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THE TRANSLATIONAL HULL IN SEMIGROUPS AND RINGS

Mario Petrich

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The translational hull of a semigroup is extensively studied and is motivated by its use in the theory of (ideal) extensions of semigroups. The construction of all extensions of a semigroup S by a semigroup Q with zero is reduced to finding certain functions from the groupoid $Q\setminus 0$ into the translational hull of S. The case of a weakly reductive S receives special attention. Extensions determined by a partial homomorphism, strict, pure, dense, and cancellative extensions are then discussed. Isomorphism properties of densely embedded ideals and applications of the latter to various semigroups of partial transformations and binary relations are presented. The translational hull of a regular Rees matrix semigroup is studied in detail and some applications thereof to semigroups of linear transformations, semigroup representations, and dense embeddings are exhibited. A development parallel to that for semigroup extensions is briefly presented for ring extensions and extensions of posets. Similarity with group extensions is stressed throughout. The bibliography of 224 items is itemized by subjects, including references to extensions of semigroups, rings, algebras, posets, and algebraic systems. No proofs are given.

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1. BACKGROUND AND DEFINITIONS

The translational hull occurs naturally when one is concerned with a construction of ideal extensions of semigroups. Since ring extensions represent a particular case of semigroup ideal extensions, the same construction, with the obvious modification that all the functions in the definition be additive, appears also in ring theory (under several names). Pursuing this further by consideration of linear associative algebras, the next obvious modification is that all the functions be also linear, and in the topological (or normed) case that they also be continuous (normed), etc. In a natural way, this makes the translational hull a semigroup, a ring, an algebra, etc. This hierarchy has been first noted by B.E. Johnson [196].

The subject of group extensions being of wider general knowledge, I will from time to time refer to the group case for a better understanding of the case at hand and for useful analogy. This analogy is quite strong and points to the numerous common features of these theories. After an extensive discussion of the translational hull and ideal extensions of semigroups, I will briefly outline the extension theory for rings and partially ordered sets.

The extension problem for groups is as follows: given two groups A and B, construct all groups G with the property that G has a normal subgroup N such that $N \cong A$, $G/N \cong B$; G is then called an extension of A by B. We may identify A with N, B with G/N. This problem has been first solved by O. Schreier, for a full discussion see Hall ([4], Ch. 15), Rédei ([12], §§ 50, 51), Specht ([13], Kap. 3.3). His solution is known as "Schreier's theorem" and group extensions as "Schreier extensions"; as a motivation for the semigroup and ring case, I will briefly outline the ideas of this solution. If G is given, one takes an arbitrary set $\{b_\beta\}$ of representatives of cosets of A in G and makes these act on A by $\varphi_\beta: a \to b_\beta^{-1} a b_\beta$ $(a \in A)$. Hence φ_β is the restriction to A of the inner automorphism of G induced by b_β. The mapping $\varphi: b_\beta A \to \varphi_\beta$ is a function from the set B of cosets of A in G into the automorphism group $\mathfrak{A}(A)$ of A. The function φ is generally not a homomorphism (if the set $\{b_\beta\}$ can be so chosen that φ is a homomorphism, G is a split extension and can be obtained as a semidirect product of A and B). Hence a function $\varphi: B \to \mathcal{G}(A)$ can be used as an ingredient in the converse problem of constructing G out of A and B. The measure of how far φ is from being a homomorphism is given by a "factor system" which is a function $\psi: B \times B \to A$. Indeed, given φ and ψ, satisfying certain conditions, every extension of A by B can be constructed.

Now let V be a semigroup. A nonempty subset S of V is an <u>ideal</u> of V if $sv, vs \in S$ for all $s \in S$, $v \in V$. The <u>Rees quotient semigroup</u>, denoted by $Q = V/S$, is the set $V \setminus S = \{v \in V \mid v \notin I\}$ together with a new symbol 0 under multiplication

$$a * b = ab \quad \text{if} \quad a, b, ab \in V \setminus S$$

and all other products are equal to 0 (the zero of Q); V is then an <u>ideal extension</u> of S by Q. The ideal extension problem for semigroups can be formulated thus: given semigroups S and Q, where Q has a zero, construct all semigroups V such that V has S as an ideal and $V/S = Q$ (with obvious identifications). In fact, we can take $V = S \cup Q^*$, where $Q^* = Q \setminus \{0\}$ and S and Q are supposed to be disjoint, and must find all associative multiplications on V which agree with existing products on S and Q^* and make S an ideal of V.

For general semigroup theory I will follow Clifford and Preston [2], Ljapin [8], Petrich [9]. Ideal extensions occur quite often for the reason that we are frequently interested in building more complex semigroups out of some of "simpler" structure and this can be sometimes achieved by constructing ideal extensions. This approach fails if the semigroup we are constructing has no proper ideals. In such a case, one may resort to various generalizations of the Schreier theory in groups; for such developments or general type extensions consult Coudron [18], D'Alarcao [19], Hancock [51], [52], Inasaridze [56] - [59], Rédei [96], Skornjakov [112], Tamura and Burnell [118], Verbeek [125], Wiegandt [129], [130], and numerous papers concerning the structure of bisimple inverse semigroups.

Let V be a semigroup and S an ideal of V. Then each $v \in V$ induces the functions λ^v and ρ^v as follows

$$\lambda^v : s \to vs, \ \rho^v : s \to sv \qquad (s \in S).$$

We write λ^v as a left and ρ^v as a right operator and note that for all $x,y \in S$, $v,u \in V$,

$$\lambda^v(xy) = (\lambda^v x)y, \ (xy)\rho^v = x(y\rho^v),$$
$$x(\lambda^v y) = (x\rho^v)y, \ (\lambda^v x)\rho^u = \lambda^v(x\rho^u)$$

which of course reflects the associative law in V. These properties suggest the following definitions.

Let S be a semigroup and let x,y denote arbitrary elements of S. A function λ on S is a <u>left translation</u> of S if $\lambda(xy) = (\lambda x)y$, a function ρ on S is a <u>right translation</u> of S if $(xy)\rho = x(y\rho)$; λ and ρ are <u>linked</u> if $x(\lambda y) = (x\rho)y$ and in such a case the pair (λ,ρ) is a <u>bitranslation</u> of S; λ and ρ are <u>permutable</u> if $(\lambda x)\rho = \lambda(x\rho)$. It is sometimes convenient to consider a bitranslation as a bioperator on S and denote it by a single letter, say ω, and write $x\omega$, ωx for any $x \in S$. A set T of bitranslations is <u>permutable</u> if for any $(\lambda,\rho),(\lambda',\rho') \in T$, λ and ρ' are permutable.

The set $\Lambda(S)$ of all left translations is a semigroup under the composition $(\lambda\lambda')x = \lambda(\lambda'x)$, similarly the set $P(S)$ of all right translations is a semigroup under the composition $x(\rho\rho') = (x\rho)\rho'$. The subsemigroup $\Omega(S)$ of the direct product $\Lambda(S) \times P(S)$ consisting of all bitranslations of S is the <u>translational hull</u> of S.

For every $s \in S$, the <u>inner left (right) translation</u> induced by s is the function $\lambda_s(\rho_s)$ defined by $\lambda_s x = sx$ $(x\rho_s = xs)$; $\pi_s = (\lambda_s,\rho_s)$ is the <u>inner bitranslation</u> induced by s. One verifies easily that

$$\pi_s\omega = \pi_{s\omega},\ \omega\pi_s = \pi_{\omega s} \quad (s \in S,\ \omega \in \Omega(S)),$$

which shows that the <u>inner part</u> $\Pi(S) = \{\pi_s \mid s \in S\}$ of $\Omega(S)$ is its ideal. The mapping

$$\pi\colon s \to \pi_s \quad (s \in S)$$

is the canonical homomorphism of S into $\Omega(S)$. Then π is $1-1$ iff S is <u>weakly reductive</u> in the sense that $ax = bx$, $xa = xb$ for all $x \in S$ implies $a = b$. This condition also insures that any two bitranslations of S are permutable. Note the analogy with groups:

automorphism group - translational hull

inner automorphisms (normal subgroup) - inner bi-translations (ideal)

trivial center - weak reductivity.

We will encounter further analogy in the case of rings and partially ordered sets.

It is to the point here to say something about the terminology, notation, and history of these concepts. For translations I mainly follow the notation introduced by Grillet [44], [45]. A bitranslation was called by Hochschild [192] in 1947 a multiplication for algebras (where the functions are then required to be linear); by Clifford [17] in 1950 a pair of linked left and right translations for semigroups; by Rédei [168] in 1954 Doppelhomothetismus for rings; by Helgason [189] in 1956 a multiplier for commutative Banach algebras; by MacLane [160] in 1958 a bimultiplication (in the last three the functions are also additive); by B.E. Johnson [196] in 1964 a double centralizer for semigroups, rings, and algebras; by Keimel [215] in 1968 a bihomothétie for ordered rings (the last two require only the linking condition which in the presence of trivial left and right annihilators implies the remaining conditions), he also called the translational hull the bicentroïde. The translations also appear under various names. It would of course be useful if some terminology would be generally accepted (not to remain behind, I came up with a bitranslation in Petrich [9] in 1967).

2. BASIC PROPERTIES OF THE TRANSLATIONAL HULL

This is a small sample of properties of the translational hull of arbitrary or special kinds of semigroups. The proofs of most of these properties as well as further results on this subject can be found in Clifford and Preston ([2], 1.3), B.E. Johnson [196] and Petrich ([9], Ch. 2), see also Gluskin [32].

Throughout this section, S denotes an arbitrary semigroup unless specified otherwise. Besides weak reductivity, the condition that occurs frequently in consideration of translations is: S is <u>globally idempotent</u> if for every $a \in S$ there exist $x,y \in S$ such that $a = xy$ (i.e., $S^2 = S$). This condition, e.g., insures that every left translation is permutable with every right translation; weak reductivity and global idempotency are independent conditions.

With the notation introduced in the previous section,

$$\lambda\lambda_a = \lambda_{\lambda a},\ \rho_a\rho = \rho_{a\rho} \quad (a \in S,\ \lambda \in \Lambda(S),\ \rho \in P(S)),$$

and hence with the notation

$$\Gamma(S) = \{\lambda_a \mid a \in S\},\ \Delta(S) = \{\rho_a \mid a \in S\},$$

we have that $\Gamma(S)$ is a left ideal of $\Lambda(S)$ and $\Delta(S)$ is a right ideal of $P(S)$.

<u>PROPOSITION</u> <u>1</u>. i) $\Gamma(S) = \Lambda(S)$ <u>iff</u> S <u>has</u> <u>a</u> <u>left</u> <u>identity</u>.

ii) $\Pi(S) = \Omega(S)$ <u>iff</u> S <u>has</u> <u>an</u> <u>identity</u>.

If A is a subsemigroup of S, the <u>idealizer</u> of

A in S is the largest subsemigroup of S containing A as an ideal, it is given by $\{s \in S \mid sa, as \in A$ for all $a \in A\}$. By $C(S)$ denote the <u>center</u> of S, i.e., $C(S) = \{c \in S \mid cs = sc$ for all $s \in S\}$.

<u>THEOREM 2</u> (<u>Gluskin</u> [32]). <u>If</u> S <u>is weakly reductive, then</u> $\Omega(S)$ <u>is the idealizer of</u> $\Pi(S)$ <u>in</u> $\Lambda(S) \times P(S)$.

<u>PROPOSITION 3</u>. <u>If</u> S <u>is weakly reductive or globally idempotent, then</u>

$$C(\Omega(S)) = \{(\lambda,\rho) \in \Omega(S) \mid \lambda s = s\rho \text{ for all } s \in S\}.$$

In this case, for $(\lambda,\rho) \in C(\Omega(S))$ and $x,y \in S$ we have $\lambda(xy) = (\lambda x)y = x(\lambda y)$, cf. centroid of an algebra, Jacobson ([5], V. 4.). Let

$$\pi_\Lambda : \Omega(S) \to \Lambda(S), \quad \pi_P : \Omega(S) \to P(S)$$

be the projection homomorphisms, and let $\tilde{\Lambda}(S)$ and $\tilde{P}(S)$ denote the corresponding images of $\Omega(S)$ in $\Lambda(S)$ and $P(S)$, respectively.

<u>PROPOSITION 4</u>. <u>If</u> S <u>is commutative, then</u> $\tilde{\Lambda}(S) = \Lambda(S)$, $\tilde{P}(S) = P(S)$.

A semigroup S is <u>left reductive</u> if $xa = xb$ for all $x \in S$ implies $a = b$, right reductivity is defined dually; S is <u>reductive</u> if it is both left and right reductive. Thus right reductivity means that the mapping $s \to \lambda_s$ is $1-1$ and is hence an isomorphism of S onto $\Gamma(S)$.

<u>PROPOSITION 5</u>. <u>The following statements are valid for a left reductive semigroup</u> S.

i) $\widetilde{P}(S)$ *is the idealizer of* $\Delta(S)$ *in* $P(S)$.

ii) *If* φ *and* ψ *are functions on* S *satisfying the linking condition* $x(\varphi y) = (x\psi)y$ *for all* $x,y \in S$, *then* $\varphi \in \Lambda(S)$.

iii) π_P *is* 1 - 1 *and hence* $\widetilde{P}(S) \cong \Omega(S)$.

COROLLARY. *If* S *is reductive, then*

i) $\Omega(S) = \{(\varphi,\psi) \mid \varphi$ *and* ψ *are functions on* S *satisfying* $x(\varphi y) = (x\psi)y$ *for all* $x,y \in S\}$,

ii) $\widetilde{\Lambda}(S) \cong \Omega(S) \cong \widetilde{P}(S)$,

and if S *is also commutative, then*

iii) $\Lambda(S) \cong \Omega(S) = C(\Omega(S)) \cong P(S)$

where all isomorphisms are projections.

We see that for a reductive semigroup S, the linking condition for two functions on S implies that they form a bitranslation. Indeed, B.E. Johnson [196] defines a "double centralizer" by this single condition. Which properties of S carry over to $\Omega(S)$? Very few; a more pertinent question is: how do properties of S influence the properties of $\Omega(S)$?

PROPOSITION 6 (Tamura [116]). S *is a right zero semigroup iff every transformation on* S *is a right translation iff the identity mapping on* S *is the only left translation.*

This shows that the translational hull of a right zero semigroup S is isomorphic to the semigroup of all transformations on the set S written as right operators and points to the phenomenon that S and $\Omega(S)$ may vary wildly in their properties.

THEOREM 7 (Ponizovski [92]). *If S is an inverse semigroup, so is* $\Omega(S)$.

This theorem fails for regular semigroups. We can regard a semilattice both as a semigroup and as a partially ordered set.

PROPOSITION 8. *If S is a semilattice, so is* $\Omega(S)$.

COROLLARY. *If S is a finite semilattice, then* $\Omega(S)$ *is a lattice.*

A homomorphism φ of a semigroup V into a semigroup V' both of which contain S is an *S-homomorphism* if φ leaves S elementwise fixed. If I is an ideal of S and φ is an I-endomorphism of S mapping S onto I, let $\lambda s = s\rho = s\varphi$ for all $s \in S$. Then (λ,ρ) is the *bitranslation on S induced by* φ.

THEOREM 9 (Petrich [9]). *A semigroup S is a semilattice iff every bitranslation on S is induced by an I-endomorphism for some ideal I of S. Such ideals I in a semilattice S are characterized by the property: the intersection of I with any principal ideal of S is a principal ideal. If an I-endomorphism exists it is unique.*

For further information on this subject see Petrich ([9], 2.4), Szász [113], Szász and Szendrei [114].

PROPOSITION 10. *If S is left reductive and right cancellative, then* $\Omega(S)$ *is right cancellative.*

COROLLARY. *If S is cancellative, so is* $\Omega(S)$.

For isomorphisms between translational hulls, we have

THEOREM 11. *Let* θ *be an isomorphism of a semigroup* S *onto a semigroup* T. *For* $\lambda \in \Lambda(S)$, $\rho \in P(S)$, *let*

$$\bar{\lambda}t = [\lambda(t\theta^{-1})]\theta, \quad t\bar{\rho} = [(t\theta^{-1})\rho]\theta \quad (t \in T).$$

Then $\lambda \to \bar{\lambda}$ *and* $\rho \to \bar{\rho}$ *are isomorphisms of* $\Lambda(S)$ *onto* $\Lambda(T)$ *and of* $P(S)$ *onto* $P(T)$, *respectively. The mapping* $\bar{\theta}: (\lambda,\rho) \to (\bar{\lambda},\bar{\rho})$ *is an isomorphism of* $\Omega(S)$ *onto* $\Omega(T)$ *with the properties*:

$$\pi_s\bar{\theta} = \pi_{s\theta}, \quad (\omega s)\theta = (\omega\bar{\theta})(s\theta), \quad (s\omega)\theta = (s\theta)(\omega\bar{\theta})$$
$$(s \in S, \ \omega \in \Omega(S)).$$

If S *is weakly reductive or globally idempotent, then* $\bar{\theta}$ *is the unique isomorphism of* $\Omega(S)$ *onto* $\Omega(T)$ *with the property* $\pi_s\bar{\theta} = \pi_{s\theta}$ *for all* $s \in S$.

COROLLARY 1. *If both* S *and* T *are weakly reductive, then every isomorphism of* $\Pi(S)$ *onto* $\Pi(T)$ *can be uniquely extended to an isomorphism of* $\Omega(S)$ *onto* $\Omega(T)$.

A result analogous to Theorem 11 is also valid for anti-isomorphisms and involutorial anti-automorphisms. Denoting by $\mathfrak{A}(S)$ the automorphism group of S (functions written on the right), we have

COROLLARY 2. *If* S *is weakly reductive and simple or* 0-*simple, then the mapping* $\theta \to \bar{\theta}$ *is an isomorphism of* $\mathfrak{A}(S)$ *onto* $\mathfrak{A}(\Omega(S))$.

Let $\Sigma(S)$ denote the group of units of $\Omega(S)$. If $(\lambda,\rho) \in \Sigma(S)$ and λ and ρ are permutable, then the mapping $\delta_{(\lambda,\rho)}$, defined by $s\delta_{(\lambda,\rho)} = (\lambda^{-1}s)\rho$ for all $s \in S$, is an automorphism of S, to be called the <u>generalized inner automorphism</u> induced by (λ,ρ). The set $\mathcal{J}(S)$ of all generalized inner automorphisms is a subgroup of $\mathcal{G}(S)$. If S has an identity and a is contained in its group of units, let $s\epsilon_a = a^{-1}sa$ for all $s \in S$ (the <u>inner automorphism</u> induced by a).

<u>THEOREM 12</u> (<u>Petrich</u> [9]). <u>If S is weakly reductive or globally idempotent, then the mapping $(\lambda,\rho) \to \delta_{(\lambda,\rho)}$ is a homomorphism of $\Sigma(S)$ onto $\mathcal{J}(S)$ with kernel $\Sigma(S) \cap C(\Omega(S))$ so that $\Sigma(S)/\Sigma(S) \cap C(\Omega(S)) \cong \mathcal{J}(S)$, and $\bar{\delta}_{(\lambda,\rho)} = \epsilon_{(\lambda,\rho)}$.</u>

In case that S has an identity, generalized inner automorphisms coincide with inner automorphisms, and if S is a group, the preceding theorem reduces to the familiar theorem in groups. For further properties of automorphisms of $\Omega(S)$ consult Petrich ([9], 2.6). The translational hull of an extension of a semigroup by another and of a semigroup with an ideal chain has been described by Ponizovski [89], [90], of an $\mathfrak{h}$-semigroup by Hall [50], of a regular Rees matrix semigroup and several special classes thereof by Petrich [83], [84]. Some related results and certain generalizations have been obtained by Delorme [20], Goralčík [41], Goralčik and Hedrlín [42], [54], Hedrlín [53], Lallement [65], Ljapin [70], Posey [94], Schein [97], Shirjaev [106], Shutov [111], Tamura [116].

3. IDEAL EXTENSIONS OF SEMIGROUPS

Let S be an ideal of a semigroup V. I will refer to V as an *extension* of S (since no other extensions will be considered) by the semigroup $Q = V/S$ with zero. It is assumed that S and Q are disjoint, the multiplication in V is denoted by another symbol if V is being constructed, but if no confusion is likely or if V is already given, it is simply denoted by juxtaposition. I will only give some highlights of the theory of extensions as developed by Clifford [17], Grillet and Petrich [48], Petrich [9]; see also Clifford and Preston ([2], 4.4).

For an extension V of a semigroup S, define

$$\tau = \tau(V:S): v \to \tau^v = (\lambda^v, \rho^v) \quad (v \in V)$$

where for every $v \in V$,

$$\lambda^v s = vx, \quad s\rho^v = sv \quad (s \in S).$$

The function $\tau = \tau(V:S)$ is the *canonical homomorphism* of V into $\Omega(S)$ (in fact, onto a semigroup of permutable bitranslations of S containing $\Pi(S)$). The image $T(V:S)$ of V under τ is the *type* of the extension V of S.

PROPOSITION 1 (Grillet-Petrich [48]). *Let V be an extension of S. Then the canonical homomorphism $\tau = \tau(V:S)$ extends the canonical homomorphism $\pi: S \to \Omega(S)$. If S is weakly reductive or globally idempotent, then τ is the unique extension of π to a homomorphism of V into $\Omega(S)$.*

PROPOSITION 2 (Grillet [44]). *A subset* T *of* $\Omega(S)$ *is a type of some extension of* S *iff* T *is a subsemigroup of* $\Omega(S)$ *containing* $\Pi(S)$ *and in which any two bitranslations are permutable.*

Hence the types of extension of a weakly reductive or globally idempotent semigroup S coincide with all subsemigroups of $\Omega(S)$ containing $\Pi(S)$ and are thus in 1 - 1 correspondence with subsemigroups of $\Omega(S)/\Pi(S)$ containing its zero. A simple Zorn's lemma argument shows that every set of permutable bitranslations is contained in a maximal one. Subsemigroups of $\Omega(S)$ generated by sets of permutable bitranslations consist of permutable bitranslations, so that maximal subsets of permutable bitranslations are types of extension. It follows that the union of all types consists of those $(\lambda,\rho) \in \Omega(S)$ for which λ and ρ are permutable. For a comparable development in rings see Rédei ([12], §53). Types of extension for the group case have been introduced by Baer, see Specht ([13], Kap. 3.3).

Conversely, I wish to construct all extensions of a semigroup S by a semigroup Q with zero. For any semigroup T, a function $\theta: Q^* \to T$ is a *partial homomorphism* if $(ab)\theta = (a\theta)(b\theta)$ whenever $a,b,ab \in Q^*$. A function φ mapping the set $\{(a,b) \in Q \times Q \mid ab = 0\}$ into S is a *ramification function* of Q into S.

PROPOSITION 3 (Yoshida [133]). *Let* S *and* Q *be disjoint semigroups,* Q *with zero. Let* θ *be a partial homomorphism of* Q^* *onto a set of permutable* *bitranslations of* S, $\theta\colon a \to \theta^a$, *and let* φ *be a ramification function of* Q *into* S, $\varphi\colon (a,b) \to [a,b]$, *satisfying*

(C1) $\theta^a\theta^b = \pi_{[a,b]}$ *if* $ab = 0$,

(C2) $[ab,c] = [a,bc]$ *if* $abc = 0,\ ab \neq 0,\ bc \neq 0$,

(C3) $[ab,c] = \theta^a[b,c]$ *if* $ab \neq 0,\ bc = 0$,

(C4) $[a,b]\theta^c = [a,bc]$ *if* $ab = 0,\ bc \neq 0$,

(C5) $[a,b]\theta^c = \theta^a(b,c]$ *if* $ab = bc = 0$.

On the set $V = S \cup Q^*$ *define the multiplication* $*$ *by*:

$$a * b = \begin{cases} a\theta^b & \text{if } a \in S,\ b \in Q^*, & \text{(M1)} \\ \theta^a b & \text{if } a \in Q^*,\ b \in S, & \text{(M2)} \\ [a,b] & \text{if } a,b \in Q^*,\ ab = 0, & \text{(M3)} \\ ab & \text{otherwise}. & \text{(M4)} \end{cases}$$

Then V *is an extension of* S *by* Q, *and conversely, every extension of* S *by* Q *can be so constructed*.

If we write $\theta^a = (\lambda^a,\rho^a)$, then e.g., (C3) can be written $[ab,c] = \lambda^a[b,c]$, etc. Even though this proposition is essentially the disguised associative law and the requirement that S be an ideal, it gives a general procedure from which various special cases can be easily deduced and shows that every extension can be expressed by two parameters θ and φ, so we write $V = \langle S,Q;\theta,\varphi\rangle$.

PROPOSITION 4. *Two extensions* $V = \langle S,Q;\theta,\varphi\rangle$ *and* $V' = \langle S,Q';\theta',\varphi'\rangle$ *are* S-*isomorphic iff there exists* *an isomorphism* ψ *of* Q *onto* Q' *such that* $\theta = \psi\theta'$ *and* $[a,b] = [a\psi, b\psi]'$ *for all* $a,b \in Q^*$ *such that* $ab = 0$.

Such extensions V and V' should be considered as essentially the same, I call them *equivalent*. This is in line with the customary definition in groups and rings; a weaker definition of equivalence of extensions can be found in Clifford and Preston [2]. Proposition 3 simplifies considerably in the following important special cases: (i) Q has no zero divisors, the extension is then given by a homomorphism θ and (C1) - (C5) are vacuous, (ii) S is weakly reductive, the corresponding theorem due to Clifford is historically the first important theorem on the subject.

THEOREM 5 (*cf.* *Clifford* [17]). *Let* S *be a weakly reductive semigroup and* Q *a semigroup with zero disjoint from* S. *Let* $\theta: Q^* \to \Omega(S)$ *be a partial homomorphism,* $\theta: a \to \theta^a$, *with the property that* $\theta^a\theta^b \in \Pi(S)$ *if* $ab = 0$. *Define the multiplication* $*$ *on* $V = S \cup Q^*$ *by* (M1), (M2), (M4) *and*

(M3′) $a * b = c$ where $\theta^a\theta^b = \pi_c$ if $a,b \in Q^*$, $ab = 0$.

Then V *is an extension of* S *by* Q, *and conversely, every extension of* S *by* Q *can be so constructed*.

In other words, for S weakly reductive, the ramification function φ is uniquely determined by the function θ according to (C1), further (C2) - (C5) are automatically fulfilled, and any two bitranslations of S are permutable. We thus may write $V = \langle S,Q;\theta\rangle$.

COROLLARY. *Two extensions* $V = \langle S,Q;\theta\rangle$ *and* $V' = \langle S,Q',\theta'\rangle$ *are equivalent iff there exists an isomorphism* ψ *of* Q *onto* Q' *such that* $\theta = \psi\theta'$.

Let $V = \langle S,Q;\theta,\varphi\rangle$; call V a *strict* extension of S if θ maps Q^* into $\Pi(S)$, *pure* if θ maps Q^* into $\Omega(S)\setminus\Pi(S)$. For $\tau = \tau(V:S)$, $\tau^a = \theta^a$ if $a \in Q^*$ and $\tau^a = \pi_a$ if $a \in S$. Thus V is strict iff its type is $\Pi(S)$, pure iff $\tau^a \in \Pi(S)$ implies $a \in S$ (definition in Grillet [45]).

PROPOSITION 6 (Grillet [45]). *For an extension* V *of an arbitrary semigroup* S, *the complete inverse image* K *of* $\Pi(S)$ *under* $\tau(V:S)$ *is the greatest subsemigroup of* V *which is a strict extension of* S; *furthermore* V *is a pure extension of* K.

COROLLARY. *Every extension of* S *by* Q *is a pure extension of a strict extension. If* Q *has no proper ideals, an extension is either strict or pure.*

Call a homomorphism φ of a semigroup S with zero 0 into a semigroup S' with zero $0'$ *pure* if $s\varphi = 0'$ iff $s = 0$. Let $V = \langle S,Q;\theta\rangle$ be a pure extension of a weakly reductive semigroup S. Consider θ as a function from Q^* into $\Omega(S)/\Pi(S)$ and extend it to all of Q by letting θ map the zero of Q onto the zero of $\Omega(S)/\Pi(S)$. Then θ is a pure homomorphism, and we deduce

PROPOSITION 7 (Grillet [45]). *Let* S *be weakly reductive and* $\theta: Q \to \Omega(S)/\Pi(S)$ *be a pure homo-*

morphism. Then $V = \langle S,Q;\theta|_{Q^*}\rangle$ is a pure extension of S by Q, and conversely, every pure extension of S by Q can be so constructed.

Keep S weakly reductive and suppose that $V = \langle S,Q;\theta\rangle$ is a strict extension. Then $\eta = \theta\pi^{-1}$: $Q^* \to S$ is a partial homomorphism satisfying

$$a * b = \begin{cases} a(b\eta) & \text{if } a \in S,\ b \in Q^*, & (M1'') \\ (a\eta)b & \text{if } a \in Q^*,\ b \in S, & (M2'') \\ (a\eta)(b\eta) & \text{if } a,b \in Q^*,\ ab = 0, & (M3'') \\ ab & \text{otherwise} & (M4'' = M4) \end{cases}$$

Conversely, let S be arbitrary and $\eta\colon Q^* \to S$ be a partial homomorphism, on $V = S \cup Q^*$ define a multiplication according to (M1″) - (M4″). Then V is an extension of S by Q said to be determined by the partial homomorphism η. Such an extension is always strict, and from above we see that every strict extension of a weakly reductive semigroup is of this form.

PROPOSITION 8 (Petrich [79]). An extension V of a semigroup S is determined by a partial homomorphism iff V has an S-endomorphism iff V has an idempotent endomorphism with range S.

The precise relationship between the two kinds of extensions just considered can be elucidated using a concept due to Clifford. Let S be a semigroup, to every $s \in S$ associate a set Z_s such that Z_s are p.d., $Z_s \cap S = \{s\}$, and let $V = \bigcup_{s\in S} Z_s$ with multiplication

$x * y = ab$ if $x \in Z_a$, $y \in Z_b$. Then V is an *inflation* of S. An extension V of S is an inflation of S iff V is an extension of S by a semigroup Q, for which $Q^2 = 0$, determined by a partial homomorphism. If $T \subseteq S$ and $a \in S \setminus T$ implies $Z_a = \{a\}$, call V an *inflation of* S *over* T.

THEOREM 9 (Grillet-Petrich [48]). *Every strict extension of* S *is determined by a partial homomorphism iff* S *is an inflation of a weakly reductive semigroup* R *over* $R \setminus R^2$. *In such a case* $\Pi(S) \cong R$.

Extensions determined by partial homomorphisms are usually easier to handle and exhibit many properties not shared by arbitrary extensions. A sufficient condition in order to have an extension of this kind is given in the next theorem. By E_S denote the set of all idempotents of S with the partial order $e \leq f$ iff $e = ef = fe$.

THEOREM 10 (Petrich [79]). *Let* S *be a regular semigroup,* Q *a semigroup with zero disjoint from* S *in which every element has either an idempotent left or right identity. Then an extension* V *of* S *by* Q *is determined by a partial homomorphism iff every element of* Q^* *has either an idempotent left or right identity* e *for which the set* $\{f \in E_S \mid f < e\}$ *admits a unique maximal element.*

The second part of Theorem 9 in Section 2 is a particular case of this theorem. For a special case see Warne [127]. The condition on Q is a mild one (regular semigroups, semigroups with one sided identity, etc.). The question which semigroups admit only extensions determined by partial homomorphism answers

THEOREM 11 (*Clifford* [17], *Grillet* - *Petrich* [48]). *Every extension of a semigroup* S *is determined by a partial homomorphism iff every extension of* S *is strict iff* S *has an identity*.

The extension theory I have heretofore discussed is based on translations. For the weakly reductive case this theory is quite satisfactory from the point of view of construction, properties, classification, etc. However, for the general case, as exhibited in Proposition 3, all we can do is to essentially rewrite the associative law in a more convenient form. Another approach to extensions, based on S-congruences, has been devised in Petrich and Grillet [88]. I will only touch upon this subject and will mainly establish a link with densely embedded ideals.

Let V be an extension of an arbitrary semigroup S. A congruence σ on V is an S-*congruence* if $\sigma|_S$ is the equality relation. The extension V is *dense* if the equality relation on V is the only S-congruence on V. If σ is any congruence on V, let $\sigma(S) = \{v \in V \mid v \,\sigma\, s \text{ for some } s \in S\}$. By $\mathfrak{T} = \mathfrak{T}(V:S)$ denote the congruence on V induced by $\tau = \tau(V:S)$.

PROPOSITION 12 (*Petrich*-*Grillet* [88]). *If* V *is an extension of* S, *then every* S-*congruence on* V *is contained in* $\mathfrak{T}(V:S)$ *and* $\mathfrak{T}(S)$ *is the largest strict extension of* S *contained in* V.

Summarizing certain properties of extensions we have

THEOREM 13 (Grillet-Petrich [48], [88]). *The following statements hold for an extension* V *of a weakly reductive semigroup* S.

i) $\mathfrak{J} = \mathfrak{J}(V:S)$ *is the largest* S-*congruence on* V.

ii) V *is a strict extension iff* $\mathfrak{J}(S) = V$.

iii) V *is a pure extension iff* $\mathfrak{J}(S) = S$.

iv) V *is a dense extension iff* $\tau(V:S)$ *is* 1 - 1 *(and thus an isomorphism into).*

v) *Two dense extensions of* S *are equivalent iff they have the same type.*

A dense extension V of S is *maximal* if for every dense extension V' of S containing V as a subsemigroup we must have $V' = V$ (i.e., maximal under inclusion); in such a case S is a *densely embedded ideal* of V. For another proof of the following important theorem see Grillet and Petrich [48].

THEOREM 14 (Gluskin [32]). *A weakly reductive semigroup* S *is a densely embedded ideal of a semigroup* V *iff* $\tau(V:S)$ *is an isomorphism of* V *onto* $\Omega(S)$.

Since $\tau(V:S)$ is the unique homomorphism of V into $\Omega(S)$ which extends $\pi: S \rightarrow \Omega(S)$ for an extension V of a weakly reductive semigroup S, it follows that V is a (maximal) dense extension of S iff there exists an isomorphism which extends π and maps V into (onto) $\Omega(S)$. If we identify S with $\Pi(S)$, then up to equivalence, the different subsemigroups of $\Omega(S)$ containing $\Pi(S)$ constitute the set of all dense extensions of S and $\Omega(S)$ is the only maximal one. It follows that any two maximal dense extensions of S are equivalent which implies that if S is a densely embedded ideal of V, then V is completely determined by S up to an S-isomorphism. Gluskin had conjectured in [32] that a semigroup which is not weakly reductive cannot be a densely embedded ideal of any semigroup. His conjecture was proved in the affirmative several years later, viz.

THEOREM 15 (Shevrin [103], [105]). *A semigroup which is not weakly reductive cannot be a densely embedded ideal of any semigroup.*

Call an extension V of a semigroup S *cancellative* if V is a cancellative semigroup. From the corollaries to Propositions 5 and 10 of Section 2, follows easily

PROPOSITION 16. *An extension V of S is cancellative iff S is cancellative and the extension is dense. Further, V is commutative and cancellative iff S is commutative and cancellative and the extension is dense.*

COROLLARY 1. *Let S be a cancellative semigroup without idempotents and* $Q = G^0$ *be a group with zero. Then there exists a cancellative extension of S by Q iff G is isomorphic with a subgroup of* $\Sigma(S)$, *the group of units of* $\Omega(S)$.

COROLLARY 2 (Heuer-Miller [55]). *Let S be a*

<u>commutative cancellative semigroup without idempotents. Then</u> $V = S^1$ (S <u>with identity adjoined) is the only cancellative extension of</u> S <u>by a group with zero iff for any</u> a,b ∈ S, aS = bS <u>implies</u> a = b.

If S has an identity, then it has no proper pure hence no dense and hence no cancellative extensions. Further results concerning cancellative extensions can be found in Heuer and Miller [55] and Petrich ([9], 3.7).

For more information on extensions consult Clifford [16], [17], Clifford and Preston ([2], 4.4), M.P. Grillet [43], P.A. Grillet [44] - [46], P.A. and M.P. Grillet [47], Grillet and Petrich [48], [49], [88], Petrich ([9], Ch. 3), [79] - [82], Tamura [117], Tamura and Graham [119], Yoshida [133]. For extensions of (i) Brandt semigroups see Lallement and Petrich [67], Warne [126], [128], (ii) $\mathfrak{n}$-semigroups see Hall [50], (iii) null semigroups see McNeil [74], [75], Yamada [131], [132], (iv) primitive inverse semigroups see Ault [14]. Dense extensions of various kinds of finite reductive completely 0-simple semigroups and their application to the theory of machines have been extensively studied by Krohn and Rhodes (LLM, RLM, GM, GGM semigroups, see Krohn, Rhodes and Tilson [7]). Material closely related to extensions can be found in Gluskin [32], Kimura, Tamura and Merkel [63], Schwarz [102], Tamura [115], Tully [122].

4. DENSELY EMBEDDED IDEALS

One of the main preoccupations of the Soviet schools of semigroups has been finding of abstract characteristics of various "concrete" semigroups of (partial) transformations. For a given semigroup of partial transformations on a set, an abstract characteristic of S is a system of axioms which an abstract semigroup T must satisfy in order that S and T be isomorphic. Note that this is precisely the opposite of the idea of representation; indeed, if T is an abstract semigroup satisfying certain conditions, then a homomorphism of T into a "concrete" semigroup of relatively familiar structure is called a representation of T by elements of S (say, functions, matrices etc.). Of particular interest is the case when this homomorphism is 1 - 1 (i.e., the representation is faithful) and even more so when it is also onto (e.g., the Rees theorem). In the last case, turning things around, T represents an abstract characteristic of S. A concrete semigroup can have many different abstract characteristics and those of simpler form are then preferable.

In the effort of obtaining abstract characteristics, the concept of a densely embedded ideal, and its variants, has turned out to be a very powerful tool. I have defined a densely embedded ideal in terms of a maximal dense extension, but the customary definition in Soviet literature is expressed in terms of homomorphisms. A semigroup A is a densely embedded ideal of a semigroup B if A is an ideal of B satisfying (nontrivial homomorphism means it is not 1 - 1):

a) every nontrivial homomorphism of B induces a nontrivial homomorphism of A (cf. essential extensions in categories),

b) for every semigroup C containing B and different from B, and containing A as an ideal, there exists a nontrivial homomorphism of C which induces a trivial homomorphism on A.

This notion was introduced by Ljapin [68] in 1953 in order to give an abstract characteristic of the semigroup F(X) of all transformations (written on the left) on a set X as follows: an abstract semigroup S is isomorphic to F(X) iff S has a densely embedded ideal which is a left zero semigroup of the same cardinality as X. This also follows from Proposition 6 of Section 2 and Theorem 14 of Section 3. He also gives a similar abstract characteristic for the semigroup of all partial 1 - 1 transformations on X. These are the first two of a long list of results of this type, a great number of which are due to Gluskin. The idea here being that one can characterize a relatively complicated semigroup V by finding in it a densely embedded ideal of rather simple structure. The interest in this subject has also been motivated by the desire to extend isomorphisms of an ideal to the whole semigroup. For the properties of a semigroup V relative to its densely embedded ideal S as well as for extension of isomorphisms of S to all of V, Theorem 14 of Section 3 is of crucial importance. Combining

Theorems 14 and 15 of Section 3, we get the fundamental result:

A semigroup S *is a densely embedded ideal of a semigroup* V *iff* S *is weakly reductive and* $\tau(V:S)$ *is an isomorphism of* V *onto* $\Omega(S)$; $\tau(V:S)$ *is the unique extension of* $\pi\colon S \to \Omega(S)$ *to a homomorphism of* V *into* $\Omega(S)$.

I will merely state a few key theorems on this subject, a more systematic treatment can be found in Petrich ([9], 3.8, 5.2 - 5.4, 6.1 - 6.4). The paper Gluskin [33] contains a concise summary of a number of important results on the subject.

In light of the above discussion, the next theorem can be proved by using Theorem 11 of Section 2.

THEOREM 1 (Gluskin [32]). *Let* V *be a dense extension of a semigroup* S, *let* S' *be a densely embedded ideal of a semigroup* V', *and let* θ *be an isomorphism of* S *onto* S'. *Then* θ *admits an extension* ψ *to a* 1 - 1 *homomorphism of* V into V', ψ *is the unique homomorphism from* V *into* V' *extending* θ, *and* ψ *is onto iff* S *is a densely embedded ideal of* V.

COROLLARY 1. *Let* S *be a densely embedded ideal of* V. *In order that* V *be isomorphic to a semigroup* V' *it is necessary and sufficient that* V' *contain a densely embedded ideal* S' *isomorphic to* S. *If so, every isomorphism of* S *onto* S' *can be uniquely extended to an isomorphism of* V *onto* V'.

COROLLARY 2. *If S is a densely embedded ideal of V, then every automorphism θ of S admits a unique extension to an automorphism ψ of V and θ and ψ have the same order.*

It follows from above that a semigroup V has a densely embedded ideal iff V has an identity. It is not known if in such a case densely embedded ideals of V form a lattice under inclusion. The next result can be viewed as a step toward the determination of the distribution of densely embedded ideals in a semigroup.

THEOREM 2 (Petrich [9]). *Let S be a globally idempotent densely embedded ideal of V. Then every ideal of V containing S is a densely embedded ideal of V.*

The case in which S is also a (0-) minimal ideal of V occurs frequently among various semigroups of partial transformations; for it we have

COROLLARY 1. *Let S be a densely embedded (0-) minimal ideal of V. Then S is contained in every (nonzero) ideal of V and every (nonzero) ideal of V is densely embedded in V.*

COROLLARY 2. *Let S and S' be densely embedded (0-) minimal ideals of V and V', and let T and T' be subsemigroups of V and V' containing S and S', respectively. Then every isomorphism of T onto T' maps S onto S' and can be uniquely extended to an isomorphism of V onto V'.*

COROLLARY 3. *Let S be a densely embedded (0-) minimal ideal of V. Then every automorphism of V restricted to S is an automorphism of S, and conversely, every automorphism of S can be uniquely extended to an automorphism of V. This association yields* $\mathcal{G}(S) \cong \mathcal{G}(V)$.

I will now discuss some of the principal examples of densely embedded ideals. The results just quoted can be readily applied to these examples furnishing additional information concerning isomorphisms and their (unique) extensions. For the remainder of this section, let X be an arbitrary nonempty set unless specified otherwise. Let W(X) denote the semigroup of all partial transformations on X written as operators on the left under composition. The zero of W(X) is the empty transformation ϕ. For $\alpha \in W(X)$, $\underline{d}\alpha$ denotes the domain of α, $\underline{r}\alpha$ the range. A subsemigroup S of W(X) is *weakly transitive* if $\bigcup_{\alpha \in S} \underline{d}\alpha = \bigcup_{\alpha \in S} \underline{r}\alpha = X$; *separative* (also called X-*simple*) if for any two distinct elements x and y of X, there exists $\alpha \in S$ such that either $\underline{d}\alpha$ contains exactly one of the elements x, y or $x, y \in \underline{d}\alpha$ and $\alpha x \neq \alpha y$.

PROPOSITION 3. *A weakly transitive separative semigroup is reductive*.

It follows that even though the mentioned conditions on a subsemigroup of W(X) are relatively mild and are usually easy to verify, we obtain in this way only a class of reductive subsemigroups of W(X). These conditions come in very handy in the following generalization of a densely embedded ideal.

A semigroup A is a _densely embedded subsemigroup_ of a semigroup B if A is a subsemigroup of B and A is a densely embedded ideal of its idealizer in B. We say simply that A is _densely embedded_ in B; this is in concordance with the terminology already used for ideals. A rather difficult proof establishes the following result. The trouble of proving it is well worth the effort since it has far reaching consequences.

THEOREM 4 (Gluskin [29]). _Every weakly transitive separative subsemigroup of_ W(X) _is densely embedded_.

Roughly speaking, every "sufficiently rich" subsemigroup of W(X) is densely embedded. To prove the same kind of theorem for a subsemigroup B of W(X), it suffices to show that the idealizer in W(X) of every weakly transitive separative subsemigroup A of B is contained in B. This is the essence of the proof of several of the following corollaries, all of which are due to Gluskin [29].

COROLLARY 1. _Every weakly transitive separative subsemigroup of the semigroup_ F(X) _of all transformations on_ X _is densely embedded. The result remains valid for the semigroup_ $E_T(A)$ _of all endomorphisms of a universal algebra_ A _with the domain_ T _of operators (which may be empty)_.

Note that for (full) transformations on X, weak transitivity of a semigroup S amounts to $\bigcup_{\alpha \in S} \underline{r\alpha} = X$, separativity is usually expressed by saying that "S separates points of X".

COROLLARY 2. *Let G be a group with the domain T of operators. Every weakly transitive subsemigroup S of $E_T(G)$ such that $\alpha x = 1$ for all $\alpha \in S$ implies $x = 1$, is densely embedded.*

COROLLARY 3. *Let V be a vector space over a division ring. Every weakly transitive subsemigroup S of the semigroup $S(\Delta,V)$ of all endomorphisms of V such that $\alpha v = 0$ for all $\alpha \in S$ implies $v = 0$, is densely embedded. In particular, every transitive subsemigroup (and thus every transitive or dense ring of linear transformations) of $S(\Delta,V)$ is densely embedded.*

For 1 - 1 partial transformations, we have simpler statements, viz.

COROLLARY 4. *Every weakly transitive subsemigroup of the semigroup V(X) of all 1 - 1 partial transformations on X, is densely embedded. The result remains valid for the semigroup D(X) of all 1 - 1 transformations on X and the semigroup T(X) of all continuous, open 1 - 1 transformations of a topological space X into itself.*

Ideals of W(X), V(X) and F(X) can be expressed by means of the rank of a partial transformation. We then see that the (partial) constants in the first two semigroups form a 0-minimal ideal and in the third a minimal ideal. For D(X) the ideals are expressed in terms of the defect of a transformation, this semigroup also has a minimal ideal. All these (0-) minimal ideals are weakly transitive and separative and thus

COROLLARY 5. *Every nonzero ideal of* $W(X)$, $V(X)$, $F(X)$, $D(X)$ *is densely embedded subsemigroup of the respective semigroup and of* $W(X)$.

With the notation of Corollary 3, ideals of $S(\Delta,V)$ can be expressed by means of the rank of a linear transformation, and the linear transformations of rank ≤ 1 form a 0-minimal ideal satisfying the hypothesis of Corollary 3, and thus

COROLLARY 6. *Every nonzero ideal of* $S(\Delta,V)$ *is densely embedded.*

The hypothesis of the last part of Corollary 4 can be sometimes weakened, viz.

THEOREM 5 (Gluskin [29]). *Let* X *be the open,* $\bar{X}$ *the closed unit ball in the n-dimensional Euclidean space. Every subsemigroup* S *of* $T(\bar{X})$ *for which* $\{\alpha|_X \mid \alpha \in S\}$ *is weakly transitive, is densely embedded in* $T(\bar{X})$. *In particular, the semigroup* $T_o(\bar{X})$ *consisting of all* $\alpha \in T(\bar{X})$ *such that the boundaries of* $\bar{X}$ *and* $\alpha\bar{X}$ *are disjoint, is a densely embedded ideal of* $T(\bar{X})$.

PROPOSITION 6 (Gluskin [29]). *Let* $\bar{X}$ *be the closed unit disc in the complex plane,* $C(\bar{X})$ *the semigroup of all conformal mappings on* $\bar{X}$, $C_o(\bar{X})$ *the set of all* $\alpha \in C(\bar{X})$ *such that the boundaries of* $\bar{X}$ *and* $\alpha\bar{X}$ *are disjoint. Then* $C_o(\bar{X})$ *is a densely embedded ideal of* $C(\bar{X})$.

Let $\mathcal{B}(X)$ be the semigroup of all binary relations on X. The set $\mathcal{R}(X)$ of all $\sigma \in \mathcal{B}(X)$ of the form $\sigma = A \times B$ for some $A, B \subseteq X$ (called _rectangular binary relations_) is a 0-minimal ideal of $\mathcal{B}(X)$. Either directly (Zaretski [134]) or using the translational hull of $\mathcal{R}(X)$ (Petrich [84]), it can be shown that $\mathcal{R}(X)$ is a densely embedded ideal of $\mathcal{B}(X)$. From Corollary 1 to Theorem 2, we conclude

THEOREM 7. Every nonzero ideal of $\mathcal{B}(X)$ _is densely embedded._

Further examples of densely embedded ideals and subsemigroups of semigroups of partial transformations can be found in Gluskin [26], [27], [29], [31], [33] - [35], of semigroups of binary relations in Petrich [84], of abstract semigroups in Petrich [87]. For example, the canonical image of a commutative cancellative semigroup in its quotient group is a densely embedded subsemigroup. Another example is provided by Theorem 9 of Section 2: in the semigroup (under intersection) of all ideals of a semilattice S the semigroup of principal ideals is densely embedded.

The concept of a densely embedded ideal has been generalized in several directions. A left ideal L of a semigroup S is an $\mathcal{L}$-_densely embedded ideal_ if it satisfies conditions a) and b) (at the beginning of this section written for left ideals). Most of the results on densely embedded ideals carry over to this case, with right reductivity playing the role of weak reductivity and the semigroup of left translations the role of the translational hull. For an abstract characterization of the semigroup of endomorphisms of a quasi (or partially) ordered set, the above concepts are not suitable. For this purpose Gluskin [32] has introduced Σ-semigroups and Σ-densely embedded ideals. A Σ-semigroup is a semigroup together with two sets of n-ary relations satisfying certain compatibility conditions relative to multiplication. A Σ-semigroup S is a Σ-densely embedded ideal of a Σ-semigroup V if V is maximal relative to being a dense extension of S satisfying certain conditions pertaining to the Σ-structure of S. These concepts and results are due to Gluskin [32].

Two more generalizations of a densely embedded ideal have been successfully used in Gluskin [35] for models. This theory has very varied applications, e.g., to quasi or partially ordered sets, complete lattices, topological and uniform spaces, abelian groups, etc. Here one frequently starts with a certain structure and considers some kind of "hull" for it, e.g., the translational hull of a semigroup or a ring, completion of a partially ordered set or a uniform space, holomorph or a divisible hull of an abelian group, etc. It is interesting that these can be put under the same roof using the concept of a (general) dense embedding. Many of these results can be formulated for categories satisfying some weak restrictions, and certain of them carry over to the injective hull of a unital module over a ring. The precise relationship among these various "hulls" for the same structure (e.g., semilattices considered as semigroups, or bands, or partially ordered sets, or S-systems) has not yet been established (for the injective hull of S-systems see Berthiaume [15] and for a related topic see Fort [24]).

Further results concerning densely embedded ideals and various generalizations thereof can be found in Gluskin [26] - [36], [38] - [40], Kalmanovich [60] - [62], Ljapin ([8], VII. 5.6 - 5.10), [68], [69], [71], Malinin [72], [73], Raben [95], Schein [98], Shutov [111], Trohimenko [120].

5. THE TRANSLATIONAL HULL OF A REGULAR REES MATRIX SEMIGROUP

Whenever a semigroup has a completely (0-) simple ideal, we are essentially dealing with an extension of a regular Rees matrix semigroup (henceforth the modifier "regular" will be omitted). This situation occurs quite often and warrants a closer look at the translational hull of a Rees matrix semigroup. While the main sources for this section are Petrich [83], [85], [87] for the translational hull and applications to representations and Gluskin [27] for semigroups of linear transformations, a more systematic treatment can be found in Petrich ([9], Chapters 4, 5, 6), [10]. The notes of Krohn, Rhodes and Tilson [7] contain many related results for finite semigroups.

Let I and M be nonempty sets, G^o a group with zero, $P = (p_{\mu i})$ a $M \times I$ - matrix over G^o with at least one nonzero entry in each row and column; on the set $D = M \times G^0 \times I$ define the multiplication $(i,a,\mu)(j,b,\nu) = (i,ap_{\mu j}b,\nu)$. The semigroup $S = D/\{(i,0,\mu) \mid i \in I, \mu \in M\}$ is a (regular) <u>Rees matrix semigroup</u>, to be denoted by $S = \mathcal{M}^0(I,G,M;P)$. For convenience its zero will be written $(i,0,\mu)$ for any $i \in I$, $\mu \in M$; P is called the <u>sandwich matrix</u> and G the <u>structure group</u> of S. For a full discussion on this subject, see Clifford and Preston ([2], 3.2), Ljapin ([8], V. §§ 4-6).

For functions φ and φ' mapping subsets (possibly empty) of I into G, define the product $\varphi \cdot \varphi'$ by: $(\varphi \cdot \varphi')i = (\varphi i)(\varphi' i)$ for all $i \in \underline{d}\varphi \cap \underline{d}\varphi'$; if $\alpha \in W(I)$, define $\varphi^{\alpha} i = \varphi(\alpha i)$ for all $i \in \underline{d}\alpha$ for which $\alpha i \in \underline{d}\varphi$. For a subsemigroup T of $W(I)$, the <u>left wreath product</u> $T \,w\ell\, G$ of T and G is the set of all (α,φ) where $\alpha \in T$ and φ maps $\underline{d}\alpha$ into G under the multiplication

$$(\alpha,\varphi)(\alpha',\varphi') = (\alpha\alpha',\varphi^{\alpha'} \cdot \varphi').$$

<u>THEOREM 1</u> (<u>Petrich</u> [83]). <u>For</u> $S = \mathcal{M}^0(I,G,M;P)$, <u>the function</u>

$$\underline{a}: \lambda \rightarrow (\alpha,\varphi) \quad (\lambda \in \Lambda(S))$$

<u>where</u>

$\underline{d}\alpha = \underline{d}\varphi = \{i \in I \mid \lambda(i,1,\mu) \neq 0\}$,
1 <u>is the identity of</u> G,
$\lambda(i,1,\mu) = (\alpha i,\varphi i,\mu)$ <u>if</u> $i \in \underline{d}\alpha$

<u>is an isomorphism of</u> $\Lambda(S)$ <u>onto</u> $W(I) \,w\ell\, G$. <u>In particular</u>,

$\lambda(i,a,\mu) = (\alpha i,(\varphi i)a,\mu)$ *if* $i \in \underline{d}\alpha$, *and* 0 *otherwise*.

One defines the *right wreath product* G wr Q of G and Q, where Q is a subsemigroup of $W'(M)$ (the prime always denotes that the functions are written on the right) with multiplication $(\psi,\beta)(\psi',\beta') = (\psi \cdot {}^{\beta}\psi',\beta\beta')$, and obtains an analogous result for right translations, viz. an isomorphism $\underline{b}$: $P(S) \to G$ wr $W'(M)$. A simple calculation shows that for $\underline{a}\lambda = (\alpha,\varphi)$, $\rho\underline{b} = (\psi,\beta)$, we have $(\lambda,\rho) \in \Omega(S)$ iff

$$i \in \underline{d}\alpha,\ p_{\mu(\alpha i)} \neq 0 \leftrightarrow \mu \in \underline{d}\beta,\ p_{(\mu\beta)i} \neq 0$$
$$\to p_{\mu(\alpha i)}(\varphi i) = (\mu\psi)p_{(\mu\beta)i}.$$

Hence we can write bitranslations of S in the form $(\alpha,\varphi;\psi,\beta)$. In fact, ψ (or φ) is completely determined by the remaining three parameters. For convenience write rank (α,φ) = rank α, rank (ψ,β) = rank β, then

THEOREM 2 (Petrich [83]). *For* $S = \mathcal{M}^0(I,G,M;P)$,

$$\Pi(S) = \{(\lambda,\rho) \in \Omega(S) \mid \text{rank}\ \underline{a}\lambda = \text{rank}\ \underline{b}\rho \leq 1\}.$$

The above description simplifies considerably in the following special cases.

COROLLARY 1. *For* $S = \mathcal{M}(I,G,M;P)$, *a Rees matrix semigroup without zero, we have*

$$\Lambda(S) \cong F(I)\ \text{w}\ell\ G, \quad P(S) \cong G\ \text{wr}\ F'(M)$$
$$\Omega(S) = \{(\lambda,\rho) \mid \underline{a}\lambda = (\alpha,\varphi),\ \rho\underline{b} = (\psi,\beta),\ p_{\mu(\alpha i)}(\varphi i) = (\mu\psi)p_{(\mu\beta)i} \text{ for all } i \in I,\ \mu \in M\}$$
$$\Pi(S) = \{(\lambda,\rho) \in \Omega(S) \mid \text{rank}\ \underline{a}\lambda = \text{rank}\ \rho\underline{b} = 1\}.$$

If the set of idempotents of such an S forms a subsemigroup, then $S \cong L \times G \times R$ where $L(R)$ is a left (right) zero semigroup defined on $I(M)$, and we have

COROLLARY 2. $\Omega(L \times G \times R) \cong F(L) \times G \times F'(R)$.

As special cases, we get an isomorphic copy of the translational hull of a left (right) group, rectangular band, left (right) zero semigroup (cf. Proposition 6 of Section 2 and the introduction to Section 4). It is sometimes useful to know when the projection homomorphism π_Λ is 1 - 1. The μ^{th} and ν^{th} rows of the sandwich matrix P are *left proportional* if $p_{\mu i} = c p_{\nu i}$ for some $c \in G$ and all $i \in I$.

THEOREM 3 (Petrich [83]). *The following are equivalent on* $S = \mathcal{M}^0(I,G,M;P)$: (i) π_Λ *is* 1 - 1, (ii) $\pi_\Lambda |_{\Pi(S)}$ *is* 1 - 1, (iii) *the identity mapping on* S *as a left translation is linked only to itself*, (iv) *no two distinct rows of* P *are left proportional*, (v) S *is right reductive*.

The wreath product admits a natural interpretation in terms of matrices. For let $\underline{M}(I,G)$ be the set of all column monomial $I \times I$-matrices over G^0 (i.e., each column contains at most one nonzero entry) under the usual row by column multiplication (treating the zeros in the obvious way).

THEOREM 4. *The function* c: $(\alpha,\varphi) \to (a_{ij})$, *where* $a_{ij} = \varphi j$ *if* $j \in d\alpha$, $\alpha j = i$, *and* 0 *otherwise, is an isomorphism of* $W(I)\ w\ell\ G$ *onto* $\underline{M}(I,G)$.

An analogous situation occurs with row-monomial matrices $\underline{M}'(G,M)$, viz. there exists an isomorphism $\underline{d}$ of G wr $W'(M)$ onto $\underline{M}'(G,M)$.

COROLLARY 1. *For* $S = \mathcal{M}^0(I,G,M;P)$, $(\lambda,\rho) \in \Omega(S)$ *iff letting* $A = \underline{c}\,\underline{a}\,\lambda$ *and* $B = \rho\,\underline{b}\,\underline{d}$, *we have* $PA = BP$.

We can write an element (i,a,μ) of S as an $I \times M$-matrix with a in the (i,μ) position and 0 elsewhere.

COROLLARY 2. *If we write the elements and, say, left translations of* $S = \mathcal{M}^0(I,G,M;P)$ *as matrices, then the value of a left translation at an element is the matrix product of the two corresponding matrices.*

Let $S = \mathcal{M}^0(I,G,M;P)$. By Theorem 11 of Section 2 every automorphism θ of S induces an automorphism $\bar{\theta}$ of $\Omega(S)$. Conversely, since $\Pi(S)$ is a 0-minimal densely embedded ideal of $\Omega(S)$, every automorphism of $\Omega(S)$ maps $\Pi(S)$ onto itself (see Section 4) and hence induces an automorphism of S. The association $\theta \rightarrow \bar{\theta}$ is an isomorphism of $\mathfrak{A}(S)$ onto $\mathfrak{A}(\Omega(S))$. An automorphism θ of S is determined by several functions between I, G, and M, see Clifford and Preston ([2], 3.4), one of which is an automorphism ω of G, uniquely determined by θ up to an inner automorphism of G. For convenience I will say that θ is *induced* by ω; a connection among ω, θ, and $\bar{\theta}$ is given in

THEOREM 5 (Petrich [9]). *Let* θ *be an automorphism of* $S = \mathcal{M}^0(I,G,M;P)$ *induced by an automorphism* ω *of* G. *Then* ω *is an inner automorphism* *of* G *iff* θ *is a generalized inner automorphism of* S *iff* $\bar{\theta}$ *is an inner automorphism of* $\Omega(S)$.

Further properties of $\Omega(S)$ concerning maximal subgroups, center etc. can be found in Petrich ([9], Ch. 4).

The 0-minimal ideals of $W(X)$, $V(X)$, $\mathcal{B}(X)$, $S(\Delta,V)$ are all reductive completely 0-simple semigroups, and as we have seen in Section 4, all are densely embedded ideals of respective semigroups. The first three have trivial structure group which by Theorem 5 implies that all automorphisms of $W(X)$, $V(X)$, and $\mathcal{B}(X)$ are inner. Many of the general properties of the translational hull now yield familiar (or unfamiliar) properties of these semigroups with little effort. The case of $S(\Delta,V)$ goes along the same lines but the proofs are much more involved. It is instructive to give the 0-minimal ideal S of any of these semigroups T a Rees matrix representation, establish an explicit isomorphism of T onto $\Omega(S)$, and prove the desired results within $\Omega(S)$. I will outline this procedure for $S(\Delta,V)$; analogous but simpler statements hold for $W(X)$, $V(X)$, $F(X)$, and some also for $\mathcal{B}(X)$. For example, one obtains that every isomorphism between $F(X)$ and $F(X')$ is induced by a bijection of X and X', etc. The first result in this direction is that of J. Schreier [101]. For further results consult Gluskin [33], [37], Petrich ([9], Chapters 4, 6), [84], Popova [93], Shutov [110], and for related material Schein [99], [100], Shneperman [107] - [109], Vazhenin [123], [124].

Let V be a left vector space over a division ring Δ, to be denoted by (Δ,V). Let M_V be an index set of all 1-dimensional subspaces V_μ of V and in each V_μ choose a nonzero vector e_μ; let I_{V^*} be an index set of all 1-dimensional subspaces V_i^* of V^* and in each V_i^* choose a nonzero vector f_i. Let D be the multiplicative group of nonzero elements of Δ, and let $G(\Delta,V) = \mathcal{M}^0(I_{V^*},D,M_V;P)$ where $p_{\mu i} = e_\mu f_i$. Let γ be the smallest cardinal greater than $\dim V$, and for each $0 < \xi \leq \gamma$, let $S_\xi(\Delta,V)$ be the semigroup of all endomorphisms (i.e., linear transformations) of V of rank $< \xi$; for $S_\gamma(\Delta,V)$ write $S(\Delta,V)$. For ξ either infinite or $\xi = \gamma$, let $R_\xi(\Delta,V)$ denote the corresponding ring and write $R(\Delta,V)$ for $R_\gamma(\Delta,V)$. Then $G(\Delta,V)$ is a Rees matrix semigroup isomorphic to $S_2(\Delta,V)$, and the latter, according to Corollary 6 to Theorem 4 of Section 4, is a densely embedded ideal of $S(\Delta,V)$, so

<u>THEOREM</u> <u>6</u> (<u>Gluskin</u> [27]). $S(\Delta,V) \cong \Omega(G(\Delta,V))$.

A direct proof establishing an isomorphism between $S(\Delta,V)$ and $\Omega(G(\Delta,V))$ can be found in Petrich [86]. The set $\{S_\xi \mid 0 < \xi \leq \gamma\}$ constitutes the set of all ideals of $S(\Delta,V)$; all nonzero ideals of $S(\Delta,V)$ are densely embedded (Corollary 6 to Theorem 4 of Section 4, or by Corollary 1 to Theorem 2 of Section 4 and the above Theorem 6).

A <u>semilinear</u> <u>transformation</u> of (Δ,V) onto (Δ',V') is a pair (ω,a) where ω is an isomorphism of Δ onto Δ', a is an additive isomorphism of V

onto V', and $(\delta v)a = (\delta\omega)(va)$ for all $\delta \in \Delta$, $v \in V$. If (ω,a) exists, write $(\Delta,V) \cong (\Delta',V')$ and define $\xi_{(\omega,a)}: \alpha \to a^{-1}\alpha a$ $(\alpha \in S(\Delta,V))$; then $\xi_{(\omega,a)}$ is the isomorphism of $S(\Delta,V)$ onto $S(\Delta',V')$ <u>induced</u> by (ω,a). A long and difficult proof establishes

<u>THEOREM 7</u> (<u>Gluskin</u> [27]). <u>For dim</u> $V > 1$, <u>every isomorphism of</u> $S(\Delta,V)$ <u>onto</u> $S(\Delta',V')$ <u>is induced by a semilinear transformation of</u> (Δ,V) <u>onto</u> (Δ',V') <u>and is thus a ring isomorphism of</u> $R(\Delta,V)$ <u>onto</u> $R(\Delta',V')$.

<u>COROLLARY 1</u>. <u>For dim</u> $V > 1$, <u>every multiplicative isomorphism of</u> $R_\xi(\Delta,V)$ <u>onto</u> $R_\xi(\Delta',V')$ <u>is additive</u>.

<u>COROLLARY 2</u>. <u>For dim</u> $V > 1$ <u>and</u> $\xi > 1$, <u>we have</u> $S_\xi(\Delta,V) \cong S_\eta(\Delta',V')$ <u>iff</u> $\xi = \eta$ <u>and</u> $(\Delta,V) \cong (\Delta',V')$.

<u>COROLLARY 3</u>. <u>The following conditions are equivalent under the hypothesis dim</u> $V > 1$. (i) $\Delta \cong \Delta$, $\dim V = \dim V'$, (ii) $(\Delta,V) \cong (\Delta',V')$, (iii) $G(\Delta,V) \cong G(\Delta',V')$, (iv) $S(\Delta,V) \cong S(\Delta',V')$, (v) $R(\Delta,V) \cong R(\Delta',V')$.

If we write the elements of $R(\Delta,V)$ as row finite $(\dim V \times \dim V)$-matrices (using a fixed basis), then every automorphism can be written as a product of an automorphism of the form $(a_{ij}) \to (a_{ij}\omega)$ where $\omega \in \Gamma(\Delta)$ and an inner automorphism. Automorphisms of $R_\xi(\Delta,V)$ are then obtained by restriction.

For further results in this direction see Fajans [22], [23], Gluskin [25] - [27], [31], Hotzel [146], Kuznecov [64], a systematic treatment can be found in

Petrich [10]. Theorem 7 can be deduced also from its ring counterpart (see Jacobson [5], Ch. IV) and a theorem of Rickart [169] which implies that for $\dim V > 1$, $R(\Delta, V)$ is a ring with unique addition in the sense of R.E. Johnson [149] (i.e., every multiplicative isomorphism is additive); see also Martindale [161], Stephenson [171]. The first theorem of the type of Theorem 7 was proved by Eidelheit [141]. Papers on semigroups of endomorphisms of a module include Mihalev [76], [77], Mihalev and Shatalova [78] and on rings Johnson and Kiokemeister [150], Wolfson [182] - [184]. A generalization of a semilinear transformation can be found in Dotson [21]. An extensive discussion of the ring and various groups of endomorphisms of a vector space can be found in Baer ([1], Chapters V, VI), see also Plotkin ([11], Ch. 4).

I will now apply some of the results at the beginning of this section to representations of semigroups. For further discussion of this subject consult Clifford and Preston ([2], Ch. 3) and Petrich [85], [87]. Let S be a semigroup, K the union of some of its $\mathcal{D}$-classes, and let $Tr(K)$ be the <u>trace</u> of K, i.e., the set $K \cup 0$ with multiplication $a * b = ab$ if $a \mathcal{R} ab$, $ab \mathcal{L} b$, and 0 otherwise. Then $K = D$ is a regular $\mathcal{D}$-class of S iff $Tr(D)$ is a completely 0-simple semigroup. In such a case, a Rees matrix representation T_D of $Tr(D)$ is obtained as follows. Let $\{R_i \mid i \in I\}$ and $\{L_\mu \mid \mu \in M\}$ be the set of all $\mathcal{R}$- respectively $\mathcal{L}$-classes of S contained in D, let

$H_{i\mu} = R_i \cap L_\mu$, suppose that $1 \in I \cap M$ and that $H_{11} = R_1 \cap L_1$ is a group with identity e. For every $i \in I$, fix an element $r_i \in H_{i1}$, and for every $\mu \in M$, fix an element $q_\mu \in H_{1\mu}$. Then $T_D = \mathcal{M}^0(I, H_{11}, M; P)$ where $p_{\mu i} = q_\mu r_i$ if $q_\mu r_i \in H_{11}$, and 0 otherwise, is the sought representation of $\mathrm{Tr}(D)$. For every $i \in I$, fix an idempotent $e_i \in R_i$ and let r_i' be the unique inverse of r_i in R_1 for which $r_i r_i' = e_i$; make an analogous selection to get $q_\mu' q_\mu = f_\mu$. For every $s \in S$, define the functions λ^s and ρ^s on T_D by: $\lambda^s 0 = 0\rho^s = 0$ and

$$\lambda^s(i,a,\mu) = (j, r_j' s r_i a, \mu) \quad \text{if} \quad sH_{i1} = H_{j1},$$
$$(i,a,\mu)\rho^s = (i, a q_\mu s q_\nu', \nu) \quad \text{if} \quad H_{1\mu}s = H_{1\nu},$$

and 0 otherwise. A long and rather complicated proof yields

<u>THEOREM 8</u> (<u>Petrich</u> [85], [87]). <u>For a regular $\mathcal{D}$-class D, the function $\chi_D : s \to (\lambda^s, \rho^s)$ $(s \in S)$ is a homomorphism of S into $\Omega(T_D)$ and $\chi_D|_D$ is 1-1. Moreover, if S is a regular semigroup, then the image of S under χ_D is a densely embedded subsemigroup of $\Omega(T_D)$.</u>

A semigroup A can be <u>densely embedded</u> in a semigroup B if there exists an isomorphism φ of A into B for which $A\varphi$ is a densely embedded subsemigroup of B. Using Theorem 8 and summing over all $\mathcal{D}$-classes of a regular semigroup, one can prove

THEOREM 9 (Petrich [85], [87]). Every regular semigroup can be densely embedded both into the direct product of the translational hulls of the traces of its different $\mathcal{D}$-classes and into the translational hull of its trace.

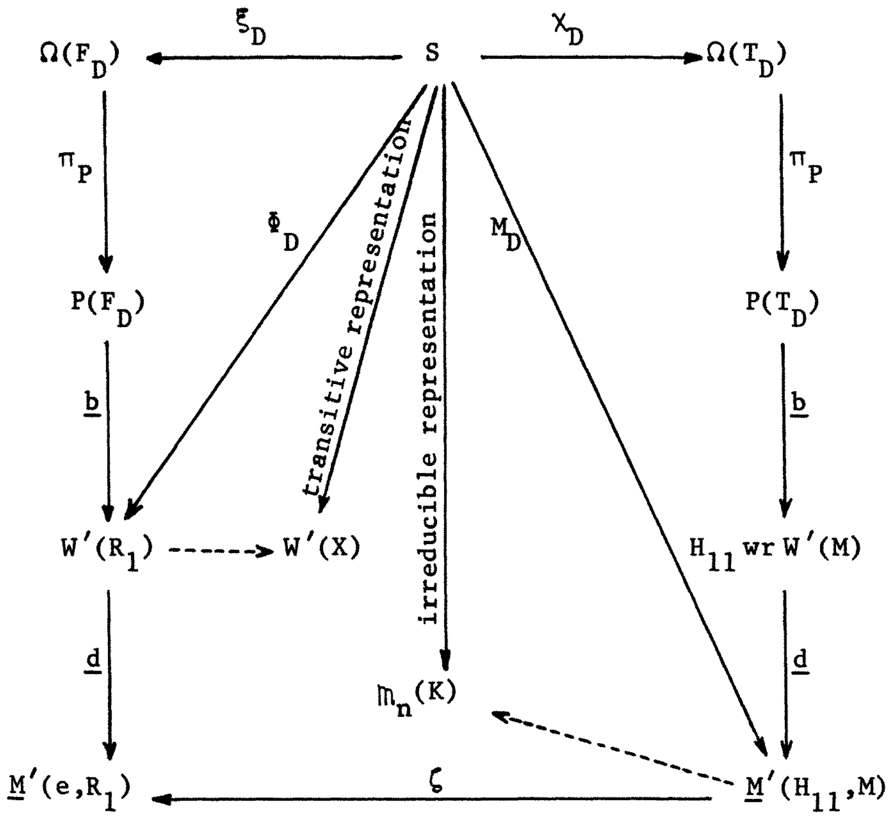
With the same notation, I define the fragment of the regular $\mathcal{D}$-class D of S by $F_D = \mathcal{M}^0(L_1, e, R_1; P)$ where $p_{r\ell} = e$ if $r\ell = e$, and 0 otherwise. Then F_D is a Rees matrix semigroup, up to an isomorphism independent of the choice of L_1 and R_1. Defining

$$\gamma^s(\ell, e, r) = (s\ell, e, r) \quad \text{if} \quad s\ell \in L_1$$

and 0 otherwise, $\gamma^s 0 = 0$, and dually δ^s, and mapping $s \to (\gamma^s, \delta^s)$, each of the statements of Theorems 8 and 9 carries over to this case (by a suitable definition of the fragment of S). Further dense embeddings of a regular semigroup S into $\Omega(Q)$ where Q is a Rees matrix semigroup with trivial structure group associated with the whole semigroup S can be found in Petrich [85], [87].

For the regular $\mathcal{D}$-class D, we can write direct M_D and dual M_D' Schützenberger representations by matrices with entries in H_{11}. Hence $M_D\colon S \to \underline{M}'(H_{11}, M)$, $M_D'\colon S \to \underline{M}(I, H_{11})$ and from Corollary 1 to Theorem 4 and Theorem 8, $P[M_D'(s)] = [M_D(s)]P$ for all $s \in S$. The function $\Phi_D\colon s \to \beta^s$ for all $s \in S$, where $r\beta^s = rs$ if $r, rs \in R_1$ is a representation $\Phi_D\colon S \to W'(R_1)$. The mapping ζ which to every matrix A in $\underline{M}'(H_{11}, M)$ associates the $R_1 \times R_1$-matrix which is obtained by substituting each nonzero entry a of A by the incidence matrix of a in the right regular representation of H_{11} and 0 by the $H_{11} \times H_{11}$ zero matrix, is an isomorphism of $\underline{M}'(H_{11},M)$ into $\underline{M}'(e,R_1)$ and $M_D\xi = \Phi_D\underline{d}$ (for $\underline{d}$ see the remark after Theorem 4).

Ponizovski [91] (see also Tully [121]) essentially proved that for a large class of semigroups S every transitive representation of S can be factored through Φ_D for some regular $\mathcal{D}$-class D of S. Lallement and Petrich [66] have established, roughly speaking, that every irreducible representation of a finite semigroup S by $n \times n$-matrices over a field factors through M_D for some regular $\mathcal{D}$-class D of S. I will illustrate these remarks and the above results by a diagram. In it each loop is a commutative diagram. $\mathfrak{M}_n(K)$ denotes the semigroup of $n \times n$- matrices over a field K and X a nonempty set. The broken lines mean "can be completed to a commutative diagram" and concern mainly finite semigroups.



6. EXTENSIONS OF RINGS

A ring R is an extension of a ring A by a ring B if R has an ideal I for which: $I \cong A$, $R/I \cong B$. With the usual identification of A with I and B with R/I, the extension problem is as follows: given rings A and B, construct all rings R having A as an ideal and such that $R/A = B$. A solution to this problem has been given by Everett [142]; this is an analogue of the Schreier theorem for group extensions and is referred to as "Everett's theorem". As in the group case, one chooses a system of representatives of the cosets of A in R, and, as in the semigroup case, makes them act on A <u>multiplicatively</u>, hence every representative induces a bitranslation of A. The only modification in the definition of the translational hull $\Omega(A)$ of the ring A is the additional requirement that for $(\lambda,\rho) \in \Omega(A)$ both λ and ρ be additive (stemming from the distributive law). This makes $\Omega(A)$ a ring in an obvious way. Most of the discussion on semigroups carries over to this case with evident changes concerning the addition. Everett's theorem is, however, quite involved in view of the long list of ring postulates the extension ring has to satisfy; in addition, because of having chosen representatives in different cosets, two "factor systems", one for addition and one for multiplication, have to be introduced. The additive group of the extension ring R is an abelian group extension of the additive group of A by the additive group of B, and hence follows the Schreier group extension theory. For a full discussion of ring extensions and a precise statement of Everett's theorem consult Rédei ([12], §§ 52-54), I will give here only some general ideas.

Two extensions R and R′ of a ring A are <u>equivalent</u> if there exists an A-isomorphism of R onto R′ (i.e., leaves A elementwise fixed) which maps the cosets of A in R onto the cosets of A in R′. Given rings A and B, a function θ of B onto a set of permutable bitranslations of A ($\theta: b \to \theta^b \in \Omega(A)$) and two functions $[\ ,\ ], \langle\ ,\ \rangle: B \times B \to A$, on $R = B \times A$

define an addition and multiplication by:

$$(a,\alpha)+(b,\beta) = (a+b,[a,b]+\alpha+\beta),$$
$$(a,\alpha)(b,\beta) = (ab,\langle a,b\rangle+\theta^{a}\beta+\alpha\theta^{b}+\alpha\beta).$$

If the three functions satisfy certain conditions, R is an extension of A by B where A is identified with $\{(0,\alpha) \mid \alpha \in A\}$ and B with the quotient R/A; write $R = E(\theta;[\ ,\ ],\langle\ ,\ \rangle)$. Conversely, every extension of A by B is equivalent to an extension of this form.

The following discussion is more reminiscent of the theory of group extensions developed by Baer (see Specht [13]) than the Schreier-Everett approach to extensions. These results are largely due to me and are as yet unpublished. Let R be an extension of a ring A by a ring B. Choose a set of representatives $\{b_\beta\}$ of the cosets of A in R and let θ map $b_\alpha + A$ onto the bitranslation of A induced by b_α (by left and right multiplication as in the case of semigroups). Let ν be the natural homomorphism of $\Omega(A)$ onto $\Omega(A)/\Pi(A)$. Then the function $\chi(R:A) = \theta\nu$ is a homomorphism of B into $\Omega(A)/\Pi(A)$ independent of the choice of representatives $\{b_\beta\}$, and is hence an invariant of the extension. Weak reductivity in semigroups corresponds to the condition $\mathfrak{A}(A) = 0$ where $\mathfrak{A}(A) = \{a \in A \mid ax = xa = 0 \text{ for all } x \in A\}$ is the <u>annihilator</u> of A. For this case, Everett's theorem simplifies to

<u>THEOREM 1</u>. <u>For the rings</u> A <u>and</u> B <u>with</u> $\mathfrak{A}(A) = 0$, <u>let</u> $\chi: B \to \Omega(A)/\Pi(A)$ <u>be a homomorphism</u>,

$\theta: B \to \Omega(A)$ (*with* $\theta: b \to \theta^b$) *be any function for which* $\theta^o = 0$, $\theta\nu = \chi$, *and define* $[\ ,\ ]$ *and* $\langle\ ,\ \rangle$ *by*:

$$\pi_{[a,b]} = \theta^a + \theta^b - \theta^{a+b},\ \pi_{\langle a,b\rangle} = \theta^a\theta^b - \theta^{ab} \quad (a,b \in B)$$

Then the ring $E(\theta;[\ ,\ ],\langle\ ,\ \rangle)$ *defined above is up to equivalence the only extension* R *of* A *by* B *for which* $\chi(R:A) = \chi$; *we denote this extension by* $E(\chi)$. *Conversely, every extension of* A *by* B *is equivalent to an extension of the form* $E(\chi)$.

We see that for $\mathfrak{A}(A) = 0$, an extension is completely determined by a single homomorphism $B \to \Omega(A)/\Pi(A)$; different choices of θ yield different but equivalent extensions, so $E(\chi)$ actually represents the equivalence class of extensions determined by χ. If we consider equivalent extensions as equal, we get

COROLLARY 1.

i) $\chi(E(\chi): A) = \chi$ *if* $\chi \in Hom(B,\Omega(A)/\Pi(A))$,

ii) $E(\chi(R:A)) = R$ *if* R *is an extension of* A.

Without identification and keeping the same notation, we have

COROLLARY 2 (*cf*. MacLane [160]). *There exists a* 1 - 1 *correspondence between the set of classes of equivalent extensions of* A *by* B *and* $Hom(B,\Omega(A)/\Pi(A))$.

Defining strict and pure extensions and $\tau = \tau(R:A): R \to \Omega(A)$ as for semigroups, analogous results to those in semigroups hold. In addition

PROPOSITION 2. *For an extension* R *of any ring* A *by a ring* B,

i) R *is a strict extension iff* $\chi(R:A)$ *is the zero homomorphism iff* $\theta: B \to \Pi(A)$,

ii) R *is a pure extension iff* $\chi(R:A)$ *is* 1-1 *iff* $\theta^b \in \Pi(A)$ *implies* $b = 0$.

COROLLARY. *For* $\mathfrak{A}(A) = 0$, *all strict extensions of* A *by* B *are equivalent to their direct sum* $A \oplus B$, *and the classes of pure extensions correspond to isomorphisms of* B *into* $\Omega(A)/\Pi(A)$.

For an extension R of A by B, let $\mathfrak{A}_R(A) = \{r \in R \mid ra = ar = 0 \text{ for all } a \in A\}$. Then

$$\mathfrak{A}_R(A) = \ker \tau(R:A) = \{(a,\alpha) \mid \theta^a = \pi_{-\alpha}\}$$

and the subring $S(R:A)$ of R generated by A and $\mathfrak{A}_R(A)$ is the maximal strict extension of A contained in R. If also $\mathfrak{A}(A) = 0$, then $S(R:A)$ is equivalent to $A \oplus \mathfrak{A}_R(A)$. For any A, if R is equivalent to $A \oplus B$, we say that A is a *direct summand* of R.

THEOREM 3. *A ring* A *is a direct summand in every strict extension iff* $\mathfrak{A}(A) = 0$.

COROLLARY (*Szendrei* [173]). *A ring* A *is a direct summand in every extension iff* A *has an identity*.

An extension R of an arbitrary ring A is *essential* if A has nonzero intersection with every nonzero ideal of R (corresponds to dense extensions in semigroups).

PROPOSITION 4. *Every pure extension of* A *is essential, the converse holds if* $\mathfrak{A}(A) = 0$.

PROPOSITION 5. *Let* $\mathfrak{A}(A) = 0$. *An extension* R *of* A *is essential iff* $\tau(R:A)$ *is* 1 - 1, *and* $\tau(R:A)$ *is then an isomorphism of* R *onto a subring of* $\Omega(A)$ *containing* $\Pi(A)$ (*cf. Proposition 2*). *Two essential extensions of* A *are equivalent iff they have the same type. Hence the classes of equivalent essential extensions can be identified with subrings of* $\Omega(A)$ *containing* $\Pi(A)$ *and are thus in a* 1 - 1 *correspondence with subrings of* $\Omega(A)/\Pi(A)$.

As in the case of semigroups, the existence of a maximal essential extension of A is equivalent to $\mathfrak{A}(A) = 0$ (announced in Shevrin [104]) and if a maximal essential extension exists, it is essentially equal to $\Omega(A)$. Defining an essential extension of a group G analogously, the corresponding result holds, viz. the existence of a maximal essential extension of G is equivalent to the triviality of its center and if a maximal essential extension exists, it is essentially the automorphism group $\mathfrak{a}(G)$ (announced in Gluskin [38]).

PROPOSITION 6. *Every extension of* A *by* B *is a subdirect product of* B *and an essential extension of* A. *If* $\mathfrak{A}(A) = 0$, *an extension* R *of* A *by* B *is a subdirect product of* B *and the type of* R. *Hence for* $\mathfrak{A}(A) = 0$, *every extension of* A *by* B *can be embedded in* $B \oplus \Omega(A)$.

The last property is reminiscent of the role of the wreath product in group extensions. As in groups, one defines a split extension of a ring by another ([,] and ⟨ , ⟩ can be chosen to be the zero functions). The usual embedding of a ring A into a ring R with identity, first published account of which is that by Dorroh [140], is a split extension of A by the ring Z of integers. However, R may lose many desirable properties A might have. Hence one may have to take some other ring instead of Z. For the case $\mathfrak{A}(A) = 0$, the canonical homomorphism $\pi: A \to \Omega(A)$ provides an embedding into a ring with identity, or one may take the subring of $\Omega(A)$ generated by $\Pi(A)$ and 1. If A is commutative and $\mathfrak{A}(A) = 0$, then $\Omega(A)$ is also commutative. It is easy to see that $\Omega(A)$ inherits from A such ring properties as "no zero divisors", "semiprime", "prime". For further information on this subject see Arhipov [135], Brown and MacCoy [137], Fuchs and Halperin [143], Fuchs and Rangaswamy [144], Funayama [145], Kohls [151], [152], Nagata [164], Szendrei [172], Tomaso [179], Weinert [180].

For the material concerning general extensions of rings consult Aumercier [136], Chew Kim Lin [138], R.E. Johnson [148], Kohls [153], MacLane [159], [160], Rédei [167], Snider [170]; for holomorphs and the translational hull see van Leeuwen [155] - [157], Petrich [165], Pollak [166], Rédei [168], Szendrei [174] - [175], Weinert and Eilhauer [181]. The translational hull of and extensions of and by a cyclic ring have been determined by Mueller and Petrich [162] - [163]. Extensions of (topological) algebras, their bitranslations and uses in analysis have been studied by Buck [185], Busby [186], [187], Fell and Goldman [188], Hochschild [192], Kohls and Lardy [194] - [195], B.E. Johnson [196] - [198], Reid [199], see also Helgason [189] - [191], Ju-Kuei Wang [193], Wendel [200], [201]. Some related subjects can be found in Dlab [139], R.E. Johnson [147], Kohls and Lardy [154], Loonstra [158], Szep [176] - [178].

7. EXTENSIONS OF PARTIALLY ORDERED SETS

This is a brief outline of the theory of extensions of partially ordered sets, mainly following Banaschewski [203] and Bruns [208], modified in such a way as to exhibit a strong analogy with the theories of extension considered heretofore. The general references for this section are Fuchs [3] and Jaffard [6].

A binary relation $\leq$ on a set Q is a <u>quasi-order</u> if $\leq$ is reflexive and transitive, a <u>partial order</u> if it is also anti-symmetric, and the set Q is then said to be a <u>quasi-ordered set</u> or a <u>partially ordered set</u> (henceforth <u>poset</u>), respectively. A subset I (possibly empty) of a quasi-ordered set Q is an <u>ideal</u> (Anfang, initial segment, lower class) of Q if for any $x,y \in Q$, $x \leq y$, $y \in I$ implies $x \in I$; for any $a \in Q$, the ideal $(a) = \{x \in Q \mid x \leq a\}$ is the <u>principal ideal</u> generated by a. Let $\Omega(Q)$ be the poset of all ideals, $\Pi(Q)$ the poset of all principal ideals of Q under set inclusion. For quasi-ordered sets Q and Q', a mapping φ of Q into Q' is an o-homomorphism if for any $x,y \in Q$, $x \leq y$ implies $x\varphi \leq y\varphi$. Call the mapping

$$\pi: a \to (a) \qquad (a \in Q)$$

the canonical o-homomorphism of Q into $\Omega(Q)$. Then π is 1 - 1 iff $\leq$ is anti-symmetric, i.e., Q is a poset. In this context, anti-symmetry corresponds to weak reductivity in semigroups. A function φ of a poset P into a poset P' is an o-isomorphism if for all $x,y \in P$, $x \leq y$ iff $x\varphi \leq y\varphi$ (which implies that φ is 1 - 1). Hence if P is a poset, then the canonical o-homomorphism $\pi: P \to \Omega(P)$ is an o-isomorphism.

I will limit myself to posets, even though some of the material that follows carries over to quasi-ordered sets. Let the following notation be fixed: E is a poset and P is a nonempty subset of E which is considered as a poset under the induced ordering; we say that E is an extension of P. Call the mapping

$$\tau = \tau(E:P): e \to (e) \cap P = \{p \in P \mid p \leq e\} \qquad (e \in E)$$

the canonical o-homomorphism of E into $\Omega(P)$; the image $T(E:P)$ of E under τ is the type of the extension. Then τ is the unique o-homomorphism of E into $\Omega(P)$ extending the canonical o-isomorphism $\pi: P \to \Omega(P)$. The poset P is join (meet) dense in E if every element of E is the join (meet) of some subset of P, in such a case, E is a join (meet) dense extension of P (superior (inferior) extension). The poset E is a _complete extension_ of P if E is a complete lattice. I will discuss (i) join dense, (ii) meet and join dense, (iii) complete join dense extensions. Let E' be another extension of P; we write $E \precsim E'$ if there exists an o-isomorphism φ of E into E' leaving P elementwise fixed, and $E \approx E'$ if φ is also onto and say that E and E' are _equivalent extensions_ of P. Then $\precsim$ is a quasi-order, $\approx$ is an equivalence relation, $\precsim$ is stable relative to $\approx$, so that $\precsim$ induces a quasi-order on the equivalence classes of $\approx$ to be denoted by the same symbol.

THEOREM 1 (_Bruns_ [208]). _An extension_ E _of_ P _is join dense iff_ $\tau(E:P)$ _is an_ o-_isomorphism of_ E _into_ $\Omega(P)$.

THEOREM 2 (_Bruns_ [208]). _For join dense extensions_ E _and_ E' _of_ P, $E \precsim E'$ _iff_ $T(E:P) \subseteq T(E':P)$.

COROLLARY 1. _Two join dense extensions of_ P _are equivalent iff they have the same type_.

COROLLARY 2. _The classes of equivalent join dense extensions of_ P _under_ $\precsim$ _form a poset_ o-_isomorphic with the power set of the difference set_ $\Omega(P) \setminus \Pi(P)$ (_and is thus a complete atomic Boolean lattice_).

For any subset A of P let Ma (Mi) be the set of all upper (lower) bounds of A in P. Hence $\text{Mi Ma } A = \bigcap_{A \subseteq (p)} (p)$ where the intersection of the empty family is P; A is _normal_ if A = Mi Ma A, let N(P) denote the poset of all normal subsets of P. For $a \in P$, we have $(a) = \text{Mi Ma } \{a\} = \text{Mi Ma } (a)$ so

(a) $\in N(P)$, and using the above formula we conclude that $\Pi(P) \subseteq N(P) \subseteq \Omega(P)$.

THEOREM 3 (Banaschewski [203], Bruns [208]). *An extension E of P is meet and join dense iff $\tau(E:P)$ is an o-isomorphism of E into N(P) (i.e., the type T(E:P) consists exclusively of normal sets).*

There are several corollaries of this theorem analogous to those of Theorem 2. The poset N(P) is the *normal* (also Dedekind-MacNeille or MacNeille) *completion* of P; we will see below that it is indeed complete. This leads us to complete extensions; to put them in proper perspective, I will consider first a more general situation.

Let $\mathfrak{F}$ be a nonempty family of subsets of a set X. If for every $x \in X$, the set $Q_x = \bigcap_{x \in F} F$ belongs to $\mathfrak{F}$, then $\mathfrak{F}$ *induces* a quasi-order $\leq$ on X by: $x \leq y$ iff $Q_x \subseteq Q_y$. Conversely, if $\leq$ is a quasi-order on X, then the family $\mathfrak{F} = \{(x) \mid x \in X\}$ induces the given quasi-order $\leq$. This correspondence is 1 - 1, in it partial orders correspond to the families for which $Q_x = Q_y$ implies $x = y$, equivalence relations to partitions. A family $\mathfrak{F}$ is a *closure system* on P if $\mathfrak{F}' \subseteq \mathfrak{F}$ implies $\bigcap_{F \in \mathfrak{F}'} F \in \mathfrak{F}$ (for $\mathfrak{F}' = \emptyset$, we get $X \in \mathfrak{F}$); $\mathfrak{F}$ is a *kernel system* on P if $\mathfrak{F}' \subseteq \mathfrak{F}$ implies $\bigcup_{F \in \mathfrak{F}'} F \in \mathfrak{F}$ (for $\mathfrak{F}' = \emptyset$, we get $\emptyset \in \mathfrak{F}$). Thus $\mathfrak{F}$ is a closure kernel system iff $\mathfrak{F}$ is a complete ring of sets. Further, $\mathfrak{F}$ is a *closure extension* of $\Pi(P)$ if $\mathfrak{F}$ is a closure system on P containing $\Pi(P)$.

PROPOSITION 4. *In the poset of all subsets of* P, N(P) *is the smallest closure system containing* $\Pi(P)$, $\Omega(P)$ *is the smallest closure kernel system containing* $\Pi(P)$.

THEOREM 5 (*Banaschewski* [203]). *An extension* E *of* P *is a complete join dense extension iff* $\tau(E:P)$ *is an* o-*isomorphism of* E *onto a closure extension of* $\Pi(P)$.

THEOREM 6 (*Banaschewski* [203], *Bruns* [208]). *The following conditions on an extension* E *of* P *are equivalent*:

i) E *is a maximal join dense extension of* P.

ii) E *is a maximal join dense complete extension of* P.

iii) $\tau(E:P)$ *is an* o-*isomorphism of* E *onto* $\Omega(P)$.

iv) *For every* $p \in P$, $A \subseteq E$, *if* $p \leq \sup_E A$ *then* $p \leq a$ *for some* $a \in A$.

The next theorem says, roughly, that for join dense extensions, "complete" pulls in the opposite direction from "meet dense" and that the normal completion is the only "extension" they have in common.

THEOREM 7 (*Banaschewski* [203], *Bruns* [208]). *The following conditions on an extension* E *of* P *are equivalent*:

i) E *is a minimal complete extension of* P.

ii) E *is a minimal join dense complete extension of* P.

iii) E *is a maximal join and meet dense extension of* P.

iv) E *is up to equivalence unique join and meet dense complete extension of* P.

v) $\tau(E:P)$ *is an* o-*isomorphism of* E *onto* N(P).

Which posets P have the property that some of the sets $\Pi(P)$, $N(P)$, $\Omega(P)$ coincide? Since $\phi \notin \Pi(P)$ and $\phi \in \Omega(P)$, these two are always different, but we have

PROPOSITION 8 (Banaschewski [203]).
$\Omega(P) = \Pi(P) \cup \{\phi\}$ *iff* P *is dually well-ordered*.

PROPOSITION 9. $N(P) = \Pi(P)$ *iff* P *is a complete lattice*.

For categorical characterizations of some of these extensions see Banaschewski and Bruns [204], and for further information Schmidt [223], [224].

To every closure system corresponds a closure operator and conversely. Hence join dense complete extensions can be identified first with closure systems and then with closure operators. If P is a partially ordered semigroup (or universal algebra) a closure operator satisfying certain compatibility conditions can be used to construct extensions of P which are themselves semigroups (or universal algebras), for this topic see Bleicher and Schneider [206], Bleicher, Schneider and Wilson [207], Burgess and McFadden [209], Clifford [210] - [212], Krishnan [217], [218]. Essentially the same idea is used in the abstract theory of ideals, see Aubert [202], Fuchs [214], Lorenzen [219]. For extensions of ordered rings and semigroups, consult Bigard et Keimel [205], Fuchs [213], Keimel [215], Kohls [216], Lugowski [220] - [222].

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Rings

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Department of Mathematics
Pennsylvania State University
University Park, Pennsylvania 16802
USA

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