varphi_i$, for every $i \in 3$ and for every word α in the alphabet $\{\varphi_i;\ i \in 3\}$ is universal.*■

Let $(A, \varphi_i;\ i \in 3)$ be an algebra from $\underset{\sim}{V}$, and let

$A \cap Z = 0$ where $Z = \{(x_i, y_j);\ i,j \in 3\}$. Denote $X = \{x_i;\ i \in 3\}$, $Y = \{y_i;\ i \in 3\}$. Set $\Psi(A, \varphi_i;\ i \in 3) = (A \cup Z,\ \cdot,\ (x_i, y_i);\ i \in 3)$ where the binary operation $\cdot$ is defined as follows:

$$(v,w)\cdot(v',w') = (v,w') \quad \text{for } v,v' \in X,\ w,w' \in Y,$$
$$a \cdot b = a \cdot z = a \quad \text{for } a,b \in A,\ z \in Z,$$
$$(x_i, y_j)\cdot a = \varphi_i(\varphi_j(a)) \quad \text{for } i,j \in 3,\ a \in A.$$

<u>LEMMA</u> 3.7: <u>$\Psi(A, \varphi_i;\ i \in 3)$ is a band.</u>

Proof: Clearly, $\cdot$ is idempotent. By a direct calculation we verify that $\cdot$ is also associative because for every $i,j \in 3$, and for every word α over $\{\varphi_i;\ i \in 3\}$ we have $\varphi_i \circ \alpha \circ \varphi_j = \varphi_i \circ \varphi_j$.■

For a homomorphism $f:(A, \varphi_i;\ i \in 3) \to (B, \varphi_i;\ i \in 3)$ define Ψf as follows:

for $z \in Z$ set $\Psi f(z) = z$,
for $a \in A$ set $\Psi f(a) = f(a)$.

<u>PROPOSITION</u> 3.8: <u>Ψ is an embedding functor from $\underset{\sim}{V}$ into the variety of bands with three nullary operations.</u>

Proof: Any homomorphism f commutes with φ_i for every $i \in 3$; using the defining identity $\underset{\sim}{V}$ we obtain that Ψf is a homomorphism. The rest is obvious.■

To prove that Ψ is full let $g:\Psi(A, \varphi_i;\ i \in 3) \to \Psi(B, \varphi_i;\ i \in 3)$ be a homomorphism for some algebras $(A, \varphi_i;\ i \in 3)$, $(B, \varphi_i;\ i \in 3)$ from $\underset{\sim}{V}$. Then $g(x_i, y_i) = (x_i, y_i)$ for every $i \in 3$ because (x_i, y_i) are nullary operations. Whence $g(z) = z$ for every $z \in Z$ because Z is a subalgebra generated by $\{(x_i, y_i);\ i \in 3\}$. Suppose that for some $a \in A$ we have $g(a) = (z, y_1)$ with $z \in X$. Then $(z, y_1) = g(a) = g(a \cdot (x_1, y_2)) = g(a) \cdot g(x_1, y_2) =$

$= (z,y_1)\cdot(x_1,y_2) = (z,y_2)$, a contradiction. Analogously, if $g(a) = (z,v)$ for some $a\in A$, $z\in X$, $v\in Y\setminus\{y_1\}$ then $(z,v) = g(a) = g(a\cdot(x_1,y_1)) = g(a)\cdot g(x_1,y_1) = (z,v)\cdot(x_1,y_1) = (z,y_1)$, and again we obtain a contradiction. Hence $g(A)\subseteq B$. For every $a\in A$, $i\in 3$ we have $g(\varphi_i(a)) = g((x_i,y_i)\cdot a) = g(x_i,y_i)\cdot g(a) = (x_i,y_i)\cdot g(a) = \varphi_i(g(a))$. Hence $g\restriction A$ is a homomorphism from $(A,\varphi_i;\ i\in 3)$ into $(B,\varphi_i;\ i\in 3)$ with $\Psi g\restriction A = g$. Thus we conclude that:

PROPOSITION 3.9: Ψ is a full embedding of $\underset{\sim}{V}$ into the variety of bands with three nullary operations.∎

THEOREM 3.10: The variety of bands with three nullary operations is universal.

Proof: Combine Theorem 3.6 and Proposition 3.9.∎

Finally, for a given semilattice S we shall investigate a category Band(S) of all bands with the semilattice S and all homomorphisms $h:B\to B'$ satisfying $h(B_s)\subseteq B'_s$ for every $s\in S$. Let $S = \{s_1,s_2,s_3,s_4,s_5\}$ be a semilattice with $s_1,s_2,s_3>s_4>s_5$.

THEOREM 3.11: The category Band(S) is universal.

Proof: For any algebra $(A,\varphi_i;\ i\in 3)\in\underset{\sim}{V}$, let $\Psi'(A,\varphi_i;\ i\in 3)$ be an extension of $\Psi(A,\varphi_i;\ i\in 3)$ by three new idempotent elements $\{u_i;\ i\in 3\}$ and let the operation $\cdot$ extend that of $\Psi(A,\varphi_i;\ i\in 3)$ as follows:

$u_i\cdot u_j = (x_i,y_j)$ for $i,j\in 3,\ i \neq j$

$u_i\cdot(x_j,y_k) = (x_i,y_k)$, $(x_j,y_k)\cdot u_i = (x_j,y_i)$ for $i,j,k\in 3$

$a\cdot u_i = a$, $u_i\cdot a = \varphi_i(a)$ for $i\in$, $a\in A$

By a routine calculation we easily obtain that $\Psi'(A,\varphi_i;\ i\in 3)$ is a band over the semilattice S. For a homomorphism $f:(A,\varphi_i;\ i\in 3)\to(B,\varphi_i;\ i\in 3)$ define $\Psi'f$ to be an extension of Ψf by $\Psi'f(u_i) = u_i$ for every $i\in 3$. Clearly $\Psi'f$ is a homomorphism and therefore it is a morphism of Band(S). Thus Ψ' is an embedding functor from $\underset{\sim}{V}$ into Band(S).

To prove that Ψ' is full, let $g:\Psi'(A,\varphi_i;\ i\in 3)\to\Psi'(B,\varphi_i;\ i\in 3)$ be a morphism of Band(S). Then obviously, $g(u_i) = u_i$ for every $i\in 3$. Hence $g(t) = t$ for every $t\in Z$ because $Z\cup\{u_i;\ i\in 3\}$ is a subalgebra of $\Psi'(A,\varphi_i;\ i\in 3)$ generated by $\{u_i;\ i\in 3\}$. Now we apply Proposition 3.9 to conclude that Ψ' is full. Theorem 3.6 now completes the proof.∎

4. Appendix.

We prove Theorem 3.1 using the following result by Z. Hedrlín and A. Pultr [8].

THEOREM 4.1[8]: <u>The variety A(1,1) of all unary algebras with two operations is universal.</u>∎

For any $(A,\varphi,\psi)\in A(1,1)$ we now define $\Lambda(A,\varphi,\psi) = (A\times 6,\mu,\nu)$, where $6 = \{0,1,2,3,4,5\}$,

for every $a\in A$, $\mu(a,0) = (\varphi(a),2)$, $\mu(a,1) = (\psi(a),3)$,
for $i = 2,3$ $\mu(a,i) = (a,i+2)$,
for $i = 4,5$ $\mu(a,i) = (a,i-2)$,
for $i = 0,1,3,4$, $\nu(a,i) = (a,i+1)$,
for $i = 2,5$ $\nu(a,i) = (a,i-2)$.

For any homomorphism $f:(A,\varphi,\psi)\to(B,\varphi,\psi)$ define $\Lambda f(a,i) = (f(a),i)$ for every $a\in A$, $i\in 6$.

LEMMA 4.2: <u>Λ is an embedding functor from the variety A(1,1) into itself.</u>∎

Next we prove that Λ is full. Let

$g:\Lambda(A,\varphi,\psi)\to\Lambda(B,\varphi,\psi)$ be a homomorphism. Since g has to map every cycle of μ onto a cycle of μ, and analogously for ν, for every $a\in A$ there exist $b_1,b_2,b_3,b_4\in B$ such that

$g(\{(a,i);\ i = 0,1,2\}) = \{(b_1,j+k);\ j = 0,1,2\}$ where either $k = 0$ or $k = 3$,
$g(\{(a,i);\ i = 3,4,5\}) = \{(b_2,j+k);\ j = 0,1,2\}$ where either $k = 0$ or $k = 3$,
$g(\{(a,2),(a,4)\}) = \{(b_3,j+k);\ j = 0,2\}$ where either $k = 2$ or $k = 3$,
$g(\{(a,3),(a,5)\}) = \{(b_4,j+k);\ j = 0,2\}$ where either $k = 2$ or $k = 3$.

These properties yield $b_1 = b_3$ and $b_3 = b_2 = b_4$. Hence there exists a mapping $h:A\to B$ such that $g(\{a\}\times 6)\subseteq\{h(a)\}\times 6$. Since $g(\{(a,3),(a,5)\})\subseteq g(\{(a,i);\ i = 3,4,5\})$ for every $a\in A$, it follows that $g(a,4) = (h(a),4)$. Then $g(\{(a,2),(a,4)\}) = \{(h(a),2),(h(a),4)\}$ and, therefore, $g(a,2) = (h(a),2)$. Since g commutes with ν, $g(a,i) = (h(a),i)$ for every $i\in 6$. Moreover, for every $a\in A$, $(\varphi(h(a)),2) = \mu(h(a),0) = \mu(g(a,0)) = g(\mu(a,0)) = g(\varphi(a),2) = (h(\varphi(a)),2)$ and $(\psi(h(a)),3) = \mu(h(a),1) = \mu(g(a,1)) = g(\mu(a,1)) = g(\psi(a),3) = (h(\psi(a)),3)$. Hence $h:(A,\varphi,\psi)\to(B,\varphi,\psi)$ is a homomorphism with $\Lambda h = g$ as was to be shown.

PROPOSITION 4.3: *Λ is a full embedding.* ∎

Finally we prove:

LEMMA 4.4: *Let (A,φ,ψ), (B,φ,ψ) be unary algebras. Then there exists no mapping $h:A\times 6\to B\times 6$ such that $h\circ\mu = \nu\circ h$ and $h\circ\nu = \mu\circ h$.*

Proof: Let $f:X\to X$, $g:Y\to Y$, $h:X\to Y$ be mappings such that $h\circ f = g\circ h$. Then h maps every cycle of f of length m onto a cycle of g of length n where n

divides m. Since every cycle of μ has length 2 and every cycle of ν has length 3 we conclude that there is no h for which $h \circ \mu = \nu \circ h$ would hold.■

The proof of Theorem 3.1 is now complete.■

To prove Theorem 3.6, we show that there exists a full embedding θ of A(1,1) into the variety $\underset{\sim}{V}$. For any $(A,\varphi,\psi) \in A(1,1)$ set $\theta(A,\varphi,\psi) = (A\times 6,\mu,\nu,\eta)$ where the operations μ,ν,η are defined for all $a \in A$ by:

$\mu(a,0) = \mu(a,1) = (a,0)$, $\mu(a,2) = \mu(a,3) = \mu(a,4) = (a,2)$, $\mu(a,5) = (\varphi(a),2)$, $\nu(a,0) = \nu(a,1) = \nu(a,5) = (a,1)$, $\nu(a,2) = \nu(a,3) = (a,3)$, $\nu(a,4) = (\psi(a),1)$, $\eta(a,0) = \eta(a,1) = \eta(a,4) = (a,0)$, $\eta(a,2) = \eta(a,3) = \eta(a,5) = (a,3)$.

For a homomorphism $f:(A,\varphi,\psi) \longrightarrow (B,\varphi,\psi)$ set $\theta f(a,i) = (f(a),i)$ for every $a \in A$, $i \in 6$.

LEMMA 4.5: *θ is an embedding functor from the variety A(1,1) into $\underset{\sim}{V}$.*

Proof: It is easy to see that μ,ν,η are idempotent. Also, every operation maps the image of any other operation bijectively onto its own. Hence $\theta(A,\varphi,\psi) \in \underset{\sim}{V}$. It is obvious that θ is an embedding functor.■

LEMMA 4.6: *θ is a full embedding from A(1,1) into $\underset{\sim}{V}$.*

Proof: Let $g:\theta(A,\varphi,\psi) \longrightarrow \theta(B,\varphi,\psi)$ be a homomorphism. Observe that $\mu(x) = \eta(x) = x$ for some $x \in A\times 6$ (or $x \in B\times 6$) if and only if $x = (a,0)$ for some $a \in A$ (or $a \in B$). Hence there exists a mapping $f:A \longrightarrow B$ such that $g(a,0) = (f(a),0)$. From $\nu(a,0) = \nu(a,1) = (a,1)$ we conclude that $g(a,1) = (f(a),1)$. Further, for $x \in A\times 6$ (or $x \in B\times 6$),

$\nu(x) = \eta(x) = x$ just when $x = (a,3)$ for some $a \in A$ (or $a \in B$). Thus $g(a,3) = (f'(a),3)$ for some mapping $f': A \to B$. From $\mu(a,3) = \mu(a,2) = (a,2)$ we obtain $g(a,2) = (f'(a),2)$. We also have $\eta(a,4) = (a,0)$, $\mu(a,4) = (a,2)$ and $\eta^{-1}(a,0) = \{(a,0),(a,1),(a,4)\}$, $\mu^{-1}(a,2) = \{(a,2),(a,3),(a,4)\} \cup \{(b,5);\ \varphi(b) = a\}$ for $a \in A$ (or $a \in B$). The intersection of these sets is the singleton $\{(a,4)\}$; whence $f = f'$ and $g(a,4) = (f(a),4)$. Analogously, $\nu(a,5) = (a,1)$, $\eta(a,5) = (a,3)$ and $\nu^{-1}(a,1) = \{(a,0),(a,1),(a,5)\} \cup \{(b,4);\ \psi(b) = a\}$, $\eta^{-1}(a,3) = \{(a,2),(a,3),(a,5)\}$ for $a \in A$ (or $a \in B$), and the intersection of these sets is the singleton $\{(a,5)\}$; whence $g(a,5) = (f(a),5)$. Finally, $(f(\psi(a)),1) = g(\psi(a),1) = g(\nu(a,4)) = \nu(g(a,4)) = \nu(f(a),4) = (\psi(f(a)),1)$ and $(f(\varphi(a)),2) = g(\varphi(a),2) = g(\mu(a,5)) = \mu(g(a,5)) = \mu(f(a),5) = (\varphi(f(a)),2)$ for every $a \in A$, thus f commutes with φ and ψ. Therefore f is a homomorphism from (A,φ,ψ) into (B,φ,ψ) with $\Theta f = g$, so that Θ is a full embedding.∎

The proof of Theorem 3.6 follows from Theorem 4.1.∎

A. Pultr and J. Sichler [16] proved that the variety of $A(1,1)$ determined by the identities $\varphi \circ \alpha \circ \varphi = \varphi$ and $\psi \circ \alpha \circ \psi = \psi$ where α is an arbitrary word in $\{\varphi,\psi\}$ is not universal. Therefore the variety $\underset{\sim}{V}$ is the minimal universal variety of unary algebras fulfilling this type of identities. Precisely, we have:

THEOREM 4.7: *Let $\underset{\sim}{V}$ be a variety of unary algebras with n operations $\varphi_0,\varphi_1,\dots,\varphi_{n-1}$ whose defining identities are all identities of the form $\varphi_i \circ \alpha \circ \varphi_i = \varphi_i$ for every $i = 0,1,\dots,n-1$ and every word α over the alphabet $\{\varphi_0,\varphi_1,\dots,\varphi_{n-1}\}$. Then $\underset{\sim}{V}$ is universal if and only if $n \geq 3$.*∎

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