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RESEARCH ARTICLE

THE LATTICES OF VARIETIES AND PSEUDOVARIETIES OF BAND MONOIDS

Shelly L. Wismath[1]

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1. Introduction

The structure of the lattice of varieties of bands has been determined by Birjukov [2], Fennemore [3] and Gerhard [6]. The structure of this lattice is used here to determine the structure of two related lattices: the lattice LBM of varieties of band monoids, and the lattice LFBM of pseudovarieties of finite band monoids.

In Section 2, a brief description of the lattice LB of varieties of bands is given. Then in Section 3, a function Mon from LB to LBM is defined, and shown to be a surjective lattice homomorphism. In Section 4, the image of LB under Mon is studied, thus determining the structure of the lattice LBM.

In the final section, pseudovarieties of finite band monoids are considered. A treatment of the connection between pseudovarieties and languages may be found in Lallement [10]. Using some results of Ash's which relate varieties, pseudovarieties and generalized varieties, it is shown that the lattices LBM of varieties of band monoids and LFBM of pseudovarieties of finite band monoids are isomorphic.

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2. The Lattice of Varieties of Bands

Fennemore has shown in [3] that the varieties of bands in $\underline{LB}$ are each determined by one identity in addition to $x^2 = x$. Because we will be considering only varieties of bands, we will denote by $\underline{V}(P=Q)$ the variety of bands satisfying the additional identity $P=Q$, where P and Q are words on the alphabet $X = \{a,d,x,y,x_1, x_2,...\}$. Following the notation of Fennemore [3], the words R_n, S_n and Q_n, for $n \geq 2$, are defined as follows:

$$R_2 = R_2(x_1x_2x_3) = x_3x_2x_1,$$

$$R_3 = R_3(x_1x_2x_3) = x_1x_2x_3,$$

$$Q_2 = Q_2(x_1x_2x_3) = x_2x_3x_1,$$

$$Q_3 = Q_3(x_1x_2x_3) = x_1x_2x_3x_1x_3,$$

$$S_2 = S_2(x_1x_2x_3) = x_3x_1x_2x_1,$$

$$S_3 = S_3(x_1x_2x_3) = x_1x_2x_3x_1x_3x_2x_3,$$

$$R_n = R_n(x_1,...,x_n) = R_{n-1}x_n, \quad \text{for } n=4,6,...$$

$$R_n = R_n(x_1,...,x_n) = x_nR_{n-1}, \quad \text{for } n=5,7,...$$

$$Q_n = Q_n(x_1,...,x_n) = Q_{n-1}x_nR_n, \quad \text{for } n=4,6,...$$

$$Q_n = Q_n(x_1,...,x_n) = R_nx_nQ_{n-1}, \quad \text{for } n=5,7...$$

$$S_n = S_n(x_1,...,x_n) = S_{n-1}x_nR_n, \quad \text{for } n=4,6,...$$

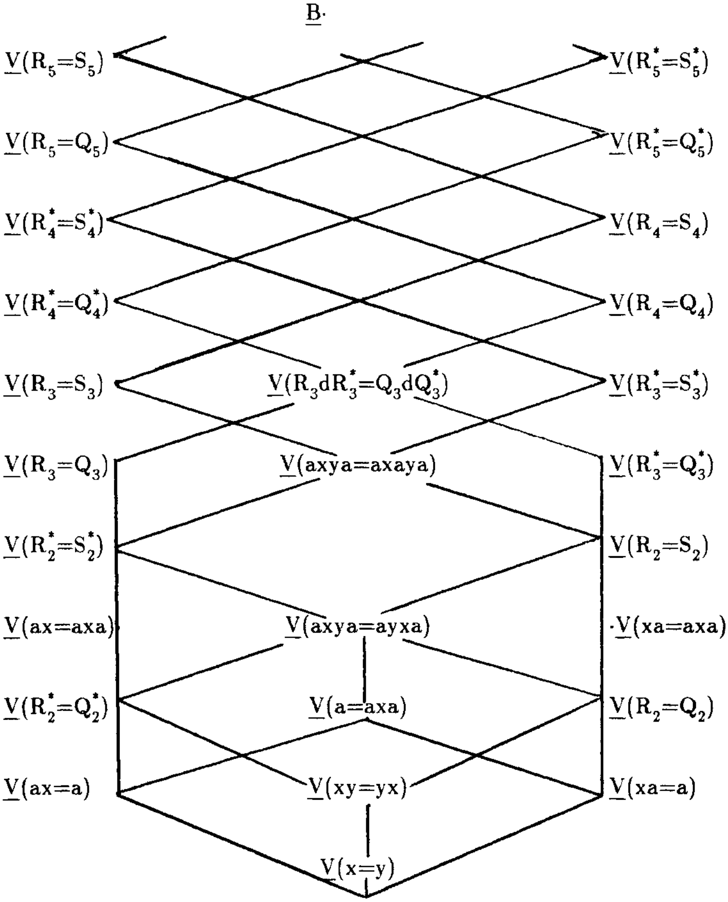
$$S_n = S_n(x_1,...,x_n) = R_nx_nS_{n-1}, \quad \text{for } n=5,7,....$$

For any word A, (variety $\underline{V}$), $A^*(\underline{V}^*)$ will denote the dual (variety) of A ($\underline{V}$).

The structure of the lattice $\underline{LB}$ is shown in Figure 1. The portion of the lattice above the variety $\underline{V}(axya=axaya)$ will be referred to as the <u>inductively defined part</u> of the lattice; the portion below and including the variety $\underline{V}(R_3dR_3^*=Q_3dQ_3^*)$ will be called the <u>base</u> of the lattice. Identities for the varieties not specifically labelled in Figure 1 may be found in Fennemore [3].

An important property of $\underline{LB}$ is its symmetry. The lattice is symmetic about a vertical line through $\underline{V}(x=y)$, in the sense that the corresponding

Figure 1: The Lattice of Varieties of Bands



varieties on either side of the line are $\underline{V}(P=Q)$ and $\underline{V}(P^*=Q^*)$, for some identity P=Q. This symmetry means that many of the results to be obtained in the following section may be "dualized": in any proof involving the words P,Q,..., replacing the words by their duals P^*,Q^*,... throughout will give a proof of the "mirror image" or dual result.

3. The Mapping Mon

Let $\underline{M}$ be the variety of all monoids, and let $\underline{V}$ be any variety of bands. Then $\underline{V} \cap \underline{M}$ is a variety of band monoids. Thus the mapping Mon taking $\underline{V}$ to $\underline{V} \cap \underline{M}$, for $\underline{V}$ in $\underline{LB}$, is a function from $\underline{LB}$ to $\underline{LBM}$. We show that Mon is a surjective lattice homomorphism. The following results will be useful. For any collection K of semigroups or monoids, we use SI[K] for the class of subdirectly irreducible members of K.

PROPOSITION 3.1: (Schein, [11]): A band monoid is subdirectly irreducible if and only if it is subdirectly irreducible as a band.

PROPOSITION 3.2: (Gerhard, [7]): Let $\underline{V} = \underline{U} \vee \underline{W}$ be any join-reducible variety in LB, with the exceptions that $\underline{V} \neq \underline{V}(R_2{=}Q_2)$ and $\underline{V} \neq \underline{V}(R_2^*{=}Q_2^*)$. Then $SI[\underline{V}] = SI[\underline{U} \cup \underline{W}]$.

PROPOSITION 3.3: Any variety in LB can be expressed as the join of a finite number of join-irreducible varieties in LB.

PROPOSITION 3.4: Mon is a lattice homomorphism.

Proof: Let $\underline{V}$ and $\underline{W}$ be any two varieties in $\underline{LB}$. Then $(\underline{V} \cap \underline{W}) \cap \underline{M} = (\underline{V} \cap \underline{M}) \cap (\underline{W} \cap \underline{M})$, so Mon preserves the meet operation. Since any variety is generated by its subdirectly irreducible members, it suffices to show for the join operator that

$$SI[(\underline{V} \vee \underline{W}) \cap \underline{M}] = SI[(\underline{V} \cap \underline{M}) \vee (\underline{W} \cap \underline{M})].$$

As long as $\underline{V} \vee \underline{W}$ is not one of the two exceptions mentioned in Proposition 3.2, we have by Proposition 3.1,

$$\begin{aligned} & SI[(\underline{V} \vee \underline{W}) \cap \underline{M}] \\ = {} & SI[\underline{V} \vee \underline{W}] \cap \underline{M} \\ = {} & (SI[\underline{V}] \cup SI[\underline{W}]) \cap \underline{M} \\ = {} & (SI[\underline{V}] \cap \underline{M}) \cup (SI[\underline{W}] \cap \underline{M}) \\ = {} & SI[\underline{V} \cap \underline{M}] \cup SI[\underline{W} \cap \underline{M}] \\ = {} & SI[(\underline{V} \cap \underline{M}) \vee (\underline{W} \cap \underline{M}]. \end{aligned}$$

The two exceptions to Proposition 3.2 occur in the base of $\underline{LB}$, for $\underline{V}(R_2{=}Q_2) = \underline{V}(xa{=}a) \vee \underline{V}(xy{=}yx)$ and its dual. It is easy to verify that Mon preserves the join operator in these two cases by direct calculation of the images of the varieties involved.

PROPOSITION 3.5: The homomorphism Mon is surjective.

Proof: Let $\underline{U}$ be any varieties of band monoids. Then $\underline{U}$ is generated by $SI[\underline{U}]$. As a collection of bands, this set generates a variety $\underline{V}$ of bands. Then $SI[\underline{U}] \subseteq \underline{V} \cap \underline{M}$, so $\underline{U} \subseteq \underline{V} \cap \underline{M}$, while conversely all the generators of $\underline{V}$ are monoids in $\underline{U}$, so $\underline{V} \cap \underline{M} \subseteq \underline{U}$. Therefore $\underline{U} = \mathrm{Mon}(\underline{V})$.

4. The Image of Mon

We begin by investigating a general property of any homomorphic image of $\underline{LB}$. For any lattice $\underline{L}$ and any $A \subseteq \underline{L}$, we use the notation of $\underline{L}(A)$ for the sublattice of L generated by A. Now let $\underline{L}$ be any lattice, with $\psi:\underline{LB} \to \underline{L}$ any surjective homomorphism. Let J be the set of join-irreducible elements of $\underline{LB}$, and for any $\underline{V} \in \underline{LB}$, let

$$J(\underline{V}) = \{\underline{W} \in J : \underline{W} \subseteq \underline{V} \text{ OR } \underline{W}^* \subseteq \underline{V}\}.$$

Let $T = \{\underline{V} \in J : \psi(\underline{V}) \neq \psi(\underline{W}) \text{ for any } \underline{W} \in J(\underline{V})\}$,

and $S = \{\psi(\underline{V}) : \underline{V} \in T\}$.

PROPOSITION 4.1: Under the above hypotheses,

(i) $\underline{L}(S) = \underline{L}$

and (ii) ψ induces an isomorphism of $\underline{LB}(T)$ onto $\underline{L}$.

Proof: (i) For any $\underline{F} \in \underline{L}$, $\underline{F} = \psi(\underline{G})$ for some $\underline{G} \in \underline{LB}$. By Proposition 3.3,

$$\underline{G} = \bigvee_{i=1} \underline{G}_i$$

for some join irreducibles $\underline{G}_1, \ldots \underline{G}_t$, $t \geq 1$. Then

$$\underline{F} = \psi\,(\bigvee_{i=1} \underline{G}_i) = \bigvee_{i=1} \psi(\underline{G}_i).$$

If the $\underline{G}_i$'s are all in T, then $\underline{F} \in \underline{L}(S)$, and we are done. If $\underline{G}_i \notin T$ for some $1 \leq i \leq t$, it is because $\psi(\underline{G}_i) = \psi(\underline{W}_i)$ for some $\underline{W} \in J(\underline{G}_i)$. If $\underline{W}$ or its dual is in T, we replace $\underline{G}_i$ by $\underline{W}_i$ or $\underline{W}_i^*$ in the expression for $\underline{F}$ above; if not, we replace $\underline{W}_i$ by this same process. Since the collections $\underline{J}(\underline{G}_i)$, $1 \leq i \leq t$, are all finite, we eventually end with $\underline{F}$ expressed as a join of elements of the form $\psi(\underline{G}_i)$ with the $\underline{G}_i$'s all in T. Hence $\underline{F} \in \underline{L}(S)$, so $\underline{L} = \underline{L}(S)$.

(ii) When ψ is restriced to the set T, it forms a bijection from T to S: it is clearly surjective, and if $\psi(\underline{V}_1) = \psi(\underline{V}_2)$ for $\underline{V}_1$, $\underline{V}_2$, $\in$ T, then $\underline{V}_1$, $\underline{V}_2$ are join-irreducibles with neither properly contained in the other or in the dual of the other, forcing $\underline{V}_1 = \underline{V}_2$. Hence on LB(T) ψ induces a lattice homomorphism.

Hence to examine the image $\underline{LBM}$ of $\underline{LB}$ under Mon, it suffices to identify T, the set of join-irreducible varieties $\underline{V}$ in $\underline{LB}$ such that Mon($\underline{V}$) $\neq$ Mon($\underline{W}$) for any join-irreducible variety $\underline{W}$ properly contained in $\underline{V}$ or its dual. We begin with the base of $\underline{LB}$, which contains eight join-irreducible varieties. These are $\underline{V}$(x=y) and $\underline{V}$(xy=yx), and $\underline{V}$(ax=a), $\underline{V}$(ax=axa), $\underline{V}(R_3=Q_3)$ and their duals. Using the notation $\underline{VM}$(P=Q) for the variety of band monoids satisfying the identity P=Q, we note that $\underline{V}$(P=Q) $\cap$ $\underline{M}$ = $\underline{VM}$(P=Q).

<u>PROPOSITION 4.2:</u> (i) $\underline{VM}$(ax=a) = $\underline{VM}$(xa=a) = $\underline{VM}$(x=y)
(ii) $\underline{VM}$(ax=axa) = $\underline{VM}(R_3=Q_3)$, <u>and dually.</u>

<u>Proof:</u> (i) Trivially, $\underline{VM}$(x=y) = $\underline{VM}$(x=1). Let M be any monoid in $\underline{VM}$(ax=a) or in $\underline{VM}$(xa=a). Then for any m $\in$ M, the substitution x=m and a=1 results in m=1, so M satisfies the identity x=1.

(ii) Because the corresponding inclusion is true for varieties of bands, we have

$$\underline{VM}(ax=axa) \subseteq \underline{VM}(R_3=Q_3).$$

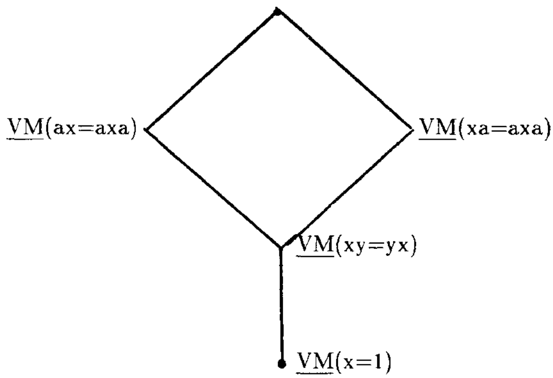
Let M be any monoid in $\underline{VM}(R_3=Q_3)$, so that M satisfies the identity $x_1x_2x_3 = x_1x_2x_3x_1x_3$. Let m,n $\in$ M. The substitution x_1 = m, x_2 = n, and x_3 = 1 produces mn = mnm from this identity. Therefore M is in $\underline{VM}$(ax=axa), and the equality follows.

<u>PROPOSITION 4.3: The image of the base of LB is as shown in Figure 2.</u>

<u>Proof:</u> Let T′ be the subset of T whose members come from the base of $\underline{LB}$. By Propositions 4.2 and 4.3, we have

$$T' \subseteq \{\underline{V}(x=y),\ \underline{V}(xy=yx),\ \underline{V}(xa=axa),\ \underline{V}(ax=axa)\}.$$

Figure 2: The Image of the Base of LB



Clearly $\underline{VM}(x=y) \subseteq \underline{VM}(xy=yx)$. We will show that $\underline{VM}(xy=yx)$, $\underline{VM}(xa=axa)$, and $\underline{VM}(ax=axa)$ are all distinct. This will show that T′ equals the set shown above and the result then follows from Proposition 4.1.

From the structure of LB, there is a semigroup A which is in $\underline{V}(ax=axa)$ but not in $\underline{V}(xa=axa)$ or in $\underline{V}(xy=yx)$. If A is a monoid, then $\underline{VM}(xa=axa) \neq \underline{VM}(ax=axa) \neq \underline{VM}(xy=yx)$. If A is not a monoid, let M be the monoid $A \cup \{1\}$. Then M does not satisfy xa=axa or xy=yx. Let $m,n \in M$, and consider the substitution x=m and a=n in the identity xa=axa. If neither m nor n is 1, then mn = nmn; if m=1, then mn = n = nmn; if n=1, then mn = m = nmn. In each case, the identity xa=axa is satisfied by M, so M is in $\underline{VM}(xa=axa)$.

We now look at the inductively defined part of the lattice LB. The join-irreducible varieties here are $\underline{V}(R_n=S_n)$ and their duals for $n \geq 3$, and $\underline{V}(R_n=Q_n)$ and their duals for $n \geq 4$. The following proposition establishes a rather technical result which will allow us to prove that $\underline{VM}(R_n=Q_n) = \underline{VM}(R^*_{n-1}=S^*_{n-1})$ for $n \geq 3$.

PROPOSITION 4.4: Let $n \geq 4$. Let M be a monoid with identity element 1, and let $a_1, \ldots, a_{n-1}$ be any elements of M. Then

$$R_n(a_3,a_2,1,a_1,a_4,a_5,\ldots,a_{n-1}) = R^*_{n-1}(a_1,\ldots,a_{n-1}),$$

and

$$Q_n(a_3,a_2,1,a_1,a_4,a_5,\ldots,a_{n-1}) = S^*_{n-1}(a_1,\ldots,a_{n-1}).$$

Proof: We use induction on n. For n=4, we have

$$\begin{aligned} R_4(a_3,\ a_2,\ 1,\ a_1) &= a_3a_2 1 a_1 \\ &= a_3a_2a_1 \\ &= R_3^*(a_1,a_2,a_3) \end{aligned}$$

and

$$\begin{aligned} Q_4(a_3,\ a_2,\ 1\ ,a_1) &= a_3a_2 1 a_3 1 a_1 a_3 a_2 1 a_1 \\ &= a_3a_2 1 a_3 1 a_1 a_3 a_2 1 a_1 \\ &= S_3^*(a_1,a_2,a_3). \end{aligned}$$

Thus the result holds for n=4.

Now assume that the result of the proposition is true for all k such that $4 \leq k < n$. Then when n is odd, we have

$$\begin{aligned} R_n(a_3,a_2,1,a_1,a_4,a_5,\ldots,a_{n-1}) &= a_{n-1}R_{n-1}(a_3,a_2,1,a_1,a_4,\ldots,a_{n-2}) \\ &= a_{n-1}R^*_{n-2}(a_1,\ldots,a_{n-2}) \\ &= R^*_{n-1}(a_1,\ldots,a_{n-1}). \end{aligned}$$

Now using the induction hypothesis again and the result just established for R_n when n is odd, we have

$$\begin{aligned} &Q_n(a_3,a_2,1,a_1,a_4,a_5,\ldots,a_{n-1}) \\ &\quad= R_n(a_3,a_2,1,a_1,\ldots a_{n-2})a_{n-1}Q_{n-1}(a_3,a_2,1,a_1,\ldots a_{n-2}) \\ &\quad= R^*_{n-1}(a_1,\ldots,a_{n-1})a_{n-1}S^*_{n-2}(a_1,\ldots,a_{n-2}) \\ &\quad= S^*_{n-1}(a_1,\ldots,a_{n-1}). \end{aligned}$$

The proof for n even is very similar.

<u>PROPOSITION</u> <u>4.5:</u> <u>For</u> <u>all</u> $n \geq 3$, $\underline{VM}(R_n=Q_n) = \underline{VM}(R^*_{n-1}=S^*_{n-1})$, <u>and</u> <u>dually</u>.

<u>Proof:</u> Since $\underline{VM}(R^*_{n-1}=S^*_{n-1})$ is contained in $\underline{VM}(R_n=Q_n)$, only the opposite inclusion need be proved. For n=3, this follows from Proposition 4.2. Now let M be any monoid in $\underline{VM}(R_n=Q_n)$, where $n \geq 4$. We must show that if $a_n,\ldots,a_{n-1}$ are any n-1 elements of M, then

$$R^*_{n-1}(a_1,\ldots,a_{n-1}) = S^*_{n-1}(a_1,\ldots,a_{n-1}).$$

Since $n \geq 4$, Proposition 5.1 says that

$$R^*_{n-1}(a_1,\ldots,a_{n-1}) = R_n(a_3,a_2,1,a_4,\ldots a_{n-1})$$

and

$$S^*_{n-1}(a_1,\ldots,a_{n-1}) = Q_n(a_3,a_2,1,a_4,\ldots a_{n-1}).$$

But since M is in $\underline{V}(R_n=Q_n)$,

$$R_n(a_3,a_2,1,a_1,\ldots,a_{n-1}) = Q_n(a_3,a_2,1,a_1,\ldots,a_{n-1}),$$

and thus

$$R^*_{n-1}(a_1,\ldots,a_{n-1}) = S^*_{n-1}(a_1,\ldots,a_{n-1})$$

Therefore for any $n \geq 3$, $\underline{VM}(R_n=Q_n) = \underline{VM}(R^*_{n-1}=S^*_{n-1})$.

We show next that the collapsing under Mon established in the previous proposition is all the collapsing that occurs for the join-irreducibles.

PROPOSITION 4.6: For $n \geq 3$, $\underline{VM}(R^*_{n-1}=S^*_{n-1}) \subsetneq \underline{VM}(R_n=S_n)$, and $\underline{VM}(R_n=S_n) \neq \underline{VM}(R^*_n=S^*_n)$.

Proof: Gerhard has constructed in [8] a subdirectly irreducible monoid B_n, for $n \geq 3$, which generates the variety $\underline{V}(P=Q)$, where in his terminology, PT_nQ, $PR_n{}^*Q$, and $P\not{S}_n{}^*Q$. In terms of Fennemore's identities, used here, this means that for n odd, B_n generates the variety $\underline{V}(R_n=S_n)$ while B_n is not in $V(R_n=Q_n)$, and dually for n even. Hence

$$\underline{VM}(R_n=Q_n) = \underline{VM}(R^*_{n-1}=S^*_{n-1}) \subsetneq \underline{VM}(R_n=S_n),$$

and dually. This establishes the required proper inclusions for the left-hand side of the lattice.

Now suppose that for n odd, B_n was in both $\underline{V}(R_n=S_n)$ and $\underline{V}(R^*_n=S^*_n)$. Then

$$\begin{aligned} B_n &\in \underline{V}(R_n=S_n) \cap \underline{V}(R^*_n=S^*_n) \\ &= \underline{V}(R^*_{n-1}=S^*_{n-1}) \vee \underline{V}(R_{n-1}=S_{n-1}). \end{aligned}$$

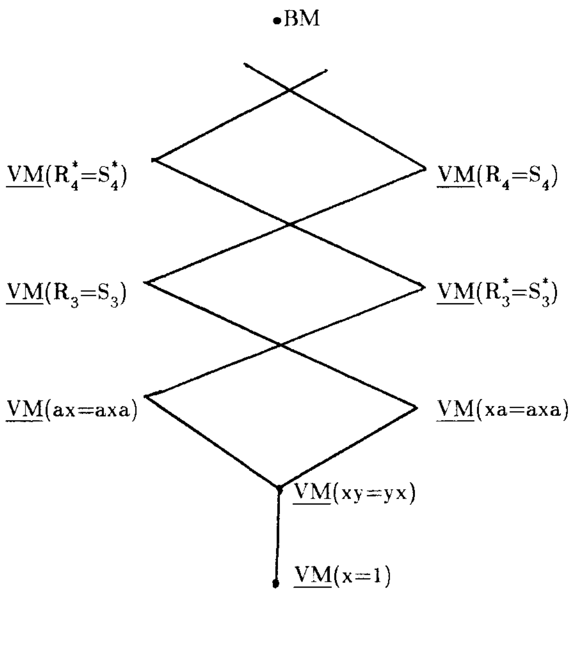
Since B_n is subdirectly irreducible, by Proposition 3.2, B_n would be in one or the other of these last two varieties. Since B_n generates $\underline{V}(R_n=S_n)$, this is impossible.

By dualizing the above argument, starting with B_n^* generating $\underline{V}(P^*=Q^*)$ where B_n generates $\underline{V}(P=Q)$, we obtain the results for the right-hand side of the lattice.

As a result of Propositions 4.3, 4.5, and 4.6, we conclude that T contains precisely the following varieties: $\underline{V}(x=y)$, $\underline{V}(xy=yx)$, $\underline{V}(ax=axa)$ and $\underline{V}(xa=axa)$ from the base, and $\underline{V}(R_n=S_n)$ and its dual, for $n \geq 3$. The lattice $\underline{LBM}$ is then isomorphic to the sublattice of $\underline{LB}$ generated by T. This proves the following:

PROPOSITION 4.7: The structure of the lattice LBM of all varieties of band monoids is as shown in Figure 3.

Figure 3: The Lattice of Varieties of Band Monoids



5. The Lattice of Pseudovarieties of Band Monoids

Having determined the structure of the lattice $\underline{LBM}$ of varieties of band monoids, we may now try to relate it to pseudovarieties of band monoids. Ash has shown in [1] that any pseudovariety is precisely the class of finite members of some generalized variety. In particular, if $\underline{V}$ is a variety in $\underline{LBM}$, then the collection Fin(V) of finite monoids in V is a pseudovariety. We denote by LFBM the lattice of pseudovarieties of finite band monoids. Then we may define a function Fin from LBM to LFBM by letting Fin take V to Fin(V), for any V in LBM. We show now that Fin is in fact a lattice isomorphism, thus determining the structure of the lattice LFBM. We do this by showing that Fin is a bijection with the property that both it and its inverse are order-preserving.

PROPOSITION 5.1: Let V and W be any two varieties in LB. Then Fin(V) $\subseteq$ Fin(W) if and only if V $\subseteq$ W.

Proof: One direction is trivial. Suppose V is not contained in W, so there is a monoid M in V which is not in W. Then there are distinct equations P=Q and H=K such that V = VM(P=Q) and W = VM(H=K). Since M is not in W, there exist $a_1,\ldots,a_n \in$ M (for some integer n) such that $H(a_1,\ldots,a_n) \neq K(a_1,\ldots,a_n)$. Let N be the free band monoid on $\{a_1,\ldots,a_n\}$. By [9], N is a finite band monoid, satisfying P=Q but not H=K; i.e., in Fin(V) but not in Fin(W).

PROPOSITION 5.2: The mapping Fin is a lattice isomorphism from LBM onto LFBM.

Proof: Let V be any pseudovariety of band monoids in LFBM. By the results of Ash, [1], we know that V must consist of Fin(W), where W is the generalized variety generated by V, and that W must be the union of some directed family D of varieties from the lattice LBM.

Suppose that the directed family D is a finite one. Then the union W of members of D is just a variety U in LBM, and we have V = Fin(U).

If D is not a finite directed family, there are only two possibilities for the union W of members of D. This union may by all of BM; in this case we have V = Fin(W) = Fin(BM). Otherwise, W must be the class of all band monoids which are contained in some proper subvariety of BM. Clearly then Fin(W) is contained in Fin(BM). But also any finite band monoid is contained in some proper subvariety of BM, [5], and so Fin(BM) is contained in Fin(W). Therefore V = Fin(W) = Fin(BM).

Hence Fin is surjective. By the previous proposition, it is also injective, and it and its inverse are both order-preserving.

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Simon Fraser University
Burnaby, B.C.
Canada

and

The University of Lethbridge
Lethbridge, Alberta
Canada

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