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CAYLEY FUNCTIONS

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An indexed function λ_a on a set S may be viewed as a partial groupoid $a \times S \to S$. In this sense the structure of a semigroup is the indexed composite of the structures of its inner translations. All the possible algebraic (orbit) structures of these functions are fully described, and those elements which may serve as index for a potential inner translation are specified.

A considerable part of group theory has developed historically through the characterization of inner translations as regular permutations[1]. The problem of characterizing inner translations of semigroups was raised by Schein [6] and solved by Goralčík and Hedrlín [2],[3],[4]. An alternative, and possibly simpler, development of the subject might still be of interest. It is the purpose of the present note to outline such an alternative development. A preliminary account of these investigations was presented at the Seventh International Symposium on Functional Equations, September 1969 (see [8]).

In the sequel f will denote a function mapping a non-empty set S into itself, i.e., Ran f $\subseteq$ Dom f = S. For any positive integer n, f^n denotes the n^{th} iterate of f. By f^o we mean the identity function on S, so that $f^o x = x$ for every $x \in S$. Suppose there is an integer s such that

1) A function f is an inner translation of a group on S if and only if f is a regular permutation; i.e., Ran f = Dom f = S, f is one-one and $f^m a = a$ for some $a \in S$ implies $f^m x = x$ for every $x \in S$. The proof for the "only if" part of the statement can be found e.g. in Carmichael [1] for finite groups; the first complete proof is apparently due to Schein [6].

Ran f^{n+1} = Ran f^n if and only if $n \geqslant s$; then s will be called the (range)-stabilizer of f. If f has a stabilizer s, then by P2 [2] the set

$$B_f = \{b \in \text{Dom } f \mid f^n b \in \text{Ran } f^s \text{ if and only if } n \geqslant s-1\} \quad (1)$$

is non-empty. Clearly Ran f^s = Ran f^{s+1} implies Ran f^s = Ran f^{s+c} for every $c \geqslant 0$.

In the following three theorems we characterize all functions f which are inner translations of some semigroup on S; we call such functions <u>Cayley</u> <u>functions</u>.[3]

<u>THEOREM</u> <u>1</u>. A function f with no stabilizer is a Cayley function if and only if there exists an element $a \in S$ such that for every $n \geqslant 0$, $f^n a \notin$ Ran f^{n+1}.

<u>THEOREM</u> <u>2</u>. Let f be a function with stabilizer s and such that $f \mid$Ran f^s is one-one. Then f is a Cayley function if and only if there is a $b_1 \in B_f$ such that $f^m b_1 = f^n b_1$ implies $f^{m+1} = f^{n+1}$.

<u>THEOREM</u> <u>3</u>. Let f be a function with stabilizer s and such that $f \mid$Ran f^s is not one-one. Then f is a Cayley function if and only if there is a $b_1 \in B_f$ such that $f^m b_1 = f^n b_1$ implies $m = n$, and f has full branches at $f^n b_1$ for every $n \geqslant s-1$.

The last condition in Theorem 3 uses the following definition: Let f be a function, a an element of S and n a positive integer. Then f has a full branch at $f^n a$ if there is a sequence x_m $(m = n, n-1, \ldots 1, 0, -1, \ldots)$ of <u>distinct</u> elements of S such that $fx_m = x_{m+1}$ for $m < n$, $fx_n = f^n a$ and $x_n \neq f^{n-1} a$. The x_m's themselves are <u>elements</u> of the full branch.

It will be useful to introduce a notation which allows us to read the same equation either as involving

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2) Some properties of functions which will be used throughout the paper are gathered in the appendix and labeled P1, P2, ...

3) Perhaps "Cayley-Suschkewitsch" functions would be more appropriate, as Suschkewitsch was the first to generalize Cayley's theorem to semigroups [7] .

binary operations (two-place functions) or as involving one-place functions, e.g. inner translations. We use Łukasiewicz notations; in particular then, the associativity equation $x\cdot(y\cdot z) = (x\cdot y)\cdot z$, or $F(x,F(y,z)) = F(F(x,y),z)$, will be written as:

$$FxFyz = FFxyz. \tag{2}$$

Moreover, if F is a binary operation on a set S, and $a \in S$, we define Fa to be the one-place function on S such that $(Fa)x = Fax$ for every $x \in S$. Then (2) may be read as $(Fx)(Fy)z = (F(Fxy))z$ for every $z \in S$, or

$$FxFy = FFxy \tag{3}$$

which is the regular representation theorem ("Cayley's theorem") for semigroups. It follows that a function f is a Cayley function if and only if there exists an element $a \in \mathrm{Dom}\, f = S$ and a semigroup-operation H on S (briefly a semigroup H) such that $f = Ha$. Some additional direct consequences of equation (3) are given by:

LEMMA 1. Let $f = Ha$, where H is a semigroup. Then

$$f^nHx = (Ha)^nHx = H(Ha)^nx = Hf^nx,$$

and in particular

$$Hf^{n-1}a = f^{n-1}Ha = f^{n-1}f = f^n = (Ha)^n.$$

If we set $f = Ha_1$ and $f^{n-1}a_1 = a_n$ for all $n > 0$, then $Ha_n = Hf^{n-1}a_1 = f^n = (Ha_1)^n$.

LEMMA 2. Let $f = Ha$ for some semigroup H. Then $f^ma = f^na$ for any $m,n \geqslant 0$ implies $f^{m+1} = f^{n+1}$.

PROOF. Let $f = Ha$. Since $f^ma = f^na$ implies $Hf^ma = Hf^na$ we have $f^{m+1} = f^{n+1}$ by virtue of Lemma 1.

LEMMA 3. Let f be a Cayley function and let $b,x \in S$ be such that $f^{m+t}x = f^mb$ for some $m \geqslant 0$, $t \geqslant 2$. Then $f = Hb$ only if $\mathrm{Ran}\, f^{m+2} = \mathrm{Ran}\, f^{m+1}$.

PROOF. Let $f = Hb$ and let $f^{m+t}x = f^mb$. Hence $Hf^{m+t}x = Hf^mb$. By Lemma 1, we have $f^{m+t}Hx = f^{m+1}$, whence $\mathrm{Ran}\, f^{m+1} \subseteq \mathrm{Ran}\, f^{m+t}$ by P1. Now if $t \geqslant 2$ P1 also yields $\mathrm{Ran}\, f^{m+t} \subseteq \mathrm{Ran}\, f^{m+1}$, whence $\mathrm{Ran}\, f^{m+2} = \mathrm{Ran}\, f^{m+1}$.

For x,y in Dom f, we consider the set N(x,y) of pairs (r,q) of non-negative integers such that $f^rx = f^qy$. If N(x,y) is non-empty, then x and y are said to be in the same f-orbit; and in that case we define δxy via:

$$\delta xy = \psi xy - \tau xy, \tag{4}$$

where $\psi xy = \min\{r \mid (r,q) \in N(x,y)\}$ and $\tau xy = \min\{q \mid f^{\psi xy}x = f^q y\}$.

PROOF OF THEOREM 1: Let f have no stabilizer, i.e., Ran $f^{n+1} \subsetneqq$ Ran f^n for every $n \geqslant 0$. Hence $f^m \neq f^n$ whenever m and n are distinct. If $f = Ha'$, we need by Lemma 2 that $f^m a' \neq f^n a'$ for distinct m and n, i.e., $f^m a' = f^n a'$ implies $m = n$. By Lemma 3, if $f^{m+t}x = f^m a'$ for some x, then $t \leqslant 1$. If $t < 1$ for any such x then we chose $a = a'$. If there exists x such that $f^{m+1}x = f^m a'$, then we chose a to be one such x. Clearly in either case $f^m a = f^n a$ implies $m = n$. Now $f^n a \in$ Ran f^{n+1} implies there exists an element y such that $f^{n+1}y = f^n a$, which is impossible since a was chosen so that $f^m x = f^n a$ implies $m \leqslant n$. This proves the necessity of the conditions of Theorem 1.

Now suppose that there is an element $a \in S$, which satisfies the conditions. We construct a groupoid H by: For every $y \in S$

$$Hxy = \begin{cases} f^{\delta ax+1}y & \text{if x and a are in the same f-orbit} \\ x, & \text{otherwise.} \end{cases}$$

Clearly, the way a was chosen assures that $\delta ax \geqslant 0$, whence $f^{\delta ax+1}$ is well-defined. Straightforward computation will show that H is a semigroup with $f = Ha$. Q.E.D.

LEMMA 4. Let $f = Hb$ where H is a semigroup. If f has stabilizer s, then $f^{s-2}b \notin$ Ran f^s.

PROOF. Suppose there exists an element $x \in S$ such that $f^{s-2}b = f^s x$. By Lemma 3 this implies that Ran f^{s-1} = Ran f^{s+1}, and hence by P1 that Ran f^{s-1} = Ran f^s, which contradicts the fact that the range-stabilizer of f is s.

Note the two possibilities: 1) $f^{s-1}b \in$ Ran f^s whence $b \in B_f$ [cf.(1)]. In this case we define b_s to be $f^{s-1}b$, 2) $f^{s-1}b \notin$ Ran f^s but $f^{s-1}fb \in$ Ran f^s. In this case we define b_s to be $f^s b$. We denote in this case fb (which is in B_f) by b_1, so that $b_s = f^{s-1}b_1$, and $Hb_n = f^n$ (Lemma 1).

In general if $x_1 \in S$ then we denote $f^n x_1$ by x_{n+1}.

LEMMA 5. Let f,H,b be as in Lemma 4. Then for every $n \geqslant 1$ there exists a sequence of elements $a_{s-n} \in$ Ran f^s

such that $fa_{s-n} = a_{s-n+1}$ and $fa_{s-1} = b_s$.

<u>PROOF</u>. By P2 $\mathrm{Ran}(f|\mathrm{Ran}\ f^s) = \mathrm{Dom}\ (f|\mathrm{Ran}\ f^s)$, whence by P3 there exist elements $x_n \in \mathrm{Ran}\ f^s$ such that $f^n x_n = b_s$ for every $n > 0$. A sequence $\{a_{s-n}\}$ may then be chosen, using Axiom of Choice if necessary. Once the element a_{s-1} has been chosen, we extend the sequence $\{a_m\}$ to all $m \geqslant 0$ by defining $a_{s+n} = f^{n+1}a_{s-1}$ for $n \geqslant 0$. Note that this implies that $a_m = b_m$ for $m \geqslant s$.

<u>PROOF OF THEOREM 2</u>. If $f = Hb$, then Lemma 4 establishes the existence of an element $b \in S$ such that $f^{s-2}b \notin \mathrm{Ran}\ f^s$, while Lemma 2 shows that $f^m b = f^n b$ implies $f^{m+1} = f^{n+1}$. If $b \in B_f$ we chose b to be the element b_1 in the statement of the theorem. If $b \notin B_f$ we chose fb as b_1. In either case b_1 clearly satisfies the conditions of the theorem. This proves the necessity.

To prove sufficiency, let f, b_1 be as in the statement of the theorem. Then for every $x \in S$ there exists a smallest integer m $(0 \leqslant m \leqslant s)$ such that $f^m x \in \mathrm{Ran}\ f^s$, and a <u>unique</u> element px - $\mathrm{Ran}\ f^s$ (since $f\ \mathrm{Ran}\ f^s$ is one-one) such that $f^m px = f^m x$. This defines a function p with $\mathrm{Dom}\ p = S$, $\mathrm{Ran}\ p = \mathrm{Ran}\ f^s$, $p^2 = p$, and

$$p|\mathrm{Ran}\ p = p^o|\mathrm{Ran}\ p = f^o|\mathrm{Ran}\ f^s.$$

Hence $p(f|\mathrm{Ran}\ f^s) = f|\mathrm{Ran}\ f^s$.

Since $f|\mathrm{Ran}\ f^s$ is not only one-one, but a permutation (by P2), the iterates $(f|\mathrm{Ran}\ f^s)^n$ are well defined for <u>all</u> integers n. It follows that $(f|\mathrm{Ran}\ f^s)^m p(f|\mathrm{Ran}\ f^s)^n p = (f|\mathrm{Ran}\ f^s)^{m+n}p$ for all integers m,n. Clearly $px = y$ implies $f^s x = f^s y$, whence by P4 $f^s p = f^s$. And since $f^s p = (f^s|\mathrm{Ran}\ p)p = (f^s|\mathrm{Ran}\ f^s)p = (f|\mathrm{Ran}\ f^s)^s p$, it follows that $f^s = (f|\mathrm{Ran}\ f^s)^s p$.

For $x \in S$ let $dx = \delta b_1 x + 1$ [see (4)]. Define a groupoid H on S by:

For every $y \in S$

$$Hxy = \begin{cases} f^{dx}y & \text{if } \delta b_1 x < s-1, \\ (f|\mathrm{Ran}\ f^s)^{dx}py & \text{if } \delta b_1 x \geqslant s-1, \\ x & \text{if } x \text{ and } b_1 \text{ are not in the same } f\text{-orbit.} \end{cases}$$

Making use of the relations between the functions f and p

indicated above one easily establishes that for a_n (as defined in Lemma 5) $Ha_n = (f|\text{Ran } f^s)^n p$, $Hb_n = f^n$ and in particular $Hb_1 = f$. Straightforward computation will show that H is a semigroup, which proves our theorem. An alternate simpler method to prove that H is a semigroup is outlined at the end of the proof for Theorem 3.

PROOF OF THEOREM 3. If $f = Hb$ for some semigroup H and some $b \in S$ then exactly as in the proof of Theorem 2, we notice that either b or $fb \in B_f$. We chose b_1 to be whichever of b,fb is in B_f, and set $f^n b_1 = b_{n+1}$. By Lemmas 4 and 5 there exists a sequence $\{a_n\}$ such that $fa_n = a_{n+1}$ for every integer n, and $a_n = b_n$ for every $n \geq s$.

But now with $f|\text{Ran } f^s$ assumed to be not one-one, we can show that f must have full branches at b_n for every $n \geq s$. To do this we first show that $Ha_{-1}b_n \neq b_{n-1}$ for every $n \geq 1$. We consider first the case $b \in B_f$. Then $f = Hb_1$. Suppose $Ha_{-1}b_n = b_{n-1}$. This would imply

$$HHa_{-1}b_n = Ha_{-1}Hb_n = Ha_{-1}f^n = (Ha_{-1}f)f^{n-1} = Hb_{n-1} = f^{n-1}.$$

Hence by P4, $(Ha_{-1}f)|\text{Ran } f^{n-1} = Ha_{-1}(f|\text{Ran } f^{n-1}) = f^o|\text{Ran } f^{n-1}$, which is impossible since $f|\text{Ran } f^{n-1}$, being an extension of $f|\text{Ran } f^s$, is not one-one. Therefore, $Ha_{-1}b_n \neq b_{n-1}$. This shows that $Ha_{-1} \neq Hb_m$ for any positive integer m, whence $a_{-1} \neq b_m$, and the a_n are distinct from any b_m and from each other. Hence the terms of the sequence $\{a_n | n \leq s-1\}$ are the elements of a full branch of f at b_s.

Now $f^{s+1}Ha_{-1}b_n = Hf^{s+1}a_{-1}b_n = Hb_s b_n = f^s b_n = b_{n+s}$, whence $\tau b_1 Ha_{-1}b_n$ (briefly, tn) exists, and $0 < tn \leq s+1$, for every $n > 1$. Therefore the terms of the sequence $\{Ha_m b_i | m < ti-1\}$ for any given $i \geq 1$ form a full branch of f at b_{i+ti-1}. Note that $(i+ti-1)$ cannot be smaller than s, since the existence of a full branch of f at b_1 for $i < s$ will cause b_1 not to be an element of B_f. Therefore $t1 = s$ or $t1 = s+1$, i.e., $Ha_{-1}b_1$ is an element of a full branch of f at either b_s or b_{s+1}. To assure the existence of a full branch of f at b_n for every $n > s$ it is necessary that the equation $n = i + ti - 1$ is solvable for every $n > s$. This follows without difficulty once we show that $t(i+1) \leqslant ti$ for every $i \geqslant 1$, which we do as follows:

From $f^{ti}Ha_{-1}b_i = b_{i+ti-1}$, we have

$$Hf^{ti}Ha_{-1}b_i = f^{ti}Ha_{-1}Hb_i = f^{ti}Ha_{-1}f^i = Hb_{i+ti-1} = f^{i+ti-1}.$$

Hence $f^{ti}Ha_{-1}b_{i+1} = f^{ti}Ha_{-1}f^ib_1 = f^{i+ti-1}b_1 = b_{i+ti}$, whence $t(i+1) \leqslant ti$.

Finally we note that the existence of full branches of f implies that for distinct m and n $f^n \neq f^m$. Hence the necessity of the condition $f^mb_1 = f^nb_1$ implies $m = n$ follows from Lemma 2. This completes the proof for the case $b \in B_f$; and the proof for the case $fb \in B_f$ is entirely similar.

PROOF OF SUFFICIENCY. Let f, b_1 be as in the statement of the theorem. Then the elements a_n, $n < s$ are the elements of a branch of f at b_s; and for every $i \geqslant 1$, we pick one full branch at b_{s+i} and label its element a_{mi}, $m < s$ so that $fa_{mi} = a_{m+1,i}$ for $m < s-1$, and $fa_{s-1,i} = a_{s+i}$. Now by the Axiom of Choice there exists a function g such that Dom g = Ran f, $gx = y$ implies $fy = x$, and which has the following properties:

$$\mathrm{Ran}(g \mid \mathrm{Ran}\, f^s) \subseteq \mathrm{Ran}\, f^s,$$

$$ga_{mi} = a_{m-1,i},$$

$$gb_n = \begin{cases} b_{n-1} & \text{if } n < s, \\ a_{s-1,n-s} & \text{if } n \geqslant s, \end{cases}$$

$$ga_n = a_{n-1} \qquad \text{for } n < s.$$

Let $dx = \delta b_1 x + 1$ [see (4)]. Define a function p on S by:

$$px = \begin{cases} a_{o,dx} & \text{if } \tau b_1 x < s-1. \\ g^q f^q & \text{for any other } x \in S, \text{ where } q > 0 \text{ is the smallest integer such that } f^q x \in \mathrm{Ran}\, f^s. \end{cases}$$

Note that for f as in the theorem $\tau b_1 x < s-1$ implies $dx \geqslant 1$, so that $a_{0,dx}$ is well defined.

We define a groupoid H by: For every $y \in S$

$$Hxy = \begin{cases} f^{dx}y & \text{if } \psi b_1 x < s-1, \\ f^{dx}py & \text{if } \psi b_1 x = s-1 \text{ and } dx \geqslant 0, \\ g^{-dx}y & \text{if } \psi b_1 x = s-1 \text{ and } dx \leqslant 0, \\ Ha_{s-t}f^{c-s} & \text{if } \psi b_o x > s-1, \text{ where } t = \tau b_1 x \\ & \qquad \text{and } c = \psi b_1 x + 1. \\ x & \text{if } x \text{ and } b_1 \text{ are not in the same} \\ & \qquad f\text{-orbit.} \end{cases}$$

$Ha_m b_i = a_{mi}$ for every $m < s$, and every b_i.

Essentially, H has been defined by specifying Hx for every $x \in S$. Note that although Dom $g^n \subsetneq$ Dom f, we have Dom $g^n p$ = Dom p = S; this is because Ran p $\subseteq$ Ran $f^s \subseteq$ Ran $f^n \subseteq$ Dom g^n.

From the above definition we have $Ha_n = f^n p$ if $n \geqslant 0$ and $Ha_n = g^{-n}p$ if $n \leqslant 0$; $Hb_n = f^n$ and in particular $Hb_1 = f$. It remains to prove that H is a semigroup. It would be possible but very laborious to prove this directly. It is much simpler to prove the equivalent condition (Hx)(Hy) = H(Hxy). To do this we require the following easily verified relations between f,g,p and k_x where k_x is the constant function with value x.

R1. $f^n p = f^n$ if and only if $n \geqslant s$.

R2. $pf^n p = pf^n$.

R3. $pg^n p = g^n p$.

R4. $f^n g^m p = \begin{cases} f^{n-m}p & \text{if } n-m \geqslant 0 \\ g^{m-n}p & \text{if } n-m \leqslant 0. \end{cases}$

R5. $Hyk_x = k_{Hyx}$.

We shall outline part of the argument (the remainder is similar). From the definition of a_{mi} it follows that $\tau b_1 a_{mi} = s-m$, $\psi b_1 a_{mi} = s+i-1$, whence $Ha_{mi} = Ha_m f^i$ $= Ha_m Hb_i = HHa_m b_i$.

Let $n < s$. Then applying R1., resp. R2, we have:

$$Hb_{s-n}Ha_n = f^{s-n}f^n p = f^s p = f^s = Hb_s = Hf^{s-n}a_n = HHb_{s-n}a_n.$$

$$Ha_{s-n}Ha_n = f^{s-n}pf^n p = f^{s-n}pf^n = Ha_{s-n,n} = Hf^{s-n}a_{on} =$$
$$= Hf^{s-n}pa_n = HHa_{s-n}a_n.$$

Let m+n-u > 0 > n-u. Then

$$\begin{aligned} Ha_{mn}Ha_{-uv} &= f^m pf^n g^u pf^v \\ &= f^m pg^{u-n} pf^v \quad \text{(using R4)} \\ &= f^m g^{u-n} pf^v \quad \text{(R3)} \\ &= f^{m+n-u} pf^v \quad \text{(R4)} \end{aligned}$$

$= Ha_{m+n-u,v} = Hf^m a_{n-u,v} = Hf^m pa_{n-u,v} = Hf^m pf^n a_{-uv} = HHa_{mn}a_{-uv}$.

Finally we note that, if x is not in the same f-orbit with b_1, neither is $fx = Hb_1x$; we have,

$$Hb_1Hx = Hb_1k_x = k_{Hb_1x} = HHb_1x \quad \text{(R5)}.$$

Theorem 3 immediately yields the following corollary, which is a generalization of Theorems III 6.8, 6.9, 6.11 in [4].

<u>THEOREM 4</u>. The semigroup H has a subsemigroup H' with a magnifying element a, i.e., a is an element such that Ran H'a = Dom H'a, and H'a is not one-one, if and only if (i) there exists an element a_{-1} such that $HaHaa_{-1} = a$ (i.e., Haa_{-1} is a right identity for the particular element a), (ii) $Ha_{-1}a \neq Haa_{-1}$, and (iii) the subsemigroups generated by a and a_{-1} are each infinite. If (i), (ii), and (iii) hold then $H' = H|(\text{Ran } Ha)^2$.

APPENDIX

In the following, f,g,h are arbitrary functions from some non-empty set S into S.

P1. Ran fg = Ran(f|Ran g) $\subseteq$ Ran f. In particular, Ran $f^{n+c} \subseteq$ Ran f^n, $c \geqslant 0$.

P2. A function f has stabilizer s > 0 if and only if

1) Dom(f|Ran f^s) = Ran(f|Ran f^s) = Ran f^s.

2) There exists an element b $\in$ Ran f such that $f^n b \notin$ Ran f^{n+1} for $0 < n < s$.

P3. Let Ran f = S. Then for every x $\in$ S, and for every positive integer n, there exists an element x_{-n} such that $f^n x_{-n} = x$.

P4. fh = h if and only if f|Ran h = h^o|Ran h, where h^o is the identity function on S; fh = f if and only if hx = y implies fx = fy for all x,y $\in$ S.

REFERENCES

1. Carmichael, R. D., Introduction to the theory of groups of finite order, Ginn & Co., Boston, 1937.
2. Goralčik, P. and Z. Hedrlín, Translations of semigroups I. Periodic and quasiperiodic transformations, (Russian) Mat. Časopis Sloven. Akad.Vied 18 (1968), 161-176. (MR 39, #340)
3. ______________________, Translations of semigroups II. Surjective transformations, (Russian)ibid. 263-272. (MR 40, #7371)
4. Goralčik, P., Translations of semigroups. III. Transformations with expanding or irregular surjective part, (Russian) ibid., 273-282. (MR 40, #7372)
5. Ljapin, E. S., Semigroups (Gosudarstv. Izdat. Fiz.-Mat.Lit., Moscow 1960, Translations of Mathematical Monographs, 3, Amer. Math. Soc., Providence, R.I. 1963).
6. Schein, B. M., On Translations in semigroups and groups, (Russian) Volžskiĭ Matem. Sbornik, 2 (1964), 163-169. (MR 34, #7691)
7. Suschkewitsch, A. K., Über die Darstellung der eindeutig nicht umkehrbaren Gruppen mittels der verallgemeinerten Substitutionen, Matem. Sbornik, 33 (1926), 371-374.
8. Zupnik, D., Cayley functions, Aequat. Math. 4, fasc. 1/2, (1970), 227-228.

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