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Gorenstein injective and projective modules

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0 Introduction

Auslander and Reiten in ([3], 1991) study the notion of right and left $\mathscr{F}$-approximations and of minimal right and left $\mathscr{F}$-approximations (with $\mathscr{F}$ some class of modules). The original ideas for these notions go back to Auslander, Smalø ([4], 1980). In Enochs, ([11], 1981) these approximations were also defined but there they were called $\mathscr{F}$-precovers, $\mathscr{F}$-preenvelopes, $\mathscr{F}$-covers and F-envelopes respectively.

In this paper we will follow Enochs' terminology. When every module has an $\mathscr{F}$-precover for some class of module $\mathscr{F}$, we can use the machinery of relative homological algebra developed by Eilenberg and Moore in [8]. If R is left noetherian and $\mathscr{E}$ is the class of injective left R-modules, then every left R-modules admits an $\mathscr{E}$-cover (i.e. an injective cover) [11]. We use this fact to define left derived functors of Hom (instead of the usual right derived functors Ext^n) and denote these functor Ext_n as in ([14], Enochs, Jenda).

We will use these extension functors to characterize Gorenstein injective modules and to investigate their basic properties.

Gorenstein injective modules are rarely finitely generated but can be what we call mock finitely generated. The minimal injective resolutions of these mock finitely generated modules have properties similar to those of finitely generated modules. In fact, many of Bass' arguments in [5] concerning minimal injective resolutions carry over verbatim to this situation. We argue that over a commutative Gorenstein ring there is a natural way to define extended Bass invariants with negative indices. In section 6 we argue that if the module is furthermore finitely generated then all of these invariants are finite.

We also study finitely generated Gorenstein projective modules and find a new way (other than taking duals or syzygies) of generating indecomposable Gorenstein projective modules from other such modules. This involves taking orthogonal complements in a free R-module with an inner product. We note that for a commutative, local Gorenstein ring, the finitely generated Gorenstein projective modules are just the maximal Cohen–Macaulay modules, so our results can be applied to their study.

In the last section we consider the question of the existence of Gorenstein injective preenvelopes of modules. We show that when R is Gorenstein, these always exist. When we consider the stable category of modules (with two linear maps equivalent if their difference can be factored through an injective module) we show that every module has a reduced Gorenstein injective envelope (in the stable sense) but which is surprisingly unique up to isomorphism without passing to the stable category.

We note that an important class of Gorenstein rings are the integral group rings $\mathbb{Z}G$ where G is a finite group. The Gorenstein injective ZG-modules are those which are divisible as Z-modules and the finitely generated Gorenstein projective modules are the lattices, i.e. those modules which are free with a finite base as Z-modules. Hence the notions of Gorenstein injective and projective modules have some relevance to the theory of modular representations of finite groups.

1 Left derived extension functors

In this section we consider one of the methods for computing left derived extension functors (or extension functors with negative indices). These functors will be used in the sequel to characterize and to study Gorenstein injective modules.

Definition 1.1 *Let $\mathscr{F}$ be a class of left R-modules for some ring R. If $\phi : F \to M$ is linear where $F \in \mathscr{F}$ and M is a left R-module, then $\phi : F \to M$ is called an $\mathscr{F}$-precover of M if* $\mathrm{Hom}(G,F) \to \mathrm{Hom}(G,M) \to 0$ *is exact for all $G \in \mathscr{F}$. If, moreover, whenever $f : F \to F$ is linear and such that $\phi \circ f = \phi$, f is an automorphism of F, then $\phi : F \to M$ is called an $\mathscr{F}$-cover of M.*

If $\mathscr{F}$ is some known class of modules, for example, the class of flat modules, then an $\mathscr{F}$-(pre)cover is called a flat (pre)cover.

$\mathscr{F}$-preenvelopes and $\mathscr{F}$-envelopes are defined dually. We follow the same conventions with their terminology. We recall

Theorem 1.2 *For a ring R, every left R-module has an injective cover if and only if R is left noetherian.* (*see* [11], *Theorem* 2.1.)

Remarks. If R is left hereditary and M a left R-module, then $E \to M$ with $E \subset M$ the largest injective submodule of M is an injective cover of M. In [7] there are some less trivial examples.

It is easy to see that the injective cover of any left R-module M is unique up to isomorphism. If $\phi : E \to M$ is an injective cover and K is an injective module, then any linear map $E \oplus K \to M$ agreeing with ϕ on E is an injective precover. Conversely if $\phi' : E' \to M$ is an injective precover and $f : E \to E'$, $g : E' \to E$ are such that $\phi' \circ f = \phi$, $\phi \circ g = \phi'$, then $\phi \circ g \circ f = \phi$ so $g \circ f$ is an automorphism of E. Hence $K = \ker(g)$ is an (injective) summand of E' and $E' = f(E) \oplus K$ is such that $\phi'|f(E)$ is an injective cover of M. Consequently, an injective precover $\phi' : E' \to M$ is an injective cover if and only if $\ker(\phi')$ contains no non-zero summands of E'.

Definition 1.3 *If N is a left R-module then a complex*

$$\cdots \to E_1 \to E_0 \to N \to 0$$

is called an injective resolvent of N if each E_i is an injective left R-module and if for any injective left R-module E, the functor $\mathrm{Hom}(E, -)$ *leaves the sequence exact.*

We note that a complex as above is an injective resolvent if and only if $E_0 \to N$, $E_1 \to \ker(E_0 \to N)$ and $E_i \to \ker(E_{i-1} \to E_{i-2})$ for $i \geqq 2$ are injective precovers. If all these maps are injective covers then we say that the complex is a minimal injective resolvent of N. Then noting that a minimal injective resolvent is unique up to isomorphism, we denote E_i in such a complex by $E_i(N)$ (recalling that $E^i(N)$ is the i-th term in a minimal injective resolution of N).

The usual argument shows that an injective resolvent of a left R-module N is unique up to homotopy and so can be used to compute derived functors.

If M and N are left R-modules and $0 \to M \to E^0 \to E^1 \to \cdots$ is an injective resolution and $\cdots \to E_1 \to E_0 \to N \to 0$ is an injective resolvent, then the 3rd quadrant double complex $(\mathrm{Hom}(E^i, E_j))_{i,j}$ is such that the two associated spectral sequences collapse. This implies that we can compute derived functors of Hom using either the injective resolution of M or the injective resolvent of N when it exists. These left derived functors will be denoted $\mathrm{Ext}_n^R(M,N)$ or simply $\mathrm{Ext}_n(M,N)$ (instead of the usual $\mathrm{Ext}_R^n(M,N)$).

From the definition of $\mathrm{Ext}_0(M,N)$ it is clear that there is a natural transformation $\mathrm{Ext}_0(M,N) \to \mathrm{Ext}^0(M,N) = \mathrm{Hom}(M,N)$. The image of $\mathrm{Ext}_0(M,N)$ in $\mathrm{Hom}(M,N)$ consists of those linear maps $M \to N$ which can be factored through an injective left R-module.

We will let $\overline{\mathrm{Ext}}_0(M,N)$ and $\overline{\mathrm{Ext}}^0(M,N)$ denote the kernel and cokernel of the natural transformation $\mathrm{Ext}_0(M,N) \to \mathrm{Ext}^0(M,N)$.

If E is an injective left R-module then for any left R-module M, $\mathrm{Ext}^i(M,E) = 0$ for $i \geqq 1$ and $\mathrm{Ext}^0(M,E) = \mathrm{Hom}(M,E)$. From the definition of the groups $\mathrm{Ext}_i(M,E)$ it is also easy to see that $\mathrm{Ext}_i(M,E) = 0$ for $i \geqq 1$ and that $\mathrm{Ext}_0(M,E) = \mathrm{Hom}(M,E)$. Since in this case the natural map $\mathrm{Ext}_0(M,E) \to \mathrm{Ext}^0(M,E)$ is the identity we also get $\overline{\mathrm{Ext}}^0(M,E) = \overline{\mathrm{Ext}}_0(M,E) = 0$.

The groups $\overline{\mathrm{Ext}}_0(M,N)$ have occurred in the literature. They are what Hilton [17] calls the i-homotopy (i for injective) groups and which he denotes $\bar{\pi}(M,N)$. Hilton also defines higher homotopy groups $\bar{\pi}_n(M,N)$ for $n \geqq 1$. From his definition, it is not hard to see that $\bar{\pi}_1(M,N) \cong \overline{\mathrm{Ext}}_0(M,N)$ and that $\bar{\pi}_n(M,N) \cong \mathrm{Ext}_{n-1}(M,N)$ for $n \geqq 2$. However, at the time of Hilton's definitions, injective covers were not yet available and so he did not have the alternate method for computing these groups (i.e. using injective resolvents).

Remark. With a sign changing trick in mind, the functors Ext_n are also called negative extension functors. However, we note that Ext_0 and $\mathrm{Ext}^{-0} = \mathrm{Ext}^0 = \mathrm{Hom}$ are not isomorphic functors in general.

We will now state the basic properties of the functors Ext_n.

Proposition 1.4 *If $0 \to M' \to M \to M'' \to 0$ is an exact sequences of left R-modules then for any left R-module N there is a long exact sequence*

$$\begin{aligned} \cdots &\to \mathrm{Ext}_1(M'',N) \to \mathrm{Ext}_1(M,N) \to \mathrm{Ext}_1(M',N) \\ &\to \mathrm{Ext}_0(M'',N) \to \mathrm{Ext}_0(M,N) \to \mathrm{Ext}_0(M',N) \to 0 \end{aligned}$$

Proof. If $\cdots E_1 \to E_0 \to N \to 0$ is an injective resolvent of N, let $E_\bullet$ be the complex $\cdots \to E_1 \to E_0 \to 0$ then $0 \to \mathrm{Hom}(M'',E_\bullet) \to \mathrm{Hom}(M,E_\bullet) \to \mathrm{Hom}(M',E_\bullet) \to 0$ is an exact sequence of complexes. The associated long exact homology sequence is the desired sequence. □

We note that the long exact sequence has the obvious naturality properties.

Proposition 1.5 *If $0 \to N' \to N \to N'' \to 0$ is a complex of left R-modules such that for any injective left R-module E,*

$$0 \to \mathrm{Hom}(E,N') \to \mathrm{Hom}(E,N) \to \mathrm{Hom}(E,N'') \to 0$$

is exact, then for any left R-module M there is a long exact sequence

$$\begin{aligned} \cdots &\to \mathrm{Ext}_1(M,N') \to \mathrm{Ext}_1(M,N) \\ &\to \mathrm{Ext}_1(M,N'') \to \mathrm{Ext}_0(M,N') \to \mathrm{Ext}_0(M,N) \\ &\to \mathrm{Ext}_0(M,N'') \to 0\,. \end{aligned}$$

Proof. If $0 \to M \to E^0 \to E^1 \to \cdots$ is an injective resolution of M and $E^\bullet$ is the complex $0 \to E^0 \to E^1 \to E^1 \to E^2 \to \cdots$ then we have the exact sequence

$$0 \to \mathrm{Hom}(E^\bullet,N') \to \mathrm{Hom}(E^\bullet,N) \to \mathrm{Hom}(E^\bullet,N'') \to 0$$

of complexes. Taking the associated long exact sequence of homologies, the claim follows. □

Again we note that we have the obvious naturality in the above.

Proposition 1.6 *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of left R-modules, then for any left R-module N there is a long exact sequence*

$$\cdots \ \mathrm{Ext}_1(M',N) \to \overline{\mathrm{Ext}}_0(M'',N) \to \overline{\mathrm{Ext}}_0(M,N)$$

$$\to \overline{\mathrm{Ext}}_0(M',N) \to \overline{\mathrm{Ext}}^0(M'',N) \to \overline{\mathrm{Ext}}^0(M,N)$$

$$\to \overline{\mathrm{Ext}}^0(M',N) \to \mathrm{Ext}^1(M'',N) \to \cdots$$

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc} \cdots \to \mathrm{Ext}_1(M',N) & \to & \mathrm{Ext}_0(M'',N) & \to & \mathrm{Ext}_0(M,N) & \to & \mathrm{Ext}_0(M',N) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{Ext}^0(M'',N) & \to & \mathrm{Ext}^0(M,N) & \to & \mathrm{Ext}^0(M',N) & \to & \mathrm{Ext}^1(M'',N) \to \cdots \end{array}$$

with exact rows. Chasing this diagram, the result follows. □

Definition 1.7 *The sequence above will be called the extended long exact sequence of extension functors.*

Definition 1.8 *For a left R-module N, if* $0 \to N \to E^0(N) \to E^1(N) \to \cdots$ *is a minimal injective resolution of N, and*

$$\cdots \to E_1(N) \to E_0(N) \to N \to 0$$

is a minimal injective resolvent, then the complex

$$\cdots \to E_1(N) \to E_0(N) \to E^0(N) \to E^1(N) \to \cdots$$

(with $E_0(N) \to E^0(N)$ *the composition* $E_0(N) \to N \to E^0(N)$*) is called a complete minimal injective resolution of N.*

Proposition 1.9 *If M is a left R-module and* $\mathrm{Hom}(M,-)$ *is applied to a complete minimal injective resolution of the left R-module N, then the homology groups are* $\mathrm{Ext}_i(M,N)$, $\overline{\mathrm{Ext}}_0(M,N)$, $\overline{\mathrm{Ext}}^0(M,N)$ *and* $\mathrm{Ext}^i(M,N)$ *at* $\mathrm{Hom}(M,E_i(N))$, $\mathrm{Hom}(M,E_0(N))$, $\mathrm{Hom}(M,E^0(N))$, *and* $\mathrm{Hom}(E,E^i(N))$ *respectively where* $i \geqq 1$.

Proof. This follows from the definitions and from a diagram chasing argument. □

Remark. The homology groups in the Proposition above can also be computed by forming the complex

$$\cdots \to P_1 \to P_0 \to E^0(M) \to E^1(M) \to \cdots$$

from a projective resolution $\cdots \to P_1 \to P_0 \to M \to 0$ of M and the injective resolution

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots,$$

applying the functor $\mathrm{Hom}(-, N)$, and computing homology.

Remark. Some notational difficulties could be avoided by relabeling the complex $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ as $\cdots \to E^{-2} \to E^{-1} \to E^0 \to E^1 \to \cdots$ (This procedure is followed with complete resolutions of Z when computing Tate homology and cohomology (see K. Brown [6])). Then we would get exactly one derived functor of Hom for each integer n (instead of two for $n = 0$). We have avoided this practice since it is not consistent with the established notation for derived functors.

2 Gorenstein injective modules

In this section R will always denote a left noetherian ring.

In [1], Auslander defines the Gorenstein dimension of a module. We call the modules having this dimension 0 the Gorenstein projective modules. Auslander shows that over a commutative Gorenstein local ring R, a finitely generated module M is Gorenstein projective if and only if there is an exact sequence

$$\cdots \to P^{-1} \to P^0 \to P^1 \cdots$$

of finitely generated projective R-modules such that $M = \ker(P^0 \to P^1)$ and such that the dual sequence

$$\cdots \to P^{1*} \to P^{0*} \to P^{-1*} \to \cdots$$

is also exact. We note that this implies that if $\mathrm{Hom}(-, P)$ is applied to the sequence above and if P is a projective module, we still get an exact sequence. This suggests we make the following:

Definition 2.1 *A left R-module N is said to be Gorenstein injective if and only if there is exact sequence*

$$\cdots \to E^{-1} \to E^0 \to E^1 \to \cdots$$

of injective left R-modules such that $N = \ker(E^0 \to E^1)$ and such that for any injective left R-module E, $\mathrm{Hom}(E, -)$ leaves the complex above exact.

We note that if N is Gorenstein injective, then the complex

$$0 \to N \to E^0 \to E^1 \to \cdots$$

is an injective resolution of N and $\cdots \to E^{-2} \to E^{-1} \to N \to 0$ is an injective resolvent of N. Hence we get

Proposition 2.2 *A left R-module N is Gorenstein injective if and only if for any module Q which is projective or injective all of the groups* $\mathrm{Ext}_i(Q,N)$, $\mathrm{Ext}^i(Q,N)$ *for* $i \geqq 1$ *and* $\overline{\mathrm{Ext}}_0(Q,N)$, $\overline{\mathrm{Ext}}^0(Q,N)$ *vanish* (*i.e.* $= 0$).

Proof. If N is Gorenstein injective and if

$$\cdots \to E^{-1} \to E^0 \to E^1 \to \cdots$$

is as in Definition 2.1, and then for any left R-module Q we apply $\mathrm{Hom}(Q,-)$ to the complex we get a complex whose homology groups are the Exts we want to argue vanish if Q is projective or injective. If Q is projective, they vanish since the original complex is exact. If Q is injective, they vanish because of our hypothesis on the original complex.

Conversely suppose N is such that the extension groups in our hypothesis all vanish. Let $0 \to N \to E^0 \to E^1 \to \cdots$ be an injective resolution and $\cdots \to E_1 \to E_0 \to N \to 0$ be an injective resolvent of N. Then $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ is a complex and the extension groups can be computed by applying $\mathrm{Hom}(Q,-)$ and computing homology. If $Q = R$ these groups vanish and so we see the complex is exact. If Q is injective and we apply $\mathrm{Hom}(Q,-)$ we also get an exact sequence. Then noting that $N = \ker(E^0 \to E^1)$ we see that N is Gorenstein injective by the Definition 2.1. □

The argument in the Proposition above also gives

Corollary 2.3 *A left R-module N is Gorenstein injective if and only if its complete injective resolution*

$$\cdots \to E_1(N) \to E_0(N) \to E^0(N) \to E(N) \to \cdots$$

is exact and remains exact whenever $\mathrm{Hom}(E,-)$ *is applied to it for any injective module E.*

Proof. Immediate. □

It is also convenient to restate this result as

Corollary 2.4 *A left R-module N is Gorenstein injective if and only if its minimal injective resolvent*

$$\cdots \to E_1(N) \to E_0(N) \to N \to 0$$

is exact and if $\mathrm{Ext}^i(E,N) = 0$ *for* $i \geqq 1$ *and E any injective left R-module.*

Proof. It is easy to check that these conditions are equivalent to the conditions on the complete injective resolution of N of the previous Corollary. □

We note that Corollary 2.3 and Definition 2.1 say that if N is Gorenstein injective, then each of $\ker(E^i(N) \to E^{i+1}(N))$ for $i \geqq 0$, $\ker(E_{i+1}(N) \to E_i(N))$ for $i \geqq 1$ and $\ker(E_0(N) \to E^1(N))$ is Gorenstein injective. In particular for any such N, $E^0(N)/N$ and $\ker(E_0(N) \to N)$ are Gorenstein injective. Also $E_0(N) \to N$ is surjective.

Proposition 2.4 *If N is a Gorenstein injective left R-module and L is a left R-module having finite injective or projective dimension, then* $\mathrm{Ext}^i(L,N) = \mathrm{Ext}_i(L,N) = 0$ *for* $i \geqq 1$ *and* $\overline{\mathrm{Ext}}^0(L,N) = \overline{\mathrm{Ext}}_0(L,N) = 0$.

Proof. Use induction on the dimension, Proposition 2.2 and the extended long exact sequences. □

For a ring R and R-modules M and N, $\underline{\mathrm{Hom}}_R(M,N)$ (or simply $\underline{\mathrm{Hom}}(M,N)$) denotes the equivalence classes of maps $f : M \to N$ with f and g equivalent if and only if $f - g$ can be factored through an injective module. $\underline{\mathrm{Mod}}$ is the category whose objects are the left R-modules and whose morphisms are these equivalence classes (denoted [f]). Note that $h : M \to N$ can be factored through an injective if and only if it can be factored $M \to E^0(M) \to N$ and if and only if it can be factored $M \to E_0(N) \to N$. Also note that $\underline{\mathrm{Hom}}(M,N)$ is precisely $\overline{\mathrm{Ext}}^0(M,N)$. $\underline{\mathrm{Hom}}(M,M)$ is denoted $\underline{\mathrm{End}}(M)$. We recall that a module N is said to be reduced if it has no non-zero injective submodules.

Theorem 2.5 *Let N be a reduced Gorenstein injective R-module and let $N \subset E$ be an injective envelope. Then $L = E/N$ is a reduced Gorenstein injective module and $E \to N$ is an injective cover. For any $f : N \to N$ let*

$$\begin{array}{ccccccccc} 0 & \to & N & \to & E & \to & L & \to & 0 \\ & & \downarrow f & & \downarrow & & \downarrow g & & \\ 0 & \to & N & \to & E & \to & L & \to & 0 \end{array}$$

be a commutative diagram. Then the map $[f] \mapsto [g]$ from $\underline{\mathrm{End}}\,N$ to $\underline{\mathrm{End}}\,L$ is well-defined and is an isomorphism.

Proof. From earlier remarks we know L is Gorenstein injective. Now let $\bar{E}$ be an injective module. Then

$$\mathrm{Hom}(\bar{E},E) \to \mathrm{Hom}(\bar{E},L) \to \mathrm{Ext}^1(\bar{E},N)$$

is exact. But $\mathrm{Ext}^1(\bar{E},N) = 0$ since N is Gorenstein injective. Hence $E \to L$ is an injective precover. Since $\ker(E \to L) = N$ contains no non-zero injective submodules, it is in fact an injective cover.

The argument that the map $[f] \to [g]$ is a well-defined injective homomorphism is standard. Since $E \to L$ is an injective cover, we easily see that it is also surjective. □

Remark 2.6 From the above we deduce the easy but useful conclusion that $N = 0$ if and only if $L = 0$.

Corollary 2.7 *If N is reduced and Gorenstein injective, the set of $f \in \operatorname{End} N$ such that $[f] = 0$ is a two sided ideal of $\operatorname{End} N$ contained in the Jacobson radical of $\operatorname{End} N$.*

Proof. The set of such f is clearly a two sided ideal. We need to argue that for any such f, $1+f$ is a unit of $\operatorname{End} N$, i.e. is an automorphism of N. Letting $N \subset E$ be an injective envelope and letting $N \to E \xrightarrow{g} N$ be a factorization of f, let $\bar{g} : E \to E$ agree with g. The diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & E & \to & L & \to & 0 \\ & & \downarrow {\scriptstyle 1+f} & & \downarrow {\scriptstyle 1+\bar{g}} & & \downarrow {\scriptstyle 1} & & \\ 0 & \to & N & \to & E & \to & L & \to & 0 \end{array}$$

is then commutative. Since $E \to L$ is an injective cover, $1 + \bar{g}$ is an automorphism of E. Hence $1 + f$ is an automorphism of N. □

We recall that a module N is strongly indecomposable if $\operatorname{End} N$ is a local ring.

Corollary 2.8 *N is the direct sum of n non-zero indecomposable modules if and only if L is too. Moreover N is (strongly) indecomposable if and only if L is (strongly) indecomposable.*

Proof. This results from the isomorphism $\underline{\operatorname{End}}\, N \cong \underline{\operatorname{End}}\, L$, the fact that $\ker(\operatorname{End} N \to \underline{\operatorname{End}}\, N)$ and $\ker(\operatorname{End} L \to \underline{\operatorname{End}})$ are contained in the Jacobson radicals of $\underline{\operatorname{End}}\, N$ and $\underline{\operatorname{End}}\, L$ respectively and the familiar connection between decompositions of a module and idempotents in the endomorphism ring of that module. □

Corollary 2.9 *If M and N are reduced Gorenstein injective left R-modules, a linear map $f : M \to N$ is an isomorphism if and only if $[f]$ is an isomorphism.*

Proof. If f is an isomorphism, then $[f^{-1}] = [f]^{-1}$. Conversely, suppose $[f]$ is an isomorphism and that $[f]^{-1} = [g]$. Then $[g \circ f] = [\mathrm{id}_M]$. Then since the image of $g \circ f$ in $\underline{\operatorname{End}}\, M$ is a unit, $g \circ f$ is a unit in $\underline{\operatorname{End}}\, M$. Similarly $f \circ g$ is a unit in $\underline{\operatorname{End}}\, N$, so f is an automorphism of M. □

Proposition 2.10 *If L is a reduced Gorenstein injective module and $E \to L$ is its injective cover, then $E \to L$ is surjective, $K = \ker(E \to L)$ is reduced and Gorenstein injective and $K \subset E$ is an injective envelope.*

Proof. As noted earlier, $E \to L$ is surjective. To argue that $K \subset E$ is essential, note that if $\bar{E} \subset E$ is injective and $K \cap \bar{E} = 0$ then $\bar{E}$ is isomorphic to a submodule of $E/K \cong L$. Since L is reduced, this means $\bar{E} = 0$. □

Remark 2.11 If N is a reduced Gorenstein injective module, let

$$\text{(A)} \quad \cdots \to E_1(N) \to E_0(N) \to E^0(N) \to E^1(N) \to \cdots$$

be its complete minimal injective resolution. Let $K_i = \ker(E_i(N) \to E_{i-1}(N))$ for $i \geqq 1$, $K_0 = \ker(E_0(N) \to E^0(N))$ and $K^i = \ker(E^i(N) \to E^{i+1}(N))$ for $i \geqq 0$. Then from the above, the complex (A) is a complete minimal injective resolution of each of the modules K_i and K^i for $i \geqq 0$. Also for any i, $K_i = 0$ if and only if $N = 0$ and $K^i = 0$ if and only if $N = 0$.

Corollary 2.12 *If $N \neq 0$ is reduced and Gorenstein injective, then N has infinite injective and projective dimensions.*

Proof. Using the notation above, we see that if any $K^i = 0$ for $i \geqq 0$, then $N = 0$, hence $\text{inj.dim}\, N = \infty$.

Also for $i \geqq 1$,

$$0 \to K_i \to E_i(N) \to E_{i-1}(N) \to \cdots \to E_0(N) \to N \to 0$$

is a partial minimal injective resolution of K_i, so $\text{Ext}^1(N, K_0) \cong \text{Ext}^{i+2}(N, K_i)$.

Since $0 \to K_0 \to E_0(N) \to N \to 0$ does not split, $\text{Ext}^1(N, K_0) \neq 0$. Hence $\text{Ext}^{i+2}(N, -) \neq 0$ for any $i \geqq 0$ and so $\text{proj.dim}\, N = \infty$. □

Theorem 2.13 *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of left R-modules. If N' and N'' are Gorenstein injective then so is N. If N' and N are Gorenstein injective, then so is N''. If N and N'' are Gorenstein injective then N' is Gorenstein injective if and only if* $\text{Ext}^1(E, N') = 0$ *for all injective left R-modules E.*

Proof. If N' is Gorenstein injective then $\text{Ext}^1(E, N') = 0$ for all injective E by Proposition 2.2. Also if $\text{Ext}^1(E, N') = 0$ for all injective E then

$$0 \to \text{Hom}(E, N') \to \text{Hom}(E, N) \to \text{Hom}(E, N'') \to 0$$

is exact for all injective E. But if this is so we get the extended long exact sequence of Sect. 2. When we have this sequence, then by Proposition 2.2 we see that if any two of N', N or N'' are Gorenstein injective, so is the third 0. □

3 Resolutions and resolvents

We again assume all rings R are left noetherian. We show how conditions on a ring R guarantee that injective covers occur in minimal injective resolutions and how other conditions guarantee that injective envelopes occur in minimal injective resolvents.

We will use these results to show that for some rings R there is an n_0 so that if we know the tail

$$E^n(M) \to E^{n+1}(M) \to \cdots$$

of a minimal injective resolution for $n > n_0$ we can reconstruct the tail

$$E^{n_0}(M) \to E^{n_0+1}(M) \to \cdots \to E^n(M) \to E^{n+1}(M) \to \cdots$$

(so that if $n_0 = 0$, we can reconstruct the whole minimal injective resolution). We have a similar result concerning minimal injective resolvents.

We note that in the previous section we had similar but stronger results when M was reduced and Gorenstein injective.

Proposition 3.1 *Let R be a left noetherian ring such that* $\text{proj.dim}\, E \leqq n$ *for all injective left R-modules E. Then if M is a left R-module and we let*

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^i(M) \to C^{i+1} \to 0$$

be exact for all $i \geqq 0$, then $E^i(M) \to C^{i+1}$ is an injective precover for all $i \geqq n$ and is an injective cover if $i > n$. Also, C^i for $i > n$ is reduced.

Proof. The fact that $\text{Ext}^i(E,M) = 0$ for $i > n$ immediately gives that $E^i(M) \to C^{i+1}$ is an injective precover for $i \geqq n$. In particular $E^n(C) \to C^{n+1}$ is a precover. If $E \subset C^{n+1}$ is injective then we can complete

$$\begin{array}{ccc} & & E \\ & \swarrow & \downarrow \\ E^n(C) & \to & C^{n+1} \end{array}$$

If $E \neq 0$ this would contradict the minimality of the resolution, hence C^{n+1} is reduced.

But now $C^{n+1} = \ker(E^{n+1} \to C^{n+2})$. Since $E^{n+1} \to C^{n+2}$ is a precover, it has a summand E (say with $E^{n+1} = E \oplus E'$) with $E \to C^{n+2}$ a cover and E' in the kernel of $E^{n+2} \to C^{n+2}$. But then $E' \subset C^{n+1} = \ker(E^{n+1} \to E^{n+2})$. Since C^{n+1} is reduced, $E' = 0$ and so $E^{n+1} \to C^{n+2}$ is a cover. The same argument then works for $E^i \to C^{i+1}$ when $i \geqq n$. □

Corollary 3.2 *If M, N are left R-modules and*

$$0 \to M \to E^0(M) \to \cdots \to E^i(M) \to C^{i+1} \to 0$$

and

$$0 \to N \to E^0(N) \to \cdots \to E^i(N) \to D^{i+1} \to 0$$

are exact for all $i \geqq 0$, then if $i \geqq n$ and if $f : C^{i+1} \to D^{i+1}$ is any homomorphism, there exist maps $E^n(M) \to E^n(N), \ldots, E^i(M) \to E^i(N)$ such that

$$\begin{array}{ccccccc} E^n(M) & \to \cdots \to & E^i(M) & \to & C^{i+1} & \to & 0 \\ \downarrow & & \downarrow & & \downarrow f & & \\ E^n(N) & \to \cdots \to & E^i(N) & \to & D^{i+1} & \to & 0 \end{array}$$

is commutative.

Furthermore if f is an isomorphism, then each of the map $E^{n+1}(M) \to E^{n+1}(N), \ldots, E^i(M) \to E^i(N)$ above is too.

Proof. This follows immediately from the preceeding proposition and the definitions of precovers and covers. □

We note that there is an analogous result for projective resolutions (see Proposition 5.1 of [12]) of finitely generated modules over local Gorenstein rings. A different proof of this result can be given using the methods of Sect. 5.

Remark. 3.3 If M is finitely generated and R is commutative Gorenstein of finite dimension n, then using the methods of section 6 it can be argued that each C^{i+1} for $i \geqq n$ is the direct sum of a finite number of indecomposable modules.

Corollary 3.4 *If the resolution*

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots$$

is eventually periodic, then the complex

$$E^{n+1}(M) \to E^{n+2}(M) \to \cdots$$

is periodic (cf. *Eisenbud* [10] *for a similar result concerning minimal projective resolutions over hypersurface rings*).

Proof. Any isomorphism $C^m \to C^{m+p}$ for $m > n+1$ and $p > 0$ induces an isomorphism $C^{n+1} \to C^{n+1+p}$ by Corollary 3.2. □

The following proposition is analogous to proposition 3.1 but using injective resolvents in place of injective resolutions. There is an interesting difference in the indices for which the results hold.

Proposition 3.5 *Let R be left noetherian and suppose R is of finite injective dimension n over itself (on the left). For a left R-module, let*

$$\cdots \to E_2 \to E_1 \to E_0 \to M \to 0$$

be a minimal injective resolvent and let $C_i = \operatorname{coker}(E_{i+2} \to E_{i+1})$. *Then $C_i \to E_i$ is an injective envelope for $i \geqq n-1$, $C_{n-2} \to E_{n-2}$ is an injection and C_i is reduced for $i \geqq n-1$.*

Proof. We can compute $\operatorname{Ext}_i(R,M)$ using either an injective resolution of R or an injective resolvent of M. We have the injective resolution

$$0 \to R \to E^0(R) \to \cdots \to E^n(R) \to 0\,.$$

Since $\operatorname{Hom}(-,M)$ is left exact, we see that $\operatorname{Ext}_i(R,M) = 0$ for $i \geqq n-1$. Computing using an injection resolvent of M,

$$\cdots \to E_2 \to E_1 \to E_0 \to M \to 0$$

we see that this means that

$$\cdots \to E_{n+1} \to E_n \to E_{n-1} \to E_{n-2}$$

is exact. This gives that $C_i = \operatorname{coker}(E_{i+2} \to E_{i+1}) \to E_i$ is an injection if $i \geqq n-2$. Now we claim that if $i \geqq n-1$ then C_i is reduced. For by the definition of the injective resolvent, $E_i \to C_{i-1}$ is an injective cover with kernel

C_i. But we know the kernel of an injective cover has no non-zero injective submodules.

So to show that for $i \geqq n-1$, $C_i \subset E_i$ is an injective envelope note that if $E \subset E_i$ is injective and $C_i \cap E = 0$ then $E_i \to C_{i-1}$ maps E isomorphically into C_{i-1}. Since C_{i-1} is reduced, $E = 0$ and so $C_i \subset E_i$ is an injective envelope. □

Corollary 3.6 *If M and N are left R-modules and*

$$0 \to C_i \to E_i(M) \to \cdots \to E_0(M) \to M \to 0$$

and

$$0 \to D_i \to E_i(N) \to \cdots \to E_0(N) \to N \to 0$$

are exact sequences with $i \geqq n-2$, then if $f : C_i \to D_i$ is any homomorphism, then there exists a commutative diagram

$$\begin{array}{ccccccc} 0 \to & C_i & \to & E_{i-1}(M) & \to \cdots \to & E_{n-2}(M) \\ & \downarrow f & & \downarrow & & \downarrow \\ 0 \to & D_i & \to & E_{i-1}(N) & \to \cdots \to & E_{n-2}(N) \end{array}$$

Furthermore if f is an isomorphism, then so are the maps $E_i(M) \to E_i(N)$ for $i \geqq n-1$.

Proof. The proof is analogous to that of Corollary 3.2. □

4 The existence of Gorenstein injective modules

We show that if R is a Gorenstein ring, then minimal injective resolutions and resolvents can be used to generate Gorenstein injective modules.

Definition (Iwanaga [8]) *A ring R is said to be n-Gorenstein ($n \geqq 0$) if R is right and left noetherian and if R has finite self injective dimension at most n on either side. R is said to be Gorenstein if it is n-Gorenstein for some n.*

Examples. Any regular local ring is Gorenstein. If R is n-Gorenstein and G is a finite group, the group algebra RG is also n-Gorenstein ([9], Eilenberg, Nakayama). If Q is a finite quiver whose connected components either have no cycles or are cycles with no multiple edges then the path algebra RQ over a Gorenstein ring is Gorenstein ([13], Enochs, Herzog). From this it follows that the algebra of lower triangular matrices over a Gorenstein ring is Gorenstein and that the group algebra RZ is Gorenstein when R is Gorenstein. We recall

Theorem ([8], Iwanaga) *If R is n-Gorenstein and M is an R-module (left or right) then the following are equivalent*:

a) $\operatorname{proj.dim} M < \infty$
b) $\operatorname{proj.dim} M \leqq n$
c) $\operatorname{inj.dim} M < \infty$
d) $\operatorname{inj.dim} M \leqq n$

Theorem 4.2 *If R is n-Gorenstein and $0 \to M \to E^0(M) \to \cdots$ is a minimal injective resolution of the left R-module M and $C^i = \ker(E^i(M) \to E^{i+1}(M))$ for $i \geqq 0$, then C^i is Gorenstein injective for $n \geqq i$ and is reduced if $i > n$.*

Proof. We consider C^{n+1}. By Proposition 3.1, C^{n+1} is reduced and

$$0 \to C^{n+1} \to E^{n+1}(M) \to \cdots \to E^j(M) \to C^{j+1} \to 0$$

for $j \geqq n+1$ is not only a (partial) minimal injective resolution of C^{n+1} but also a (partial) minimal injective resolvent of C^{j+1}. Now we paste this sequence together with a minimal injective resolvent of C^{n+1} and get

$$\cdots \to E_1(C^{n+1}) \to E_0(C^{n+1}) \to E^{n+1}(M) \to \cdots \to E^j(M) \to C^{j+1} \to 0$$

If we take j sufficiently large and apply Prop. 3.5 we see that this sequence is exact. Since $j > n+1$ is arbitrary, we see that the long exact sequence

$$\cdots \to E_1(C^{n+1}) \to E_0(C^{n+1}) \to E^{n+1}(M) \to \cdots$$

is exact. Also $\mathrm{Hom}(E, -)$ leaves this complex exact for any injective left R-module E. It is obviously a complete minimal injective resolution of C^{n+1}. Hence C^{n+1} is a Gorenstein injective module. For $i = n$, we have $0 \to C^n \to E^n(M) \to C^{n+1} \to 0$ exact with C^{n+1} Gorenstein injective. Since $\mathrm{Ext}^1(E, C^n) = \mathrm{Ext}^{n+1}(E, M) = 0$ for any injective module E, an appeal to Proposition 2.11 gives that C^n is Gorenstein injective. □

Theorem 4.3 *If R is n-Gorenstein and*

$$\cdots \to E_1(M) \to E_0(M) \to M \to 0$$

is a minimal injective resolvent with $C_i = \mathrm{coker}(E_{i+2}(M) \to E_{i+1}(M))$ for $i \geqq 0$ then if $i \geqq n-2$, C_i is Gorenstein injective and if $i > n-2$ it is also reduced.

Proof. The proof is analogous to the proof of the preceeding Theorem. □

Corollary 4.4 *For a left R-module N the following are equivalent:*

1) $\mathrm{Ext}^1(N, M) = 0$ *for all Gorenstein injective modules M;*
2) $\mathrm{Ext}^i(N, M) = 0$ *for a fixed $i \geqq 1$ and all Gorenstein injective modules M;*
3) $\overline{\mathrm{Ext}_0}(N, M) = 0$ *for all Gorenstein injective modules M;*
4) $\overline{\mathrm{Ext}_0}(N, M) = 0$ *for all Gorenstein injective modules M;*
5) $\mathrm{Ext}_1(N, M) = 0$ *for all Gorenstein injective modules M;*
6) $\mathrm{Ext}_i(N, M) = 0$ *for a fixed $i \geqq 1$ and all Gorenstein injective modules M;*
7) *N has finite injective dimension;*
8) *N has finit projective dimension;*
9) *N has finite injective dimension at most n;*
10) *N has finite projective dimension at most n.*

Proof. We argue that 1) $\Rightarrow$ 2),3),4),5) and 6). We can assume M is reduced. We consider the complete minimal injective resolution and let $M' = \operatorname{coker}(E_0(M) \to E^0(M))$. As shown in Proposition 2.10, this resolution is also a complete minimal injective resolution of M'. But

$$0 = \operatorname{Ext}^1(N, M^1) = \operatorname{Ext}^2(N, M) \text{ so } \operatorname{Ext}^2(N, M) = 0 .$$

Variations of this procedure gives 2),3),4),5), and 6) and also each of the implications 2) $\Rightarrow$ 1), 3) $\Rightarrow$ 1),...,6) $\Rightarrow$ 1).

We now argue 1) $\Rightarrow$ 9). Let $0 \to N \to E^0(N) \to \cdots \to E^{n-1}(N) \to C^n \to 0$ be exact. By theorem 4.2, C^n is Gorenstein injective. By 6) then, $\operatorname{Ext}_n(N, C^n) = 0$. Computing $\operatorname{Ext}_n(N, C^n)$ from the complex

$$\operatorname{Hom}(E^{n+1}(N), C^n) \to \operatorname{Hom}(E^n(N), C^n) \to \operatorname{Hom}(E^{n-1}(N), C^n)$$

we see that C^n is a retract of $E^n(C)$, i.e. C^n is injective and hence $C^n = E^n(C)$.

Now 7),8),9),10) are equivalent by Iwanaga and 8) $\Rightarrow$ 1) by Proposition 2.4. □

5 Gorenstein projective modules

In this section we will briefly indicate how a modification of our approach gives results about Gorenstein projective modules.

In this section we let R be left and right noetherian and all modules be finitely generated.

For a left R-module M, a linear map $M \to P$ is called a projective preenvelope if for any linear map $M \to P'$ with P' projective can be factored through $M \to P$. This is easily equivalent to $P^* \to M^*$ being onto (with $M^* = \operatorname{Hom}(M, R)$). It is not hard to check that if $\phi_1, \ldots, \phi_s$ generate M^*, then

$$M \to R^s \quad (\text{with } x \to (\phi_1(x), \ldots, \phi_s(x))$$

is a projective preenvelope of M.

Definition 5.1 *A complex*

$$0 \to M \to P^0 \to P^1 \to \cdots$$

is called a projective resolvent of M if each P^i is a projective module and if for each projective module P, the functor $\operatorname{Hom}(-, P)$ *makes the complex exact (or equivalently, $\cdots \to P^{1*} \to P^{0*} \to M^* \to 0$ is exact with $M^* =$* $\operatorname{Hom}(M, R)$).

Using projective preenvelopes, such as resolvent for M can be constructed. If

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

is a projective resolution of M then

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

is called a complete resolution of M. We note that these coincide with the complete resolutions used in group homology and cohomology (see Brown, Cohomology of Groups [6], p. 131).

Now by example 3.5 of [14], left derived functors of Hom$(-,-)$ can be defined and computed. They are denoted $\mathrm{Ext}_n(M,N)$ (these should not be confused with the $\mathrm{Ext}_n(M,N)$'s of Sects. 2, 3 and 4 and are computed using either a projective resolvent of M or a projective resolution of N). Here there is a natural transformation

$$\mathrm{Ext}_0(M,N) \to \mathrm{Ext}^0(M,N)$$

with kernel and cokernel denoted $\overline{\mathrm{Ext}}_0(M,N)$ and $\overline{\mathrm{Ext}}^0(M,N)$. Then as in Sect. 2, it can be seen that there are extended long exact sequences which arise whenever $0 \to M' \to M \to M'' \to 0$ is an exact sequence such that $0 \to M''^* \to M^* \to M'^* \to 0$ is exact or whenever $0 \to N' \to N \to N'' \to 0$ is exact.

Definition 5.2 *A left R-module M is said to be Gorenstein projective if* $\mathrm{Ext}_i(M,Q) = \mathrm{Ext}^i(M,Q) = 0$ *for* $i \geqq 1$ *and* $\overline{\mathrm{Ext}}_0(M,Q) = \overline{\mathrm{Ext}}^0(M,Q) = 0$ *for all modules Q which are either injective or projective.*

Then there are results concerning Gorenstein projective modules analogous to those of Sect. 2 concerning Gorenstein injective modules.

Although modules M will have projective precovers and preenvelopes, they may not have covers and envelopes. If, for example, R is local, then they will have covers and envelopes and all the results of Sects. 3, 4 and 5 will have counterparts for Gorenstein projective modules.

However, even without this assumption, weakened versions of these results hold. This usually means substituting preenvelope and precover for envelope and cover and dropping hypotheses and conclusions concerning reduced modules (in this setting, meaning no non-zero projective summands).

Remark. If $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ is the complete resolution of M, then M is Gorenstein projective if and only if this resolution and its algebraic dual is exact. From this it follows that M^* is also Gorenstein projective. It also follows that M is reflexive.

Remark. If G is a finite group and $R = \mathbb{Z}G$ then it is well known that R is 1-Gorenstein. If M is a finitely generated left R-module then it can be argued that M is Gorenstein projective if and only if M is a free $\mathbb{Z}$-module. Similarly M (without finiteness assumptions) is Gorenstein injective if and only if it is divisible as a $\mathbb{Z}$-module. This and stronger results are in [16].

We now assume R is a local ring and right and left noetherian. We let G be a Gorenstein projective module, then as noted above, G^* is also Gorenstein projective, and it's not hard to see that $G \to R^\beta$ is a projective envelope if and only if $(R^\beta)^* \cong R^\beta \to G^*$ is a projective cover. If we use the inner product

$$(r_1, \ldots, r_\beta) \cdot (s_1, \ldots, s_\beta) = \sum r_i s_i$$

on R^β and if $G \subset R^\beta$ is a submodule of R^β, we will let $G^\perp = \{(s_1,\ldots,s_\beta) | (s_1,\ldots,s_\beta) \in R^\beta$ and $(r_1,\ldots,r_\beta)(s_1,\ldots,s_\beta) = 0$ for all $(r_1,\ldots,r_\beta) \in G\}$. Then $G^\perp$ is a sub-module of R^β as a right R-module. For a right submodule $H \subset R^\beta$, ${}^\perp H$ is defined in a similar fashion. Clearly $G \subset {}^\perp(G^\perp)$ for any such G.

Theorem 5.5 *If $G \subset R^\beta$ is a projective envelope of the reduced Gorenstein projective module G, then ${}^\perp(G^\perp) = G$. Furthermore $G^\perp$ (up to isomorphism) does not depend on the embedding $G \subset R^\beta$. Also $G^\perp$ is reduced and Gorenstein projective.*

Proof. By the remarks above, $R^\beta \to G^*$ is a projective cover. Its kernel is $G^\perp$, so by lemma 5.3, $G^\perp$ is reduced and Gorenstein projective and $G^\perp \subset R^\beta$ is a projective envelope. Repeating the argument, ${}^\perp(G^\perp)$ is a reduced Gorenstein projective module and ${}^\perp(G^\perp) \subset R^\beta$ is an envelope. Clearly $({}^\perp(G^\perp)^\perp = G^\perp$.

Taking the dual of

$$0 \to G^\perp \to R^\beta \to G^* \to 0$$

we get

$$0 \to G \to R^\beta \to G^{\perp *} \to 0$$

exact. The corresponding exact sequence for ${}^\perp(G^\perp)$ is

$$0 \to {}^\perp(G^\perp) \to R^\beta \to ({}^\perp(G^\perp))^{\perp *} = G^{\perp *} \to 0\,.$$

Then the commutative diagram

$$\begin{array}{ccccccc} 0 \to & G & \to & R^\beta & \to & G^{\perp *} & \to 0 \\ & \downarrow & & \| & & \| & \\ 0 \to & {}^\perp(G^\perp) & \to & R^\beta & \to & G^{\perp *} & \to 0 \end{array}$$

gives $G = {}^\perp(G^\perp)$.

We see that $G^\perp$ does not depend on the embedding $G \to R^\beta$ by noting that for two such embeddings $f : G \to R^\beta$ and $g : G \to R^\alpha$ we have an isomorphism $R^\beta \cong R^\alpha$ over G, so $\alpha = \beta$. Then the two exact sequences

$$0 \to K \to R^\beta \xrightarrow{f^*} G^* \to 0$$

and

$$0 \to L \to R^\beta \xrightarrow{g^*} G^* \to 0$$

are isomorphic, so $K \cong L$. But K and L are the $G^\perp$'s for the corresponding embeddings.

Remark. The 0-Gorenstein rings are the quasi-Frobenius rings. Over quasi-Frobenius rings all modules are Gorenstein injective. If $I \subset R$ is a left (right) ideal of the quasi-Frobenius ring R and the inner product on R is multiplication, the result ${}^\perp(I^\perp) = I(({}^\perp I)^\perp = I)$ is well-known.

We note that if G is Gorenstein projective and

$$\cdots \to P_1 \to P_0 \to G \to 0$$

is a minimal projective resolution of G and $0 \to G \to P^0 \to P^1 \cdots$ is a minimal projective resolvent of G, then the complete projective resolution

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

can be regarded as the complete resolution of any of the C_i's with $C_i = \ker(P_i \to P_{i-1})$ for $i \geqq 1$ or the terms C_0 or C^i for $i \geqq 0$ defined in a similar manner.

The proof would use the following two lemmas:

Lemma 5.3 *If G is Gorenstein projective and reduced and $R^\beta \to G$ is a projective cover with kernel K, then K is reduced, Gorenstein projective and $K \subset R^\beta$ is a projective envelope.*

Lemma 5.4 *If K is reduced and Gorenstein projective and $K \subset R^\beta$ is a projective envelope and $G = R^\beta/K$ then G is reduced and Gorenstein projective and $R^\beta \to G$ is a projective cover.*

6 Mock finitely generated modules

When R is commutative and noetherian, we define the mock finitely generated modules. We are mainly interested in Gorenstein injective modules which are mock finitely generated. Using earlier results we can see that there is a natural way of defining generalized Bass' invariants with negative indices of modules. In this section we will show that if R is Gorenstein and the module is finitely generated, then these invariants are finite.

In this section R will always be commutative and noetherian. The functors $\mathrm{Ext}_i(M, N)$ will be those of Sect. 2.

Definition 6.1 *An R-module N is said to be mock finitely generated if for any finitely generated R-module M each of $\mathrm{Ext}^i(M,N)$, $\mathrm{Ext}_i(M,N)$ for $i \geqq 1, \overline{\mathrm{Ext}}_0(M,N)$ and $\overline{\mathrm{Ext}}^0(M,N)$ are finitely generated R-modules.*

Remark. We chose this terminology since the modules we will study which are mock finitely generated will rarely be finitely generated.

Using the extended long exact sequence we have

Proposition 6.2 *If $0 \to N' \to N \to N'' \to 0$ is an exact sequence of R-modules left exact by $\mathrm{Hom}(E, -)$ for any injective module E then if any two of N', N or N'' is mock finitely generated, then so is the third.*

Proof. Immediate.

We are mainly interested in mock finitely generated modules which are Gorenstein injective. If N is Gorenstein injective and

$$0 \to N \to E^0(N) \to C \to 0 \quad \text{and} \quad 0 \to K \to E_0(N) \to N \to 0$$

and exact, then again using the extended long exact sequence and the fact that

$$\mathrm{Ext}^i(M,E) = \mathrm{Ext}_i(M,E) = \overline{\mathrm{Ext}}_0(M,E) = \overline{\mathrm{Ext}}^0(M,E) = 0$$

for all $i \geqq 1$ and all injective modules E we see that N is mock finitely generated if and only if C is, and if and only if K is.

Proposition 6.3 *If R is an n-Gorenstein and N is Gorenstein injective, then N is mock finitely generated if and only if for every finitely generated Gorenstein projective module M, each of* $\operatorname{Ext}_i(M,N)$, $\overline{\operatorname{Ext}}_0(M,N)$, $\overline{\operatorname{Ext}}^0(M,N)$, *and* $\operatorname{Ext}^i(M,N)$ *are finitely generated (for all $i \geqq 1$).*

Proof. If M is finitely generated and if

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

is exact with each P_i finitely generated and projective, then by the counterpart of Proposition 4.2 (or by Auslander [1]) K is Gorenstein projective. But each of $\operatorname{Ext}_i(P,N)$, $\operatorname{Ext}^i(P,N)$, $\overline{\operatorname{Ext}}_0(P,M)$, $\overline{\operatorname{Ext}}^0(P,N)$ is 0 when P is projective by Proposition 2.2. Since there is an extended long exact sequence of the Ext's for each short exact sequence in the first variable of the Ext's, we see that in order for each of $\operatorname{Ext}_i(M,N)$, $\operatorname{Ext}^i(M,N)$, $\overline{\operatorname{Ext}}_0(M,N)$, $\overline{\operatorname{Ext}}^0(M,N)$ to be finitely generated ($i \geqq 1$), it is necessary and sufficient that these module by finitely generated when M is replaced by K. □

Corollary 6.4 *For a Gorenstein injective module N the following are equivalent:*

a) *N is mock finitely generated*

b) $\operatorname{Ext}^1(M,N)$ *if finitely generated for all finitely generated Gorenstein projective modules M.*

Proof. a) $\Rightarrow$ b) is trivial. To argue b) $\Rightarrow$ c), note that if M is as in b) and if $0 \to K \to R^\alpha \to M \to 0$ is exact with a α finite, then K is finitely generated and Gorenstein projecitve and $\operatorname{Ext}^i(K,N) \cong \operatorname{Ext}^{i+1}(M,N)$ for $i \geqq 1$. So it is easy to get $\operatorname{Ext}^i(M,N) = 0$ for all $i \geqq 1$.

If $0 \to M \to R^\beta \to C \to 0$ is exact with β finitely and $M \to R^\beta$ is a projective preenvelope, then C is Gorenstein projective and finitely generated. Then using the extended long exact sequence and Proposition 2.2 it is easy to get $\overline{\operatorname{Ext}}^0(M,N) = \overline{\operatorname{Ext}}_i(M,N) = 0$ ($i \geqq 1$). □

Theorem 6.5 *If R is n-Gorenstein and N is finitely generated and*

$$0 \to N \to E^0(N) \to E^1(N) \to \cdots \to E^{i-1}(N) \to C^i \to 0$$

is exact, then C^i is mock finitely generated for $i \geqq n$.

Proof. By Theorem 4.2, any such C^i is Gorenstein injective. Let M be finitely generated. Then $\operatorname{Ext}^1(M,C^i) \cong \operatorname{Ext}^{i+1}(M,N)$ and so $\operatorname{Ext}^1(M,C^i)$ is finitely generated. By Corollary 6.4 this completes the proof. □

Proposition 6.6 *If N is mock finitely generated and Gorenstein injective and $0 \to N \to E_0(N) \to L \to 0$ and $0 \to K \to E_0(N) \to N \to 0$ are exact, then K and L are mock finitely generated.*

Proof. Letting M be a finitely generated module, we consider the extended long exact sequences associated with the two short exact sequences (in the second variable). Since all terms in the sequence with an injective module as the second variable vanish, we see that each of L and K is mock finitely generated. □

We note that K and L above are also Gorenstein injective.

Now we let $C = C^n$ in Theorem 6.5 above. Taking a minimal injective resolution of C and a minimal injective resolvent of C, we relabel the $E_i(C)$ in the obvious way and have the complex

$$\cdots \to E^{-2}(C) \to E^{-1}(C) \to E^0(C) \to E^1(C) \to \cdots$$

Given any prime ideal $P \subset R$, we use Bass' procedure in [5] and apply the functor $\mathrm{Hom}(R/P, -)$ to this complex. The homology modules over the resulting complex are vector spaces over $k(P)$ (the field of fractions of R/P). Again following Bass' ideas, we call the dimensions of these vector spaces the generalized Bass invariants of N. For the obvious reasons, the dimension of the k-th homology module is called the $k + (n + 1)$ generalized Bass invariant of N. For $k \geqq 0$ it is immediate that these are the usual Bass invariants of N.

Now we note that the procedure we have described "commutes" with taking localizations, i.e. applying the functor $S^{-1}-$for any multiplicative $S \subset R$.

If we then appeal to Theorem 6.5 and Proposition 6.6, we see that if N is finitely generated, then the generalized Bass invariants of N will all be finite.

7 Covers and envelopes by Gorenstein injective and projective modules

In this section we consider the question of the existence of certain envelopes, preenvelope, covers and precovers.

For a ring R, we let Mod be the category of left R-modules and let $\underline{\mathrm{Mod}}$ be the stable category of left R-modules described in Sect. 2.

Our main result is

Theorem 7.1 *If R is a Gorenstein ring, then in* $\underline{\mathrm{Mod}}$ *every module M has a reduced Gorenstein injective envelope. If $[f] : M \to K$ and $[g] : M \to L$ are two such envelopes, then any $h : K \to L$ such that $[f] \circ [h] = [g]$ is an isomorphism in* Mod.

Proof. Let R be n-Gorenstein. Let M be a left R-module and let

$$0 \to M \to E^0(M) \to \cdots \to E^n(M) \to C \to 0$$

be a partial minimal injective resolution. Then by Theorem 4.2, C is Gorenstein injective. Let

$$0 \to K \to E_n(C) \to \cdots \to E_0(C) \to C \to 0$$

be a partial minimal injective resolvent with $K = \ker(E_n(C) \to E_{n-1}(C))$.

Then by Corollary 2.3, this sequence is exact. The diagram

$$\begin{array}{ccccccccccccc}
0 & \rightarrow & M & \rightarrow & E^0(M) & \rightarrow & \cdots & \rightarrow & E^n(M) & \rightarrow & C & \rightarrow & 0 \\
 & & \vdots\downarrow f & & \vdots\downarrow & & & & \vdots\downarrow & & \| & & \\
0 & \rightarrow & K & \rightarrow & E_n(C) & \rightarrow & \cdots & \rightarrow & E_0(C) & \rightarrow & C & \rightarrow & 0
\end{array} \tag{1}$$

can be completed to a commutative diagram. By Proposition 2.10, K is reduced and Gorenstein injective. We want to show that $[f] : M \rightarrow K$ is the desired envelope.

Now suppose L is reduced and Gorenstein injective and that $M \rightarrow L$ is linear.

Then we can complete the diagram (with exact rows)

$$\begin{array}{ccccccccccccc}
0 & \rightarrow & M & \rightarrow & E^0(M) & \rightarrow & \cdots & \rightarrow & E^n(M) & \rightarrow & C & \rightarrow & 0 \\
 & & \downarrow & & \vdots\downarrow & & & & \vdots\downarrow & & \vdots\downarrow & & \\
0 & \rightarrow & L & \rightarrow & E^0(L) & \rightarrow & \cdots & \rightarrow & E^n(L) & \rightarrow & D & \rightarrow & 0
\end{array} \tag{2}$$

to a commutative diagram. Then using the map $C \rightarrow D$ and Corollary 2.3 we see that we can complete the diagram

$$\begin{array}{ccccccccccccc}
0 & \rightarrow & K & \rightarrow & E_n(C) & \rightarrow & \cdots & \rightarrow & E_0(C) & \rightarrow & C & \rightarrow & 0 \\
 & & \vdots\downarrow & & \vdots\downarrow & & & & \vdots\downarrow & & \downarrow & & \\
0 & \rightarrow & L & \rightarrow & E^0(L) & \rightarrow & \cdots & \rightarrow & E^n(L) & \rightarrow & D & \rightarrow & 0
\end{array} \tag{3}$$

to a commutative diagram.

Using (1) and (3) we get a composition $M \rightarrow K \rightarrow L$ by pasting the two diagrams together.

Since this composite diagram and (2) have the same map $C \rightarrow D$ it is a standard argument (lifting maps uniquely up to homotopy) that the original map $M \rightarrow L$ and the compositions $M \rightarrow K \rightarrow L$ give the same element of $\underline{\mathrm{Hom}}(M, L)$.

This shows that our map $M \rightarrow K$ (in $\underline{\mathrm{Mod}}$) is a preenvelope. To show that it is an envelope, let $[f] : M \rightarrow K$ be as above and let $[g] : K \rightarrow K$ be such that $[g] \circ [f] = [f]$. We want to show that $[g]$ in an isomorphism. By Corollary 2.9 we must show that g is an isomorphism.

We complete the diagram

$$\begin{array}{ccccccccccccc}
0 & \rightarrow & M & \rightarrow & E^0(M) & \rightarrow & \cdots & \rightarrow & E^n(M) & \rightarrow & C & \rightarrow & 0 \\
 & & \downarrow f & & \downarrow & & & & \downarrow & & \| & & \\
0 & \rightarrow & K & \rightarrow & E_0(C) & \rightarrow & \cdots & \rightarrow & E_0(C) & \rightarrow & C & \rightarrow & 0 \\
 & & \downarrow g & & \vdots\downarrow & & & & \vdots\downarrow & & \vdots\downarrow h & & \\
0 & \rightarrow & K & \rightarrow & E_0(C) & \rightarrow & \cdots & \rightarrow & E_0(C) & \rightarrow & C & \rightarrow & 0
\end{array}$$

to a commutative diagram. Then again by homotopy $[g] \circ [f] = [f]$ implies $[h] = [\mathrm{id}_C]$. Then by Corollary 2.9, h is an isomorphism. Hence $g : K \rightarrow K$ is an isomorphism. If $M \to K$ and $M \to L$ are two such envelopes and if

$$\begin{array}{ccc} & & K \\ & \nearrow & \downarrow \\ M & & \\ & \searrow & \downarrow \\ & & L \end{array} \quad \text{and} \quad \begin{array}{ccc} & & L \\ & \nearrow & \downarrow \\ M & & \\ & \searrow & \downarrow \\ & & K \end{array}$$

are commutative diagrams in Mod then by the above $K \to L \to K$ and $L \to K \to L$ are isomorphisms in Mod. Hence L and K are isomorphic in Mod by Corollary 2.9. □

Remark. If R is commutative and M finitely generated, then by the results of Sect. 6, K can be seen to be mock finitely generated.

Theorem 7.2 *If R is a Gorenstein ring then every left R-module M has a Gorenstein injective preenvelope.*

Proof. We suppose R is n-Gorenstein and consider the diagram

$$\begin{array}{ccccccccccc} 0 & \to & M & \to & E^0(M) & \to \cdots \to & E^n(M) & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & K & \to & E_n(C) & \to & E_0(C) & \to & C & \to & 0 \end{array}$$

constructed as above. We regard this diagram as a double complex and form the associated complex

$$0 \to M \to K \oplus E^0(M) \to \cdots \to E_0(C) \oplus C \to C \to 0$$

With the first filtration on the double complex we see that E_1 term is 0. Hence the associated complex above is exact. Note that the complex has $0 \to C \to C \to 0$ an exact subcomplex, so the quotient complex

$$0 \to M \to K \oplus E^0(M) \to \cdots \to E_0(C) \to 0$$

is also exact. Letting

$$0 \to M \to K \oplus E^0(M) \to L \to 0$$

be exact, we see that L has finite injective dimension.

Hence if N is Gorenstein injective, $\mathrm{Ext}^1(L, N) = 0$ by Proposition 2.4. This implies that

$$\mathrm{Hom}(M \oplus E^0(M), N) \to \mathrm{Hom}(K, N) \to 0$$

is exact and so that

$$M \to K \oplus E^0(M)$$

is a Gorenstein injective preenvelope of M. □

Remarks. Theorem 4.2 and the dual of Theorem 1.1 of [2] give another proof of this result. In [16] it is shown that in fact every left R-module has a Gorenstein injective envelope.

A Gorenstein injective resolution of M is a complex $0 \to M \to K^0 \to K^1 \to \cdots$ with each K^i Gorenstein injective and such that $\mathrm{Hom}(-,K)$ leaves the complex exact for all Gorenstein injective modules (so the complex is in fact exact). It can be shown that if R is n-Gorenstein every M has a Gorenstein injective resolution of the form $0 \to M \to K^0 \to \cdots \to K^n \to 0$.

Some of the results in this paper were announced at the China-Japan First International Symposium on Ring Theory in Guilin, China during October 20–25, 1991. A small portion of this paper (essentially Proposition 3.1 and its Corollaries) have appeared in the proceedings of that conference. Those results are included here for completeness.

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