Semigroup Forum Vol. 34 (1986) 185–201

RESEARCH ARTICLE

ON THE MODULUS OF ONE-PARAMETER SEMIGROUPS

by

Ingo Becker

and

Günther Greiner

Communicated by R. Nagel

1. INTRODUCTION

Given strongly continuous one-parameter semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on a Banach lattice E, we say that $(S(t))_{t\geq 0}$ dominates $(T(t))_{t\geq 0}$ provided that

(1.1) $$|T(t)f| \leq S(t)|f| \quad \text{for all } f \in E,\ t \geq 0 .$$

In this case the semigroup $(S(t))_{t\geq 0}$ necessarily consists of positive operators, i.e. it is a *positive semigroup*. It is well-known that positive semigroups have interesting properties, e.g. their spectral properties and asymptotic behavior can be analysed very well (see Nagel [11]). Some of the results on positive semigroups have consequences for a (non-positive) semigroup $(T(t))_{t\geq 0}$ provided that (1.1) is

In case the Banach lattice is σ-order complete and $D \subseteq D(A)$ is a core for A (i) and (ii) are equivalent to

(iii) $\operatorname{Re} \langle (\operatorname{sign} \bar{f}) Af , \varphi \rangle \leq \langle |f| , B'\varphi \rangle$ *for all* $f \in D$ *and for all* $\varphi \in D(B')_+$.

For semigroups having a bounded generator the existence of a modulus and the description of its generator is given by Derndinger [3]. For details on the center $\mathcal{Z}(E)$ of a Banach lattice (i.e. the set of all multiplication operators) we refer to Nagel [11], Chapter C-I. We only recall that for order complete Banach lattices the center is a projection band in the Banach lattice $\mathcal{L}^r(E)$, thus $\mathcal{L}^r(E) = \mathcal{Z}(E) \oplus \mathcal{Z}(E)^{\perp}$. Here, $\mathcal{L}^r(E)$ denotes the set of all regular operators on E , i.e. the linear hull of all positive operators. We give a new proof of Derndinger's result which is based on Chernoff's product formula (see Goldstein [5], Thm. I.8.4).

PROPOSITION 1.2. *Let E be an order complete Banach lattice and let $(T(t))_{t\geq 0}$ be a semigroup whose generator A is a regular operator. Then $(T(t))_{t\geq 0}$ possesses a modulus whose generator $A_{\#}$ is obtained as follows: If A is decomposed such that $A = M + B$, where $M \in \mathcal{Z}(E)$, $B \in \mathcal{Z}(E)^{\perp}$ then*

$$A_{\#} = \operatorname{Re} M + |B| .$$

Proof: We apply Chernoff's product formula to the function $F : [0,\infty) \to \mathcal{L}(E)$, $F(t) := |T(t)|$. In order to differentiate at 0 we first note that

$$(1.2) \qquad |F(t) - |Id - tA|| \leq |T(t) - (Id + tA)| = \Big|\sum_{n\geq 2} \frac{t^n}{n!} A^n\Big| \leq \sum_{n\geq 2} \frac{t^n}{n!} |A|^n \quad ,$$

true. This is obvious for stability results (i.e., convergence to zero as $t \to \infty$). Further examples are ergodic theorems. In fact, Kipnis [9] and Kubokawa [10] used this method in order to prove ratio ergodic theorems and local ergodic theorems for (non-positive) contraction semigroups on L^1-spaces.

In this note we mainly discuss the problem whether for a strongly continuous one-parameter semigroup $(T(t))_{t\geq 0}$ there exists a minimal dominating semigroup. In case such a semigroup exists, we call it the *modulus* of $(T(t))_{t\geq 0}$. We will give some general conditions in Sec.2 and discuss some concrete examples in Sec.3. In particular, we prove that a modulus always exists in case the semigroup is dominated and the underlying Banach lattice has order continuous norm or in case the semigroup is order contractive and the Banach lattice is an L^p-space. Moreover, we will describe the modulus for semigroups corresponding to retarded differential equations.

In the rest of this section we recall some results on domination of semigroups which will be needed in Sec.2 and Sec.3. At first we state the characterization of domination in terms of the resolvents and the generators respectively (see Arendt [1] or Sec.C-II.3 of Nagel [11]).

PROPOSITION 1.1. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach lattice E with generator A and let $(S(t))_{t\geq 0}$ be a positive semigroup with generator B. The following assertions are equivalent:*

(i) $(S(t))_{t\geq 0}$ *dominates* $(T(t))_{t\geq 0}$ (*i.e.* (1.1) *holds*).

(ii) *There is* $\lambda_0 \in \mathbb{R}$ *such that*

$$|R(\lambda,A)f| \leq R(\lambda,B)|f| \quad \text{for all} \quad f \in E \,,\ \lambda \geq \lambda_0 \,.$$

hence $$\lim_{t\to 0} \frac{F(t) - |Id + tA|}{t} = 0.$$

Moreover, from the decomposition $A = M + B \in \mathcal{Z}(E) \oplus \mathcal{Z}(E)^{\perp}$ it follows that $Id + tM$ and $t\cdot B$ are orthogonal, hence

$$(1.3) \qquad |Id + tA| = |Id + tM| + t\cdot|B| .$$

Thus we conclude

$$\lim_{t\to 0} \frac{F(t) - Id}{t} \overset{(1.2)}{=} \lim_{t\to 0} \frac{|Id + tA| - Id}{t} \overset{(1.3)}{=} |B| + \lim_{t\to 0} \frac{|Id + tM| - Id}{t}$$
$$= |B| + \text{Re } M \ .$$

The last equality follows from the fact that $\mathcal{Z}(E)$ is isomorphic to a space $C(K)$ and that the function $\phi : C(K) \to C(K)$, $f \to |f|$ is (Fréchet-) differentiable at 1 with derivative $D\phi(1)f = \text{Re } f$ for all $f \in C(K)$.

Chernoff's product formula now yields that the semigroup $(S(t))_{t\geq 0}$ generated by $\text{Re } M + |B|$ satisfies

$$S(t)f = \lim_{n\to\infty} |T(\tfrac{t}{n})|^n f \quad (f \in E) .$$

This obviously implies that $(S(t))_{t\geq 0}$ is the modulus of $(T(t))_{t\geq 0}$.

□

Kipnis [9] and Kubokawa [10] have shown that all strongly continuous contraction semigroups on L^1-spaces possess a modulus. But Kipnis gives an example of a strongly continuous semigroup on ℓ^p $(1 \leq p < \infty)$ which does not have a modulus. We briefly sketch this example:

<u>EXAMPLE</u> 1.3. Let $E = \ell^p(\mathbb{N})$ $(1 \leq p < \infty)$ and let the operator A be given by the infinite matrix

$$A := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & \text{etc.} \end{bmatrix} .$$

If we take the maximal domain

$$D(A) := \left\{ f \in \ell^p(\mathbb{N}) \; ; \; Af \in \ell^p(\mathbb{N}) \right\} ,$$

then A generates the semigroup $(T(t))_{t\geq 0}$, which is given by

$$T(t) = \begin{bmatrix} \cos t & -\sin t & 0 & 0 & 0 & 0 & \\ \sin t & \cos t & 0 & 0 & 0 & 0 & \\ 0 & 0 & \cos 2t & -\sin 2t & 0 & 0 & \\ 0 & 0 & \sin 2t & \cos 2t & 0 & 0 & \\ 0 & 0 & 0 & 0 & \cos 3t & -\sin 3t & \\ 0 & 0 & 0 & 0 & \sin 3t & \cos 3t & \text{etc.} \end{bmatrix} .$$

This semigroup does not possess a dominating semigroup.

In fact, $|\cos kt| + |\sin kt|$ $(k \in \mathbb{N})$ is an eigenvalue of $|T(t)|$.

Thus a dominating semigroup $(S(t))_{t\geq 0}$ must satisfy

$$\|S(t)\| = \|S(t/n)^n\| \geq \left(|\cos \tfrac{k\cdot t}{n}| + |\sin \tfrac{k\cdot t}{n}| \right)^n .$$

For $n \to \infty$ we deduce that $\|S(t)\| \geq e^{kt}$. Since $k \in \mathbb{N}$ is arbitrary we obtain a contradiction to the boundedness of $S(t)$.

2. EXISTENCE OF THE MODULUS OF A ONE-PARAMETER SEMIGROUP

Only for few Banach lattices E it is true that every operator $T \in \mathcal{L}(E)$ possesses a modulus : Roughly speaking E has to be isomorphic either to an L^1-space or a space $C(K)$ with K Stonian (see Cartwright-Lotz [2]). The analogue problem in the setting of one-parameter semigroups (i.e. the existence of a minimal dominating semigroup for every one-parameter semigroup on a fixed Banach lattice) is not very interesting. In fact, since every infinite dimensional L^1space contains ℓ^1 as a complemented sublattice, Kipnis' example shows that on an infinite-dimensional L^1-space there always exist one-parameter semigroups which do not have any dominating semigroup. Moreover, on spaces $C(K)$, K Stonian, every one-parameter semigroup has a bounded generator (see Sec.A-II.3 of Nagel [11]), thus Derndinger's result can be applied. These observations show that one has to restrict the class of semigroups in order to obtain interesting results on the existence of the modulus of a semigroup.

THEOREM 2.1. Let $(T(t))_{t\geq 0}$ be a strongly continuous one-parameter semigroup on a Banach lattice E with order continuous norm. If there is a dominating strongly continuous semigroup then there exists a modulus of $(T(t))_{t\geq 0}$.

Proof: Let $(S(t))_{t\geq 0}$ be a semigroup which dominates $(T(t))_{t\geq 0}$. We fix $t \geq 0$ and denote the set of all partitions of the interval $[0,t]$ by Π_t

(i.e. $\Pi_t := \left\{(t_0, t_1, t_2, \dots, t_n) : n \in \mathbb{N}, t_0=0 \leq t_1 \leq \dots \leq t_n=t \right\}$).

For $\pi = (t_0, t_1, t_2, \dots, t_n) \in \Pi_t$ we define

$$T_\pi(t) := |T(t_n-t_{n-1})| \circ |T(t_{n-1}-t_{n-2})| \circ \dots \circ |T(t_2-t_1)| \circ |T(t_1)| .$$

Note that E is order complete and $T(t_k-t_{k-1})$ is dominated by $S(t_k-t_{k-1})$, hence $|T(t_k-t_{k-1})|$ exists (see Schaefer [12], Chap. IV, Prop.1.2). The net $(T_\pi(t))_{\pi\in\Pi_t}$ is upward directed and we have

$$T_\pi(t) \leq S(t_n-t_{n-1}) \circ S(t_{n-1}-t_{n-2}) \circ \dots \circ S(t_2-t_1) \circ S(t_1) = S(t)$$

for every $\pi \in \Pi_t$.

Since E has order continuous norm it follows that for $f \in E_+$

$$T_\#(t)f := \lim_{\pi\in\Pi_t} T_\pi(t)f = \sup_{\pi\in\Pi_t} T_\pi(t)f \quad \text{exists.}$$

Extending $T_\#(t)$ to a linear operator on E , we have $|T(t)| \leq T_\#(t) \leq S(t)$. Thus in order to show that $(T_\#(t))_{t\geq 0}$ is the modulus of $(T(t))_{t\geq 0}$ we have to show that $T_\#(t+s) = T_\#(t)T_\#(s)$ for all $s,t \geq 0$ and that $t \mapsto T_{\#}(t)$ is strongly continuous at $t = 0$.

Given $t,s \geq 0$, $\pi_1 \in \Pi_t$, $\pi_2 \in \Pi_s$ we have

$T_{\pi_1}(t)\circ T_{\pi_2}(s) = T_{\pi_3}(t+s)$ for a suitable partition $\pi_3 \in \Pi_{t+s}$.

Moreover, the partitions π_3 obtained in this way are cofinal in Π_{t+s}, hence we have for $f \in E_+$

$$\begin{aligned} T_{\#}(t)\circ T_{\#}(s)f &= \sup_{\pi_1\in\Pi_t} [\, T_{\pi_1}(t)\, (\sup_{\pi_2\in\Pi_s} T_{\pi_2}(s)f)\,] \\ &= \sup \left\{ T_{\pi_1}(t)\circ T_{\pi_2}(s)f \;:\; \pi_1 \in \Pi_t,\ \pi_2 \in \Pi_s \right\} \\ &= \sup \left\{ T_{\pi}(t+s)f \;:\; \pi \in \Pi_{t+s} \right\} \\ &= T_{\#}(t+s)f . \end{aligned}$$

(Here we used that $T_\pi(t)$ is order continuous which follows from the assumption that E has order continuous norm.)

It remains to prove strong continuity at $t = 0$.

From $|T(t)f| \leq T_{\#}(t)|f| \leq S(t)|f|$ we deduce for $f \in E_+$

$$T_{\#}(t)f - f \leq S(t)f - f$$

and
$$f - T_{\#}(t)f \leq f - |T(t)f| \leq |f - T(t)f|.$$

It follows that $|T_{\#}(t)f - f| \leq |S(t)f - f| + |T(t)f - f|$.

Since both, $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are strongly continuous it follows that

$$\lim_{t\to 0} |T_{\#}(t)f - f| = 0.$$

□

<u>REMARK</u> 2.2: As can be seen from the proof of Thm. 2.1 it suffices to assume that E is order complete and there is a dominating semigroup consisting of order continuous operators.

In the result above we had to assume that there exists a dominating semigroup. In many concrete examples this condition can be verified easily (see also Section 3). In the following we show that for certain semigroups on L^p-spaces a dominating semigroup always exists. Then Thm. 2.2 implies that these semigroups possess a modulus.

PROPOSITION 2.3. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on an L^p- space $(1 \leq p < \infty)$ and assume that it is quasi-contractive with respect to the regular norm (i.e. there exists $w \in \mathbb{R}$ such that $\|T(t)\|_r := \| |T(t)| \| \leq e^{wt}$ for all $t \geq 0$). Then a dominating semigroup exists.*

Proof: Considering $(e^{-wt}T(t))_{t\geq 0}$ if necessary we may assume that $w = 0$. We use the notations introduced in the proof of Thm. 2.1. By assumption the operators $T_\pi(t)$ are contractions. Thus for $0 \leq f \in E = L^p(\mu)$ the set
$\left\{ T_\pi(t)f \ : \ \pi \in \Pi_t \right\}$ is normbounded (by $\|f\|$) and upward directed.
The Monotone Convergence Theorem implies that

$$T_{\#}(t)f := \lim_{\pi\in\Pi_t} T_\pi(t)f \qquad \text{exists.}$$

Moreover, $\|T_{\#}(t)\| \leq 1$ and $|T(t)| \leq T_{\#}(t)$ for every $t \geq 0$.
As in the proof of Thm. 2.1 we conclude that $(T_{\#}(t))_{t\geq 0}$ has the semigroup property. To prove strong continuity we make use of the inequality
$\|f\|^p + \|g\|^p \leq \|f+g\|^p$, valid for $0 \leq f,g \in L^p(\mu)$.
In fact, since we have $|T(t)f| \leq T_{\#}(t)|f|$ for all $f \in E$, it follows that

$$\| |T(t)f| \|^p + \|T_{\#}(t)|f| - |T(t)f|\|^p \leq \|T_{\#}(t)|f|\|^p \leq \|f\|^p.$$

Hence $\lim_{t\to 0} \|T_{\#}(t)|f| - |T(t)f|\| \leq \lim_{t\to 0} \left[\|f\|^p - \||T(t)f|\|^p\right]^{1/p} = 0$.
It follows that

$$\lim_{t\to 0} T_{\#}(t)f = \lim_{t\to 0} |T(t)f| = f \quad \text{for every } f \in E_+ .$$

Thus $t \mapsto T_{\#}(t)$ is strongly continuous at $t = 0$.

□

In the final result of this section we show that adding a multiplication operator to a generator does not cause difficulties when one is looking for a modulus of the corresponding semigroup. We recall that the center $\mathscr{Z}(E)$ of a Banach lattice is the set of all operators on E which are dominated by a multiple of the identity operator. In case $E = L^p(\mu)$ the center consists of all bounded multiplication operators. $\mathscr{Z}(E)$ is always a commutative subalgebra of $\mathscr{L}(E)$ and if E is order complete then for fixed $f \in E$ the operator $g \mapsto (\text{sign } f)g$ is an element of the center. It follows that for $M \in \mathscr{Z}(E)$ and $f \in E$ we have $(\text{sign } \bar{f})Mf = M((\text{sign } \bar{f})f) = M|f|$. Thus if A is any operator and $M \in \mathscr{Z}(E)$ then

$$\begin{aligned} \text{Re}((\text{sign } \bar{f})(A+M)f) &= \text{Re}((\text{sign } \bar{f})Af) + \text{Re}(M|f|) \\ &= \text{Re}((\text{sign } \bar{f})(A+\text{Re}M)f) \quad (f \in D(A)). \end{aligned}$$

So we obtain the following result as an immediate consequence of Prop. 1.1 .

PROPOSITION 2.4. *Let A be the generator of a strongly continuous one-parameter semigroup on an order complete Banach lattice E and take $M \in \mathscr{Z}(E)$. Then the semigroup generated by A possesses a modulus if and only if the semigroup generated by $A + M$ has a modulus. In case the moduli exist, their generators satisfy*

$$(A + M)_{\#} = A_{\#} + \text{Re}M \ .$$

3. SOME EXAMPLES

At first we consider second order differential operators on $L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$. Let A be defined as follows :

$$(3.1)\quad Af = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} f + c\cdot f$$

$$f \in D(A) := C_c^\infty(\mathbb{R}^n) .$$

We assume that the coefficients satisfy the following conditions :

(3.2)- a_{ij} are real-valued bounded differentiable functions on $\mathbb{R}^n$ which have bounded second order derivatives;

- $(a_{ij}(x))_{1\le i,j\le n}$ is symmetric and uniformly elliptic on $\mathbb{R}^n$ (i.e. $m\cdot\|\xi\|^2 \le \sum_{i,j} a_{ij}(x)\xi_i\xi_j$ for all $x,\xi \in \mathbb{R}^n$ and a suitable constant $m > 0$);
- b_i are bounded differentiable functions on $\mathbb{R}^n$ having bounded derivatives;
- c is a bounded continuous function on $\mathbb{R}^n$.

It is well-known that under these assumptions the closure of A is the generator of an analytic semigroup on $L^p(\mathbb{R}^n)$ (see Fattorini [4]). In case all coefficients are real-valued the corresponding semigroup is positive. For complex-valued coefficients b_i and c we have the following result:

The semigroup generated by the closure of A has a modulus, its generator is the closure of $A_\#$ defined as follows:

$$A_\# f = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f + \sum_{i=1}^{n} \mathrm{Re}(b_i) \frac{\partial}{\partial x_i} f + \mathrm{Re}(c)\cdot f$$

$$D(A_\#) := C_c^\infty(\mathbb{R}^n) .$$

We mentioned above that the closure of $A_\#$ generates a positive semigroup. Moreover we have

$$\operatorname{Re}(\operatorname{sign}\bar f\,\frac{\partial f}{\partial x_i})\,(x) = \begin{cases} \frac{\partial|f|}{\partial x_i}\,(x) & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

which implies that $\operatorname{Re}(\operatorname{sign}\bar f\,[b_i\,\frac{\partial f}{\partial x_i}]) = \operatorname{Re}(\operatorname{sign}\bar f\,[(\operatorname{Re} b_i)\frac{\partial f}{\partial x_i}])$.

Using the identity $\operatorname{Re}(\operatorname{sign}\bar f\; c\cdot f) = \operatorname{Re}(\operatorname{sign}\bar f\;(\operatorname{Re} c)\cdot f)$ we obtain

(3.3) $\operatorname{Re}(\operatorname{sign}\bar f\; Af) = \operatorname{Re}(\operatorname{sign}\bar f\; A_{\#}f)$, $f \in D(A) = D(A_{\#})$.

From Prop. 1.1 (iii) we conclude that every semigroup which dominates the semigroup $(T(t))_{t\geq 0}$ generated by $\bar A$ also dominates the semigroup $(T_{\#}(t))_{t\geq 0}$ generated by $\overline{A_{\#}}$. Since $\overline{A_{\#}}$ generates a positive semigroup it follows that $(T_{\#}(t))_{t\geq 0}$ is the modulus of $(T(t))_{t\geq 0}$.

In the same way one can determine the modulus of Schrödinger semigroups with complex potentials. We only sketch the result. On the space $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ we consider the Schrödinger operator B given by

(3.4) $Bf = \Delta f - Vf$, $f \in C_c^\infty(\mathbb{R}^n)$.

Here, Δ denotes the Laplacian and we assume that the potential V satisfies $V \in L^p_{loc}(\mathbb{R}^n)$ and $\operatorname{Re}V \geq c$ for some constant $c \in \mathbb{R}$. Kato has shown (see Kato [6] or Nagel [11], C-II, Ex.4.7) that under these assumptions the closure of B generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $L^p(\mathbb{R}^n)$. The semigroup $(S(t))_{t\geq 0}$ is positive if V is real-valued. As above one can show that for complex-valued V the semigroup $(S(t))_{t\geq 0}$ has a modulus. Its generator is the Schrödinger operator obtained by replacing in (3.4) the potential V by its real part.

The second class of examples we consider are semigroups corresponding to retarded differential equations. In the following we use the ideas of Kerscher and Nagel (see Kerscher-Nagel [7] or Nagel [11], Sec. B-IV.3).

Let F be a Banach lattice. Then $E := C([-1,0],F)$, the space of all continuous F-valued functions is a Banach lattice as well (in the canonical way). We consider a semigroup $(S(t))_{t\geq 0}$ with generator B on F and an operator $\mu \in \mathcal{L}(E,F)$. The operator $A_{B,\mu}$ on E, defined as

(3.5) $A_{B,\mu}f = f'$ with $D(A_{B,\mu}) := \Big\{ f \in C^1([-1,0],F) : f(0) \in D(B),$
$f'(0) = Bf(0) + \mu(f) \Big\}$

generates a semigroup $(T_{B,\mu}(t))_{t\geq 0}$ on E. For $f \in D(A_{B,\mu})$ and $t \geq 0$ $x(t) := (T_{B,\mu}(t)f)(0)$ gives the solutions of the F-valued retarded differential equation

$$\dot{x}(t) = Bx(t) + \mu(x_t) \quad , \quad x_0 = f \ ,$$

where $x_t \in E$ denotes the function $s \mapsto x(t+s)$, $-1 \leq s \leq 0$.
Now the following holds.

<u>LEMMA</u> 3.1. <u>If B generates a positive semigroup on F and if μ is a positive operator then the semigroup $(T_{B,\mu}(t))_{t\geq 0}$ is positive. The converse is true if we assume that μ has no mass at zero</u>.
Here, μ has no mass at zero if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mu(f)\| \leq \epsilon \cdot \|f\|$ for all $f \in E$ with $\mathrm{supp}(f) \subseteq [-\delta,0]$.
For the proof we refer to Kerscher-Nagel [7].

We introduce the following notations:

(3.6)(i) $\epsilon_\lambda \in C[-1,0]$ is given by $\epsilon_\lambda(s) := e^{\lambda s}$ for $\lambda \in \mathbb{C}$, $s \in [-1,0]$;

(ii) $f \otimes u \in E$ denotes the function $s \mapsto f(s) \cdot u$ for $u \in F$ and $f \in C[-1,0]$;

(iii) $H_\lambda \in \mathcal{L}(E)$ is defined by $H_\lambda g(s) := \int_s^0 e^{\lambda(s-t)} g(t)\, dt$,

$g \in E$, $s \in [-1,0]$, $\lambda \in \mathbb{C}$;

(iv) $\mu_\lambda \in \mathcal{L}(F)$ is defined by $\mu_\lambda(u) := \mu(\epsilon_\lambda \otimes u)$, $u \in F$, $\lambda \in \mathbb{C}$.

Now the spectra of $A_{B,\mu}$ and $B+\mu_\lambda$ coincide and one can describe the resolvent of $A_{B,\mu}$:

(3.7) $$R(\lambda, A_{B,\mu})g = \epsilon_\lambda \otimes R(\lambda, B+\mu_\lambda)(g(0) + \mu H_\lambda g) + H_\lambda g ,$$
$$\lambda \in \rho(B+\mu_\lambda) \quad , \quad g \in E .$$

With this explicit representation of the resolvent one can prove the following (see also Kerscher-Nagel [8]):

PROPOSITION 3.2. *Let* $(S_1(t))_{t\geq 0}$, $(S_2(t))_{t\geq 0}$ *be semigroups on* F *with generators* B_1 , B_2 *resp. and let* μ_1 , $\mu_2 \in \mathcal{L}(E,F)$. *If* $(S_1(t))_{t\geq 0}$ *dominates* $(S_2(t))_{t\geq 0}$ *and if* μ_1 *dominates* μ_2 *then* $(T_{B_1,\mu_1}(t))_{t\geq 0}$ *dominates* $(T_{B_2,\mu_2}(t))_{t\geq 0}$.

Proof: For $\lambda \in \mathbb{R}$ the operator $\mu_{1\lambda}$ dominates $\mu_{2\lambda}$. So for $u \in F$ and sufficiently large λ we obtain from

$(\lambda - B + \mu) = (\lambda - B)(Id - R(\lambda,B)\circ\mu)$ that

$$|R(\lambda,B_2+\mu_{2\lambda})u| = \left|\sum_{n=0}^{\infty} [R(\lambda,B_2)\mu_{2\lambda}]^n R(\lambda,B_2)u\right|$$
$$\leq \sum_{n=0}^{\infty} [R(\lambda,B_1)\mu_{1\lambda}]^n R(\lambda,B_1)|u|$$
$$= R(\lambda,B_1+\mu_{1\lambda})|u| .$$

The operator H_λ is positive for $\lambda \in \mathbb{R}$ and so we have for $f \in E$:

$$\begin{aligned}|R(\lambda, A_{B_2,\mu_2})f| &= |\epsilon_\lambda \otimes R(\lambda, B_2+\mu_{2\lambda})(f(0) + \mu_2 H_\lambda f) + H_\lambda f| \\ &\leq \epsilon_\lambda \otimes |R(\lambda, B_2+\mu_{2\lambda})(f(0) + \mu_2 H_\lambda f)| + |H_\lambda f| \\ &\leq \epsilon_\lambda \otimes R(\lambda, B_1+\mu_{1\lambda})|f(0) + \mu_1 H_\lambda f| + H_\lambda |f| \\ &\leq \epsilon_\lambda \otimes R(\lambda, B_1+\mu_{1\lambda})(|f(0)| + \mu_1 H_\lambda |f|) + H_\lambda |f| \\ &= R(\lambda, A_{B_1,\mu_1})|f| \, .\end{aligned}$$

Now the assertion follows from Prop. 1.1(ii).

□

Kerscher and Nagel ([8]) conjecture the following:

If $B_\#$ generates the modulus of the semigroup generated by B, if μ has no mass at zero and if $|\mu|$ exists then

$$(T_{B_\#,|\mu|}(t))_{t\geq 0} \text{ is the modulus of } (T_{B,\mu}(t))_{t\geq 0} \, .$$

We now will show that this is true if F is finite-dimensional. In this case B is a real $n\times n$-matrix, $B = (b_{ij})_{1\leq i,j\leq n}$ and μ is given by a $n\times n$-matrix $(\mu_{ij})_{1\leq i,j\leq n}$ of measures, i.e. $\mu_{ij} \in C([-1,0])' = M([-1,0])$. Without loss of generality we can assume that μ_{ij} has no mass at zero (otherwise we change b_{ij}). Then the following holds:

<u>PROPOSITION</u> 3.3. <u>Let</u> $B = (b_{ij})$ <u>be a real</u> <u>$n\times n$-matrix and</u> $\mu = (\mu_{ij})$ <u>be a $n\times n$-matrix of measures having no mass at zero. The semigroup on</u> $C([-1,0],\mathbb{R}^n) = (C([-1,0]))^n$ <u>generated by</u> $A_{B,\mu}$ <u>has a modulus and its generator is given by</u> $(A_{B,\mu})_\# = A_{B_\#,|\mu|}$. (<u>Here</u> $|\mu| = (|\mu_{ij}|)$ <u>and</u> $B_\# = (b_{ij\#})$ <u>is given by</u> $b_{ij\#} = |b_{ij}|$ <u>if</u> $i \neq j$ <u>and</u> $b_{ii\#} = b_{ii}$).

Proof: We give all details for the one-dimensional case. The necessary modifications for the n-dimemsional case are obvious.

At first we assume $b\ (= b_{11}) = 0$. Let $(S(t))_{t\geq 0}$ be a semigroup which dominates $(T_{0,\mu}(t))_{t\geq 0}$ and let C be its generator. We recall that $(T_{0,\mu}(t))_{t\geq 0}$ satisfies (see Sec. B-IV.3 of Nagel [11])

$$(3.8)\quad (T_{0,\mu}(t)f)(x) = \begin{cases} f(x+t) & \text{if } x+t \leq 0 \\ f(0) + \int_0^{x+t} \langle T_{0,\mu}(s)f,\mu\rangle ds & \text{if } x+t \geq 0 \end{cases} .$$

We claim that for fixed $\alpha < 1$, $f \in E_+$, f strictly positive there exists $t_o = t_o(\alpha, f) > 0$ such that

$$(3.9)\qquad T_{0,\alpha|\mu|}(t)f \leq S(t)f \quad \text{for all } 0 \leq t \leq t_o .$$

In fact, if $x+t \leq 0$ we have

$$(T_{0,\alpha|\mu|}f)(x) \overset{(3.8)}{=} f(x+t) \overset{(3.8)}{=} (T_{0,\mu}(t)f)(x) \leq (S(t)f)(x)$$

since $(S(t))_{t\geq 0}$ dominates $(T_{0,\mu}(t))_{t\geq 0}$. Moreover, since f is strictly positive, we have $\langle f, |\mu|\rangle > 0$ hence there exists $h_o \in E$, $|h_o| \leq f$ such that

$$\alpha \cdot \langle f, |\mu|\rangle = \alpha \cdot \sup \left\{ \langle h,\mu\rangle : |h| \leq f \right\} < \langle h_o,\mu\rangle .$$

Additionally, since μ has no mass at zero we can require that $h_o(0) = f(0)$. By continuity there is a $t_o > 0$ such that $\mathrm{Re}\langle T_{0,\mu}(s)h_o,\mu\rangle \geq \alpha\cdot\langle T_{0,\alpha|\mu|}(s)f, |\mu|\rangle$ for $0 \leq s \leq t_o$. Then for $t \leq t_o$, $x+t \geq 0$ we obtain

$$\begin{aligned} (S(t)f)(x) &\geq (T_{0,\mu}(t)h_o)(x) \overset{(3.8)}{=} h_o(0) + \int_0^{x+t} \langle T_{0,\mu}(s)h_o,\mu\rangle ds \\ &\geq f(0) + \int_0^{x+t} \langle T_{0,\alpha|\mu|}(s)f,\alpha|\mu|\rangle ds \\ &\overset{(3.8)}{=} (T_{0,\alpha|\mu|}(t)f)(x) . \end{aligned}$$

By now the claim (3.9) is established. Hence for $f \in D(A_{0,\alpha|\mu|})_+$, f strictly positive and $\varphi \in D(C')_+$ we have

$$\begin{aligned} \langle A_{0,\alpha|\mu|}f , \varphi \rangle &= \lim_{t\to 0} \left\langle \frac{T_{0,\alpha|\mu|}(t)f - f}{t} , \varphi \right\rangle \\ &\overset{(3.9)}{\leq} \lim_{t\to 0} \left\langle \frac{S(t)f - f}{t} , \varphi \right\rangle \end{aligned}$$

$$= \lim_{t \to 0} \left\langle f \, , \, \frac{S(t)'\varphi - \varphi}{t} \right\rangle$$

$$= \langle f \, , \, C'\varphi \rangle \, .$$

Consequently

$\langle (\lambda - A_{0,\alpha|\mu|})f \, , \, \varphi \rangle \geq \langle f \, , \, (\lambda - C'\varphi) \rangle$ for $f \in D(A_{0,\alpha|\mu|})_+$,
f strictly positive and $\varphi \in D(C')_+$.

Taking $f = R(\lambda, A_{0,\alpha|\mu|})g$, $\varphi = R(\lambda, C)'\psi$ ($g \in E_+$, g strictly positive , $\psi \in E'_+$) we obtain for sufficiently large $\lambda \in \mathbb{R}$:

$$\langle R(\lambda, C)g \, , \, \psi \rangle = \langle g \, , \, R(\lambda, C)'\psi \rangle \geq \langle R(\lambda, A_{0,\alpha|\mu|})g \, , \, \psi \rangle \, .$$

Since every positive element can be approximated by strictly positive elements we conclude that $R(\lambda, C) \geq R(\lambda, A_{0,\alpha|\mu|})$ for sufficiently large $\lambda \in \mathbb{R}$. From (3.7) it follows that the right hand side tends towards $R(\lambda, A_{0,|\mu|})$ as $\alpha \to 1$. Thus from Prop. 1.1 we conclude that $(S(t))_{t \geq 0}$ dominates $(T_{0,|\mu|}(t))_{t \geq 0}$. By Prop. 3.1 $(T_{0,|\mu|}(t))_{t \geq 0}$ dominates $(T_{0,\mu}(t))_{t \geq 0}$, hence it is the modulus.

In case $b \neq 0$ we consider the operator $S \in \mathscr{L}(E)$, $(Sf)(x) := e^{bx} f(x)$. A simple calculation yields

(3.10) $\quad S^{-1} \circ (A_{b,\mu} - b \cdot Id) \circ S = A_{0,\nu}$ with $\nu := S'\mu$.

Since S is a lattice isomorphism we have

$$\begin{aligned}(A_{b,\mu} - b \cdot Id)_\# &= (S \circ A_{0,\nu} \circ S^{-1})_\# = S \circ (A_{0,\nu})_\# \circ S^{-1} \\ &= S \circ A_{0,|\nu|} \circ S^{-1} = A_{b,|\mu|} - b \cdot Id\end{aligned}$$

(note that $|\mu| = |S'\nu| = S'|\nu|$). Thus from Prop. 2.6 we deduce

$$(A_{b,\mu})_\# = A_{b,|\mu|} \, .$$

In the n-dimensional case $(n \geq 2)$ the transformation $S : (C[-1,0])^n \to (C[-1,0])^n$ is given by

$$(Sf)_i(x) := (e^{b_{ii}x} \, f_i(x)) \qquad (1 \leq i \leq n)$$

where b_{ii} is the i-th diagonal element of the matrix B . Then the argument given above shows that we only have to consider the case

$b_{11} = b_{22} = \dots = b_{nn} = 0.$

□

References

1. Arendt, W., Kato's inequality, a characterization of generators of positive semigroups, Proc. Royal Irish Acad. Sect. A 84 (1984) 155 -174

2. Cartwright, D.I.; Lotz, H.P. Some characterizations of AM- and AL-spaces, Math.Z. 142, 97-103 (1975)

3. Derndinger, R., Betragshalbgruppen normstetiger Operatorhalbgruppen, Arch.Math. 42, 371-375 (1984)

4. Fattorini, H.O., The Cauchy Problem, Reading (Mass.), Addison-Wesley 1983

5. Goldstein, J., Semigroups of Linear Operators and Applications, New York, Oxford University Press 1985

6. Kato, T., L^p-Theory of Schrödinger operators with a singular potential, In : R. Nagel; U. Schlotterbeck; M.P.H. Wolff (eds.) : Aspects of Positivity in Functional Analysis, Amsterdam, North Holland 1986

7. Kerscher, W.; Nagel, R., Asymptotic behavior of one-parameter semigroups of positive operators, Acta Appl. Math. 2, 297-309 (1984)

8. Kerscher, W.; Nagel, R., Positivity and stability for Cauchy problems with delay, Preprint, Universität Tübingen 1986

9. Kipnis, C., Majoration des semigroupes de L^1 et applications, Ann. Inst. H. Poincaré 10 (1974), 369-384

10. Kubokawa, Y., Ergodic theorems for contraction semigroups, J. Math. Soc. Japan 27 (1975), 184-193

11. Nagel, R. (ed.), One-Parameter Semigroups of Positive Operators Lecture Notes in Mathematics 1184, Springer-Verlag (1986)

12. Schaefer, H.H., Banach Lattices and Positive Operators, New York, Springer-Verlag (1974)

Ingo Becker
Günther Greiner

Mathematisches Institut

Auf der Morgenstelle 10

7400 Tübingen
Federal Republic of Germany

Received in final form May 15, 1986