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RESEARCH ARTICLE

ON FINITE $\mathcal{J}$-TRIVIAL MONOIDS

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In this note I give several characterizations of the family $\underline{J}$ of finite monoids with one-element $\mathcal{J}$-classes. The proof uses techniques from the theory of finite automata and depends upon a theorem of Imre Simon [2] which describes the family of recognizable sets whose syntactic monoids are in $\underline{J}$.

For details concerning the automata-theoretic notions used here, the reader is referred to the books by Eilenberg [1], especially Chapter VIII of Volume B.

Let $\underline{J}$ be the family of all finite monoids with one-element $\mathcal{J}$-classes. That is, $\underline{J}$ consists of all finite monoids M such that if $asb = t$ and $ctd = s$ for some $a,b,c,d,s,t \in M$, then $s = t$. $\underline{J}$ is closed under submonoids, homomorphic images and finite direct products; that is, $\underline{J}$ is an $\underline{M}$-variety [1, vol. B].

Let Σ be a finite alphabet and Σ^* the free monoid generated by Σ. A recognizable subset A of Σ^* is said to be <u>piecewise testable</u> if A is in the boolean closure of the family of subsets of Σ^* of the form $\Sigma^*\sigma_1\Sigma^*\sigma_2\cdots\sigma_p\Sigma^*$, where $\sigma_i \in \Sigma$ for $i = 1,\cdots,p$. Let $M(A)$ be the syntactic monoid of A. Simon's Theorem is: <u>A recognizable subset A of Σ^* is piecewise testable iff $M(A) \in \underline{J}$.</u>

$\mathcal{B}$ denotes the commutative semiring $\{0,1\}$ — multiplication and addition in $\mathcal{B}$ are defined by the formulas $1\cdot 1 = 1$, $1\cdot 0 = 0\cdot 1 = 0\cdot 0 = 0$, $0+0 = 0$, $0+1 = 1+0 = 1+1 = 1$. For each $n \geq 1$, $\mathcal{B}^{n\times n}$ denotes the collection of all $n \times n$ matrices over $\mathcal{B}$. $\mathcal{B}^{n\times n}$ is thus a finite monoid under matrix multiplication; $\mathcal{B}^{n\times n}$ may also be viewed as the monoid of all binary relations on a set with n elements. Let

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$\mathcal{L}_n = \{m \in \mathcal{B}^{n\times n} | m_{ii} = 1 \text{ for } i = 1,\cdots,n\}$. $\mathcal{L}_n$ is a submonoid of $\mathcal{B}^{n\times n}$; in terms of relations it is the monoid of all reflexive relations on a set with n elements. Let $\mathcal{K}_n = \{m \in \mathcal{L}_n | m_{ij} = 0 \text{ for } j < i\}$. $\mathcal{K}_n$ is also a submonoid of $\mathcal{B}^{n\times n}$; it consists of all upper triangular matrices in $\mathcal{L}_n$. Observe that $\mathcal{B}^{n\times n} \times \mathcal{B}^{n'\times n'}$ embeds in $\mathcal{B}^{(n+n')\times(n+n')}$ under the map which sends the pair of matrices (m,m') to the matrix $\left(\begin{array}{c|c} m & 0 \\ \hline 0 & m' \end{array}\right)$. When restricted to $\mathcal{K}_n \times \mathcal{K}_{n'}$, this map gives an embedding of $\mathcal{K}_n \times \mathcal{K}_{n'}$ in $\mathcal{K}_{n+n'}$.

THEOREM. Let M be a finite monoid. Then the following are equivalent:

(a) $M \in \underline{J}$.

(b) $M \prec \mathcal{K}_n$ for some $n \geq 1$.

(c) $M \prec \mathcal{L}_n$ for some $n \geq 1$.

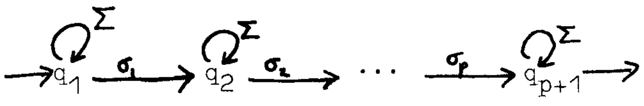
(d) $M \prec S$, where S is a monoid of transformations acting faithfully on the right of a finite partially-ordered set Q such that

$$qs \leq q \quad \text{for all } q \in Q,\ s \in S;$$

$$q \leq q' \text{ implies } qs \leq q's \text{ for all } q,q' \in S, s \in S.$$

PROOF. (a)$\Longrightarrow$(b). Let $M \in \underline{J}$. Since every $\underline{M}$-variety is generated by the syntactic monoids it contains [1, vol.B, chapter VIII] there exist finite alphabets $\Sigma_1,\cdots,\Sigma_k$ and recognizable sets $A_i \subseteq \Sigma_i^*$ such that $M \prec M(A_1) \times \cdots \times M(A_k)$. By Simon's Theorem, each A_i is piecewise testable, and thus each $M(A_i)$ in turn divides a direct product $M(B_{i1}) \times \cdots \times M(B_{ir})$, where each B_{ij}

is of the form $\Sigma_i^*\sigma_1\Sigma_i^*\sigma_2 \cdots \sigma_p\Sigma_i^*$. Since, as remarked above, $\mathcal{K}_n \times \mathcal{K}_{n'} \subseteq \mathcal{K}_{n+n'}$, it suffices to show that each such $M(B_{ij})$ divides some $\mathcal{K}_n$. A nondeterministic automaton which recognizes $B = \Sigma^*\sigma_1\Sigma^* \cdots \sigma_p\Sigma^*$ is given by the state diagram



To this automaton is associated a homomorphism $\mu : \Sigma^* \to \mathcal{K}_{p+1}$ defined by

$$(\sigma\mu)_{rs} = \begin{cases} 1 & \text{if } r = s \\ 1 & \text{if } \sigma = \sigma_r \text{ and } s = r+1 \\ 0 & \text{otherwise} \end{cases}$$

for each $\sigma \in \Sigma$, $1 \leq r$, $s \leq p+1$. It is easy to check that $w \in \Sigma^*$ is accepted by the automaton iff $(w\mu)_{1,p+1} = 1$. Thus, $B = X\mu^{-1}$, where $X = \{m \in \mathcal{K}_{p+1} | m_{1,p+1} = 1\}$. It follows that $M(B) \prec \mathcal{K}_{p+1}$.

(b)$\Longrightarrow$(c). This is immediate, since $\mathcal{K}_n \subseteq \mathcal{L}_n$ for all n.

(c)$\Longrightarrow$(d). It suffices to show that $\mathcal{L}_n$ is a monoid of transformations of the kind described in (d). Let Q be the set of all row vectors $(a_1, \cdots, a_n)$ with each $a_i \in B$. The i^{th} coordinate of $q \in Q$ is denoted q_i. A partial order on Q is defined by $q \leq q'$ iff $q_i' = 1$ implies $q_i = 1$ for $i = 1, \cdots, n$. $\mathcal{L}_n$ acts on the right of Q by ordinary matrix multiplication. The action of $\mathcal{L}_n$ on Q is faithful, for suppose $s, s' \in \mathcal{L}_n$ and $s \neq s'$. Then there is some pair of indices i, j such that $s_{ij} = 1$ and $s_{ij}' = 0$ (or vice-versa). Let

q be the vector such that $q_i = 1$ and $q_k = 0$ for all $k \neq i$. Then $(qs)_j = 1$ and $(qs')_j = 0$, so $qs \neq qs'$.

If $s \in \mathcal{L}_n$ and $q_i = 1$, then (since $s_{ii} = 1$), $qs_i = 1$. Thus $qs \leq q$. If $q \leq q'$, $s \in S$, and $(q's)_j = 1$, then there is an i such that $q'_i = 1$ and $s_{ij} = 1$. But then $q_i = 1$, so $(qs)_j = 1$. Thus $qs \leq q's$. It follows that $(Q, \mathcal{L}_n)$ is a transformation monoid with the required properties.

(d)$\Longrightarrow$(a). Let S be a monoid of transformations of the kind described in (d). Let $s,t \in S$, and suppose there are $a,b,c,d \in S$ such that $asb = t$, $ctd = s$. For any $q \in Q$, $qa \leq q$, so $qt = qasb \leq qas \leq qs$. Likewise $qt \leq qs$. Thus $qs = qt$ for all $q \in Q$. Since S acts faithfully on Q, $s = t$. Thus $S \in \underline{J}$, and consequently any divisor M of S is in $\underline{J}$.

REFERENCES

1. Eilenberg, S., Automata, Languages and Machines, Academic Press, New York, 1974 (vol. A) and 1976 (vol. B).
2. Simon, I., Piecewise Testable Events in Automata Theory and Formal Languages, 2nd GI Conference, Springer Lecture Notes in Computer Science 33 (1975) 214-222.

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