Math. Z. 214, 579–592 (1993)

Mathematische Zeitschrift

Canonical equiaffine hypersurfaces in $\mathbb{R}^{n+1}$

Changping Wang*

Technische Universität Berlin, MA 8-3, Strasse des 17. Juni 136, D-10623 Berlin, Germany
Nankai Institute of Mathematics, Nankai University, Tianjin 300071,
People's Republic of China

Received 9 April 1991; in final form 16 February 1993

1 Introduction

An important problem in affine differential geometry is to classify all the affine hyperspheres with constant sectional curvature (abbreviated CSC) Blaschke metric. This problem has been extensively studied in the recent years.

The classification has been made in dimension 2 due to works of Radon [14], Li and Penn [8], Magid and Ryan [9] and Simon [17].

The first result in high dimension for local affine hyperspheres with positive definite Blaschke metric was given by Li [7], who proved that an affine hypersphere with vanishing scalar curvature is either the elliptic paraboloid or affinely equivalent to the hypersphere defined by

$$x_1 x_2 x_3 \dots x_{n+1} = 1. \tag{1.1}$$

Then Yu [23] could show that an CSC affine hypersphere in $\mathbb{R}^4$ is either a quadric or the hypersphere (1.1) with $n=3$. Finally Vrancken et al. [21] could generalize Yu's result to all dimensions by using a technique to simplify the Fubini-Pick cubic form. Thus the classification of CSC affine hyperspheres with positive definite Blaschke metric was completed.

The classification problem becomes much more difficult if the Blaschke metric is indefinite and $n \geqq 3$. The only known result was given by Magid and Ryan [10]. They showed that an CSC affine hypersphere in $\mathbb{R}^4$ with indefinite metric and non-zero Pick invariant is affinely equivalent to one of the hyperspheres

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2) = 1, \tag{1.2}$$

$$(x_1^2 + x_2^2)(x_3^2 - x_4^2) = 1. \tag{1.3}$$

The case $n \geqq 4$ remains open. Magid and Ryan gave the following

* Supported by Technische Universität Berlin

Conjecture. *An CSC affine hypersphere in* $\mathbb{R}^{n+1}$ *with non-zero Pick invariant is affinely equivalent to one of the hyperspheres*

$$(1.4)\qquad (x_1^2+x_2^2)\dots(x_{2s-1}^2+x_{2s}^2)x_{2s+1}\dots x_{n+1}=1, \qquad 0\leqq s\leqq\left[\frac{n+1}{2}\right].$$

We note that the hyperspheres in (1.4) have flat Blaschke metric and parallel Fubini-Pick form. It motivates us to study a class of hyperspheres called canonical hyperspheres. A hypersurface $f\colon M\to\mathbb{R}^{n+1}$ is said to be canonical if its Blaschke metric G is flat and its Fubini-Pick form C is parallel with respect to G. By the Bokan-Nomizu-Simon theorem [2] we know that any canonical hypersurface is an affine hypersphere. Thus the conjecture of Magid and Ryan is equivalent to the following two assertions:

Assertion (*i*) Any CSC affine hypersphere with non-zero Pick invariant is proper and canonical.

Assertion (*ii*) Any proper canonical affine hypersphere is affinely equivalent to one of the hyperspheres given by (1.4).

Our purpose in this paper is to study the canonical hyperspheres. We reduce the classification problem of proper canonical hyperspheres to an algebraic classification problem of n mutually commutative self-adjoint linear operators in $\mathbb{R}^{n+1}$ (with an indefinite inner product) which satisfy some algebraic conditions. Then we can show the main

Theorem. *Let* $f\colon \mathbb{R}^n\to\mathbb{R}^{n+1}$ *be a proper canonical affine hypersphere with Blaschke metric* G. *If the dimension of the maximal negative definite subspace of* G *is* 1, *then* f *is affinely equivalent to the hypersphere* $(x_1^2+x_2^2)x_3\dots x_{n+1}=1$.

The key point for the proof of this theorem is the following algebraic lemma, which is also important in linear algebra.

Lemma. *Let* $\langle\,,\rangle$ *be the indefinite inner product in* $\mathbb{R}^{n+1}$ *defined by*

$$(1.5)\qquad \langle x,x\rangle=-x_1^2+x_2^2+\dots+x_{n+1}^2, \qquad x={}^t(x_1,x_2,\dots,x_{n+1})\in\mathbb{R}^{n+1}.$$

Let $A_1, A_2, \dots, A_m$ *be finite mutually commutative self-adjoint linear operators in* $\mathbb{R}^{n+1}$. *Then either there is an orthonormal basis* $\{e_1,e_2,\dots,e_{n+1}\}$ *for* $\langle\,,\rangle$ *such that we have the matrix representations*

$$\text{Type I.}\quad A_i=\begin{pmatrix} a_{i1} & & & 0\\ & a_{i2} & & \\ & & \ddots & \\ 0 & & & a_{in+1}\end{pmatrix}=(a_{i1})\oplus(a_{i2})\oplus\dots\oplus(a_{in+1}),\quad i=1,2,\dots,m,\ a_{i\alpha}\in\mathbb{R};$$

or

$$\text{Type II.}\quad A_i=\begin{pmatrix} a_{i1} & a_{i2}\\ -a_{i2} & a_{i1}\end{pmatrix}\oplus(a_{i3})\oplus\dots\oplus(a_{in+1}),\quad i=1,2,\dots,m,\ a_{i\alpha}\in\mathbb{R}$$

and $a_{j2}\neq 0$ *for some* j;

or

there exist a basis $\{x, y, e_3, \ldots, e_{n+1}\}$ *with* $\langle x, x\rangle = \langle y, y\rangle = \langle x, e_\alpha\rangle = \langle y, e_\alpha\rangle = 0$, $\langle x, y\rangle = 1$ *and* $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}$, $3 \leqq \alpha, \beta \leqq n+1$, *and a positive integer* ν, $2 \leqq \nu \leqq n+1$, *such that we have the matrix representations*

$$\text{Type III.}\quad A_i = \begin{pmatrix} a_{i1} & a_{i2} & a_{i3} & a_{i4} & \ldots & a_{i\nu} \\ 0 & a_{i1} & 0 & 0 & \ldots & 0 \\ 0 & a_{i3} & a_{i1} & & & \\ 0 & a_{i4} & & a_{i1} & & 0 \\ \vdots & \vdots & & & \ddots & \\ 0 & a_{i\nu} & & 0 & & a_{i1} \end{pmatrix} \oplus (a_{i\nu+1}) \oplus \ldots \oplus (a_{in+1}),$$

$i = 1, 2, \ldots, m$, $a_{i\alpha} \in \mathbb{R}$ *and* $a_{i\nu} \neq 0$ *for some i. Moreover, if* $\nu = 2$, *then there is j with* $a_{j2} = \pm 1$; *if* $\nu \geqq 3$, *there is j such that* $a_{j3} = 1$ *and* $a_{j2} = a_{j4} = \ldots = a_{j\nu} = 0$.

This paper is organized as follows. In § 2 we study the canonical proper hyperspheres in $\mathbb{R}^{n+1}$. In § 3 we prove the algebraic lemma. In § 4 we give the proof of the main theorem.

2 Proper canonical hyperspheres in $\mathbb{R}^{n+1}$

Let $f\colon M \to \mathbb{R}^{n+1}$ be a proper canonical affine hypersphere. By the definition we know that its Blaschke metric G is flat and its Fubini-Pick form C is parallel with respect to G. We can choose a local coordinate system $(u^1, u^2, \ldots, u^n)$ for M such that

$$G = \sum_{i,j=1}^{n} G_{ij}\, du^i du^j = \sum_{i=1}^{n} \varepsilon_i (du^i)^2, \qquad \varepsilon_i \in \{+1, -1\}. \tag{2.1}$$

Thus we have a parallel basis $\{e_1, e_2, \ldots, e_n\}$ for TM, $e_i = \dfrac{\partial f}{\partial u^i}$, satisfying

$$G_{ij} = G(e_i, e_j) = \varepsilon_i \delta_{ij}. \tag{2.2}$$

Let $\{C_{ijk}\}$ be the components of the symmetric cubic form C with respect to the basis $\{e_i\}$. Since $\{e_i\}$ and C are parallel with respect to G, we know that

$$C_{ijk} = C_{ikj} = C_{jik} = \text{const.}, \qquad \forall i, j, k. \tag{2.3}$$

We denote by e_{n+1} the equiaffine normal for M in $\mathbb{R}^{n+1}$. In general there are two possibilities to choose the equiaffine normal. But in order to fix the Blaschke metric G we need to fix e_{n+1} such that it is in the same direction of $f - f_0$, where f_0 is the center of f. Then the structure equations read:

$$de_i = \sum_{j,k=1}^{n} C_{ij}^k\, du^i e_k + \varepsilon_i\, du^i e_{n+1}, \qquad i = 1, 2, \ldots, n, \tag{2.4}$$

$$de_{n+1} = -H \sum_{i=1}^{n} du^i e_i, \tag{2.5}$$

where $C_{ij}^k = \varepsilon_k C_{ijk}$ and H is the mean curvature of M in $\mathbb{R}^{n+1}$. By the choice of e_{n+1} we know that $H<0$, so by an affine transformation we may assume that $H=-1$. The apolarity condition gives

$$\sum_{i=1}^{n} C_{ij}^i = 0, \quad j=1,2,\ldots,n. \tag{2.6}$$

We denote by $(e_1, e_2, \ldots, e_{n+1})$ the matrix for the basis $\{e_\alpha\}$, then (2.4) and (2.5) can be written by

$$d(e_1, e_2, \ldots, e_{n+1}) = (e_1, e_2, \ldots, e_{n+1})\left(\sum_{i=1}^{n} A_i du^i\right), \tag{2.7}$$

where $\{A_i\}$ are the constant matrices given by

$$A_i = \begin{pmatrix} C_{i1}^1 & \ldots & C_{ii}^1 & \ldots & C_{in}^1 & 0 \\ . & \ldots & . & \ldots & . & \ldots \\ C_{i1}^i & \ldots & C_{ii}^i & \ldots & C_{in}^i & 1 \\ . & \ldots & . & \ldots & . & \ldots \\ C_{i1}^n & \ldots & C_{ii}^n & \ldots & C_{in}^n & 0 \\ 0 & \ldots & \varepsilon_i & \ldots & 0 & 0 \end{pmatrix}, \quad i=1,2,\ldots,n, \tag{2.8}$$

where both the $(n+1)$th column and row have only one zon-zero element. By differentiating (2.7) we get

$$[A_i, A_j] = A_i A_j - A_j A_i = 0, \quad \forall i,j, \tag{2.9}$$

(cf. Li [7]). From (2.6) we have

$$\operatorname{tr}(A_i) = 0, \quad i=1,2,\ldots,n. \tag{2.10}$$

In order to rewrite (2.3) as a restricting condition on $\{A_i\}$ we introduce in $\mathbb{R}^{n+1}$ the indefinite inner product $\langle\ ,\ \rangle$,

$$\langle x, x\rangle = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \ldots + \varepsilon_n x_n^2 + x_{n+1}^2, \quad x = {}^t(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}, \tag{2.11}$$

where $\{\varepsilon_i\}$ are given by (2.1). Then we can easily verify that

$$\langle A_i x, y\rangle = \langle x, A_i y\rangle, \quad \forall x, y \in \mathbb{R}^{n+1}, \quad i=1,2,\ldots,n, \tag{2.12}$$

i.e., $\{A_i\}$ are self-adjoint with respect to $\langle\ ,\ \rangle$. Moreover, if we denote $\xi = {}^t(0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$, then $\{A_i\}$ has the property that $\{A_1\xi, A_2\xi, \ldots, A_n\xi, \xi\}$ is an orthonormal basis for $\langle\ ,\ \rangle$ with determinant 1.

Conversely, we can show that

Theorem 2.1 *Let* $\langle x, x\rangle = \sum_{i=1}^{n} \varepsilon_i x_i^2 + x_{n+1}^2$ *be an inner product in* $\mathbb{R}^{n+1}$ *with* $\varepsilon_i \in \{+1, -1\}$. *If there exist n matrices* $\{A_i\}$ *satisfying* (i) $\langle A_i x, y\rangle = \langle x, A_i y\rangle$, $\forall x, y, i$; (ii) $[A_i, A_j]=0$, $\forall i,j$; (iii) $\operatorname{tr}(A_i)=0$, $\forall i$; (iv) $\exists \xi \in \mathbb{R}^{n+1}$ *such that*

$\{A_1\xi, A_2\xi, \dots, A_n\xi, \xi\}$ is an orthonormal basis for $\langle\,,\rangle$ with determinant 1, *then there exists a canonial affine hypersphere $f: \mathbb{R}^n \to \mathbb{R}^{n+1}$ with the Blaschke metric $G=\sum_{i=1}^{n} \varepsilon_i (du^i)^2$, where $(u^1, u^2, \dots, u^n)$ are the coordinates for $\mathbb{R}^n$.*

Proof. Let $S=(A_1\xi, \dots, A_n\xi, \xi)$. We define $B_i=S^{-1}A_iS$, $i=1,2,\dots,n$. By (iv) we know that S is an orthogonal matrix with respect to $\langle\,,\rangle$, so $\{B_i\}$ also have the properties (i), (ii) and (iii). Moreover, if we denote $v={}^t(0,\dots,0,1)\in\mathbb{R}^{n+1}$, then

(2.13) $$B_i v = S^{-1}A_i(Sv) = S^{-1}(A_i\xi) = {}^t(0,\dots,0,1,0,\dots,0), \qquad i=1,2,\dots,n.$$

From (2.13) and the fact that B_i is self-adjoint with respect to $\langle\,,\rangle$ we know that

(2.14) $$B_i=\begin{pmatrix} C^1_{i1} & \dots & C^1_{ii} & \dots & C^1_{in} & 0 \\ \cdot & \dots & \cdot & \dots & \cdot & \dots \\ C^i_{i1} & \dots & C^i_{ii} & \dots & C^i_{in} & 1 \\ \cdot & \dots & \cdot & \dots & \cdot & \dots \\ C^n_{i1} & \dots & C^n_{ii} & \dots & C^n_{in} & 0 \\ 0 & \dots & \varepsilon_i & \dots & 0 & 0 \end{pmatrix}, \quad i=1,2,\dots,n,$$

for some constants $\{C^k_{ij}\}$ with $\varepsilon_j C^j_{ik}=\varepsilon_k C^k_{ij}$. From $[B_i, B_j]=0$ we get in particular $\varepsilon_i C^i_{jk}=\varepsilon_j C^j_{ik}$. Thus we know that $C_{ijk}=\varepsilon_k C^k_{ij}$ are totally symmetric. Now we consider the linear system

(2.15) $$d(e_1, e_2, \dots, e_{n+1})=(e_1, e_2, \dots, e_{n+1})\left(\sum_{i=1}^{n} B_i du^i\right)$$

with the initial value condition

(2.16) $$\det(e_1(o), e_2(o), \dots, e_{n+1}(o))=1.$$

This system is completely integrable because of $[B_i, B_j]=0$. Given any initial values satisfy (2.16) we get an unique solution $(e_1, e_2, \dots, e_{n+1})$ of (2.15), which is determined by $\{B_i\}$ up to linear transformations in $\mathrm{SL}(n+1)$. Since $C^k_{ij}=C^k_{ji}$, we have $\frac{\partial e_i}{\partial u^j}=\frac{\partial e_j}{\partial u^i}$, so we can get an unique solution $f: \mathbb{R}^n \to \mathbb{R}^{n+1}$ (up to constant vectors in $\mathbb{R}^{n+1}$) from the equation

(2.17) $$df=\sum_{i=1}^{n} e_i du^i.$$

It is clear that f is determined by $\{B_i\}$ up to equiaffine transformations in $\mathbb{R}^{n+1}$. One can easily see that f is a proper canonial affine hypersphere with Blaschke metric $G=\sum_{i=1}^{n} \varepsilon_i (du^i)^2$. Q.E.D.

Example 2.2 Let J_s be the $(n+1)\times(n+1)$ matrix

$$J_s=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\oplus\ldots\oplus\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\oplus(1)\ldots\oplus(1) \tag{2.18}$$

with s copies of $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, $s=0,1,\ldots,\left[\frac{n+1}{2}\right]$. We define $\langle\,,\,\rangle$ in $\mathbb{R}^{n+1}$ by

$$\langle x,x\rangle={}^t x J_s x=(-x_1^2+x_2^2)+\ldots+(-x_{2s-1}^2+x_{2s}^2)+x_{2s+1}^2+\ldots+x_{n+1}^2 \tag{2.19}$$

for $x={}^t(x_1,x_2,\ldots,x_{n+1})\in\mathbb{R}^{n+1}$. Let

$$\xi=\frac{1}{\sqrt{n+1}}\,{}^t((0,\sqrt{2}),\ldots,(0,\sqrt{2}),1,\ldots,1)\in\mathbb{R}^{n+1}$$

with s copies of $(0,\sqrt{2})$. Then we have $\langle\xi,\xi\rangle=1$. So we can extend ξ to an orthonormal basis $(\eta_1,\ldots,\eta_n,\xi)$ for $\langle\,,\,\rangle$ with determinant 1. We write

$$\eta_i=\frac{1}{\sqrt{n+1}}\,{}^t((\sqrt{2}b_{i1},\sqrt{2}a_{i1}),\ldots,(\sqrt{2}b_{is},\sqrt{2}a_{is}),a_{i2s+1},\ldots,a_{in+1}),\quad i=1,2,\ldots,n, \tag{2.20}$$

and define

$$A_i=\begin{pmatrix}a_{i1} & b_{i1}\\ -b_{i1} & a_{i1}\end{pmatrix}\oplus\ldots\oplus\begin{pmatrix}a_{is} & b_{is}\\ -b_{is} & a_{is}\end{pmatrix}\oplus(a_{i2s+1})\oplus\ldots\oplus(A_{in+1}). \tag{2.21}$$

then $(A_1\xi,A_2\xi,\ldots,A_n\xi,\xi)=(\eta_1,\eta_2,\ldots,\eta_n,\xi)$ is an orthonormal basis for $\langle\,,\,\rangle$. Moreover, one can easily verify that $\{A_i\}$ are mutually commutative and self-adjoint with respect to $\langle\,,\,\rangle$, i.e., ${}^tA_iJ_s=J_sA_i$, and

$$\operatorname{tr}(A_i)=(2a_{i1}+\ldots+2a_{is}+a_{i2s+1}+\ldots+a_{in+1})=(n+1)\langle\eta_i,\xi\rangle=0.$$

Thus by Theorem 2.1 we know that $(A_1,A_2,\ldots,A_n)$ define a proper canonical affine hypersphere $x\colon\ \mathbb{R}^n\to\mathbb{R}^{n+1}$ with $G=\sum_{i=1}^{s}[-(du^{2i-1})^2+(du^{2i})^2]+\sum_{i=2s+1}^{n}(du^i)^2$.

Now we show that x is affinely equivalent to the hypersphere given by (1.4). Let

$$S=(A_1\xi,\ldots,A_n\xi,\xi),\qquad B_i=S^{-1}A_iS \tag{2.22}$$

as in the proof of Theorem 2.1. We get a solution

$$(e_1,e_2,\ldots,e_{n+1})=\exp\left(\sum_{i=1}^{n}A_iu^i\right)S \tag{2.23}$$

of (2.15) with the initial value condition $\det(e_1(o), e_2(o), \ldots, e_{n+1}(o)) = \det(S) = 1$. Since x is an affine hypersphere with $H = -1$, we may assume that $x = e_{n+1}$. By (2.22) and (2.23) we have

$$x = {}^t(x_1, x_2, \ldots, x_{n+1}) = e_{n+1} = \exp\left(\sum_{i=1}^{n} A_i u^i\right)\xi. \tag{2.24}$$

Thus from (2.21) we get

$$\begin{pmatrix} x_{2j-1} \\ x_{2j} \end{pmatrix} = \frac{1}{\sqrt{n+1}} \exp\left(\sum_{i=1}^{n} \begin{pmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{pmatrix} u^i\right)\begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}, \quad j = 1, 2, \ldots, s;$$

$$x_j = \frac{1}{\sqrt{n+1}} \exp\left(\sum_{i=1}^{n} a_{ij} u^i\right), \quad j \geqq 2s+1.$$

So we have

$$(x_{2j-1}^2 + x_{2j}^2) = (x_{2j-1}, x_{2j})\begin{pmatrix} x_{2j-1} \\ x_{2j} \end{pmatrix} = \frac{2}{n+1} \exp\left(2\sum_{i=1}^{n} a_{ij} u^i\right), \quad j = 1, 2, \ldots, s,$$

and

$$\begin{aligned}(x_1^2 + x_2^2)\ldots(x_{2s-1}^2 + x_{2s}^2)x_{2s+1}\ldots x_{n+1} &= 2^s(n+1)^{-\frac{n+1}{2}} \exp\left(\sum_{i=1}^{n} \operatorname{tr}(A_i)u^i\right) \\ &= 2^s(n+1)^{-\frac{n+1}{2}}.\end{aligned}$$

Note that $s = 0$ gives the hypersphere (1.1).

Remark 2.3 From the fundamental theorem of equiaffine geometry we know that to determine all the canonical proper hyperspheres is equivalent to determine all the constant solutions $\{C_{ijk}\}$ and $\{\varepsilon_i\} \in \{+1, -1\}$ for the quadratic equation system

$$\begin{aligned} &\sum_{r=1}^{n} \varepsilon_r (C_{ikr} C_{jmr} - C_{ijr} C_{kmr}) = -\varepsilon_i \varepsilon_k (\delta_{ij}\delta_{km} - \delta_{im}\delta_{kj}) \\ &\sum_{r=1}^{n} \varepsilon_r C_{rrj} = 0 \\ &C_{ijk} = C_{jik} = C_{ikj}. \end{aligned} \tag{2.25}$$

For any $s = 1, 2, \ldots, \left[\frac{n+1}{2}\right]$ we get solutions $\{C_{ijk}\}$ and $\{\varepsilon_i\}$ for (2.25) by (2.19), (2.21), (2.22) and (2.14). Thus the assertion (ii) in § 1 claims that all solutions for (2.25) can be defined by this way.

3 The proof of the algebraic lemma

In order to prove the main theorem we establish first the algebraic lemma stated in § 1.

Let $\{A_i\}=\{A_1, A_2, \ldots, A_m\}$ be finite mutually commutative and self-adjoint linear operators in $\mathbb{R}^{n+1}$ with respect to the Lorentz inner product $\langle\ ,\ \rangle$,

$$(3.1)\qquad \langle x, x\rangle=-x_1^2+x_2^2+\ldots+x_{n+1}^2, \qquad x={}^t(x_1, x_2, \ldots, x_{n+1})\in\mathbb{R}^{n+1}.$$

Let $\mathbb{C}^{n+1}=\mathbb{R}^{n+1}\otimes\mathbb{C}$ be the complexification of $\mathbb{R}^{n+1}$. We can extend $\langle\ ,\ \rangle$ to the Hermitian Lorentz inner product in $\mathbb{C}^{n+1}$ defined by

$$(3.2)\qquad \langle z, w\rangle=-z_1\overline{w_1}+z_2\overline{w_2}+\ldots+z_{n+1}\overline{w_{n+1}}$$

for $z={}^t(z_1, z_2, \ldots, z_{n+1})$ and $w={}^t(w_1, w_2, \ldots, w_{n+1})$ in $\mathbb{C}^{n+1}$. We extend $\{A_i\}$ to the real linear operators in $\mathbb{C}^{n+1}$ by

$$(3.3)\qquad A_i(x+\sqrt{-1}\,y)=A_i x+\sqrt{-1}\,A_i y, \qquad \forall x, y\in\mathbb{R}^{n+1}.$$

It is easy to see that for any z, w in $\mathbb{C}^{n+1}$ we have

$$(3.4)\qquad \langle A_i z, w\rangle=\langle z, A_i w\rangle, \qquad i=1, 2, \ldots, m,$$

i.e., $\{A_i\}$ are real self-adjoint operators in $\mathbb{C}^{n+1}$.

Proposition 3.1 *Let $W\subset\mathbb{C}^{n+1}$ be a complex invariant subspace of $\{A_i\}$ with* $\dim W\geqq 1$. *Then for any j, given any eigenvalue λ_j of A_j there exists a common eigenvector $\eta\in W$ of $\{A_i\}$ such that $A_j\eta=\lambda_j\eta$.*

Proof. We may assume that $j=1$. *Let* $x_1\in W$ be an λ_1-eigenvector of A_1, i.e., $A_1x_1=\lambda_1x_1$. We define

$$W_1=\operatorname{span}_{\mathbb{C}}\{x_1, A_2x_1, \ldots, A_2^k x_1, \ldots\}.$$

Then W_1 is a subspace of W with $\dim W_1\geqq 1$. Since $A_2W_1\subset W_1$, A_2 has an eigenvector $x_2\in W_1$ and $x_2=p_2(A_2)x_1$ for some polynomial p_2. Similarly A_3 has an eigenvector $x_3=p_3(A_3)x_2=p_3(A_3)p_2(A_2)x_1$ in the invariant subspace $W_2=\operatorname{span}_{\mathbb{C}}\{x_2, A_3x_2, \ldots, A_3^k x_2, \ldots\}$. Since $\{A_i\}$ are finite, by repeating this process we will get an eigenvector x_m of A_m, $x_m=p_m(A_m)x_{m-1}=\ldots$ $=p_m(A_m)p_{m-1}(A_{m-1})\ldots p_2(A_2)x_1$. Since $\{A_i\}$ are commutative and x_i is an eigenvector of A_i, $i=1, 2, \ldots, m$, we know that x_m is a common eigenvector of $\{A_i\}$ such that $A_1x_m=\lambda_1x_m$. Q.E.D.

Proposition 3.2 *Let $\xi, \eta\in\mathbb{C}^{n+1}$ be λ-, μ-eigenvalue of A_j respectively. If $\lambda\neq\bar{\mu}$, then $\langle\xi, \eta\rangle=0$.*

Proposition 3.3 *Let $V\subset\mathbb{R}^{n+1}$ be an invariant subspace of $\{A_i\}$. Let $\eta\in V\otimes\mathbb{C}\subset\mathbb{C}^{n+1}$ be a common eigenvector of $\{A_i\}$. If $\langle\eta, \eta\rangle\neq 0$, then there is a real common eigenvector $x\in V$ ($x=\operatorname{Re}\eta$ or $\operatorname{Im}\eta$) such that $A_i x=\lambda_i x$, $\lambda_i\in\mathbb{R}$, $i=1, 2, \ldots, m$, and $\langle x, x\rangle\neq 0$. Moreover, we have the orthonogal decomposition $V=\mathbb{R}x\oplus x^\perp$, where $x^\perp=\{y\in V\,|\,\langle y, x\rangle=0\}$ which is also an invariant subspace of $\{A_i\}$.*

Proposition 3.2 and 3.3 follow immediately from the fact that $\{A_i\}$ are self-adjoint and that $\langle\ ,\ \rangle$ is a Hermitian inner product.

Proposition 3.4 *Suppose that* $\langle z, z\rangle = \langle z, w\rangle = \langle w, w\rangle = 0$ *for* z, w *in* $\mathbb{C}^{n+1}$, *then* z *and* w *are linearly dependent.*

Proof. Let $z = {}^t(z_1, z_2, \ldots, z_{n+1})$ and $w = {}^t(w_1, w_2, \ldots, w_{n+1})$. Then we can find $(k_1, k_2) \neq 0$ in $\mathbb{C}^2$ such that $k_1 z_1 + k_2 w_1 = 0$. Since $\langle k_1 z + k_2 w, k_1 z + k_2 w\rangle = 0$, we conclude from (3.2) that $k_1 z + k_2 w = 0$. Q.E.D.

It is clear that we have an irreducible orthonogal decomposition

$$\mathbb{R}^{n+1} = V_1 \oplus V_2 \oplus \ldots \oplus V_\gamma, \tag{3.5}$$

where V_k, $k = 1, 2, \ldots, \gamma$, are irreducible invariant subspaces of $\{A_i\}$ with $\dim V_k \geqq 1$.

Proposition 3.5 *There exists at most one* k *such that* $\dim V_k \geqq 1$.

Proof. Since $\langle\ ,\ \rangle$ is nondegenerate in each V_k and the maximal negative definite subspace of $\langle\ ,\ \rangle$ has dimension 1, we know that all but one V_k are positive definite subspaces. Since any positive definite invariant subspace of $\{A_i\}$ with dimension greater than 1 is reducible, so there is at most one k with $\dim V_k > 1$. Q.E.D.

Thus we can find an orthonormal subbasis $\{e_{\nu+1}, e_{\nu+2}, \ldots, e_{n+1}\}$ for $\mathbb{R}^{n+1}$ consisting of common eigenvectors of $\{A_i\}$ such that we have the orthonogal decomposition

$$\mathbb{R}^{n+1} = V_0 \oplus \mathbb{R}\, e_{\nu+1} \oplus \ldots \oplus \mathbb{R}\, e_{n+1}, \tag{3.6}$$

where V_0 is an irreducible non-positive definite invariant subspace of $\{A_i\}$ of dimension ν.

If $\dim V_0 = 1$, then $\{A_i\}$ is of Type I as in the lemma in § 1, i.e., $\{A_i\}$ can be simultaneously diagonalized. In the rest of this section we will always assume that $\dim V_0 \geqq 2$. It follows from Proposition 3.3 that

Proposition 3.6 *Let* $\eta \in V_0 \otimes \mathbb{C}$ *be any common eigenvector of* $\{A_i\}$, *then* $\langle \eta, \eta\rangle = 0$.

Proposition 3.7 *Each* $A_j: V_0 \to V_0$ *has either* (i): *only two non-real dual eigenvalues* λ_j *and* $\bar{\lambda}_j$; *or* (ii): *only one real eigenvalue* λ_j.

Proof. Let λ_j and μ_j $(\lambda_j \neq \mu_j)$ be two eigenvalues of $A_j: V_0 \to V_0$. By Proposition 3.1 we have two common eigenvectors ξ and η of $\{A_i\}$ in $V_0 \otimes \mathbb{C}$ such that $A_j \xi = \lambda_j \xi$, $A_j \eta = \mu_j \eta$. Then Proposition 3.6 implies that $\langle \xi, \xi\rangle = \langle \eta, \eta\rangle = 0$. Thus by Proposition 3.4 we have $\langle \xi, \eta\rangle \neq 0$. So by Proposition 3.2 we have $\lambda_j = \bar{\mu}_j$. Therefore, $A_j: V_0 \to V_0$ has either (i): only two non-real dual eigenvalues λ_j and $\bar{\lambda}_j$; or (ii): only one eigenvalue λ_j. In case (ii) λ_j must be real since A_j is real. Q.E.D.

Definition 3.8 $\{A_i\}$ is said to be (i): *of Type* II if there is j such that $A_j: V_0 \to V_0$ has two non-real dual eigenvalues λ_j and $\bar{\lambda}_j$; (ii): *of Type* III if each $A_i: V_0 \to V_0$ has only one real eigenvalue λ_i.

Proposition 3.9 *If* $\{A_i\}$ *is of Type* II, *then* $\dim V_0=2$. *Moreover, there exists a basis* $\{e_1,e_2\}$ *for* V_0 *with* $\langle e_1,e_1\rangle=-\langle e_2,e_2\rangle=-1$ *and* $\langle e_1,e_2\rangle=0$ *such that*

$$A_i(e_1,e_2)=(e_1,e_2)\begin{pmatrix} a_i & b_i \\ -b_i & a_i\end{pmatrix}, \quad i=1,2,\ldots,m; \tag{3.7}$$

where a_i, $b_i\in\mathbb{R}$ *and at least one* b_i *is non-zero.*

Proof. Let $\eta\in V_0\otimes\mathbb{C}$ be a common eigenvector of $\{A_i\}$. We may assume that $A_i\eta=\lambda_i\eta$, $i=1,2,\ldots,m$. We write $\eta=x+\sqrt{-1}\,y$ with $x,y\in V_0$. Since some of λ_i are non-real, we know that x and y are linearly independent. By Proposition 3.6 we have $\langle\eta,\eta\rangle=\langle x,x\rangle+\langle y,y\rangle=0$. Then it follows from Proposition 3.4 that $\langle\eta,\bar\eta\rangle=\langle x,x\rangle-\langle y,y\rangle+2\sqrt{-1}\langle x,y\rangle\neq 0$. By letting $\xi=\rho\eta$ with $\rho^4=\frac{\langle\bar\eta,\eta\rangle}{\langle\eta,\bar\eta\rangle}$ we have $\langle\operatorname{Re}\xi,\operatorname{Im}\xi\rangle=0$. Since $\langle\xi,\xi\rangle=0$, $\langle\operatorname{Re}\xi,\operatorname{Re}\xi\rangle=-\langle\operatorname{Im}\xi,\operatorname{Im}\xi\rangle$ is non-zero. Thus we may assume $\xi=e_1+\sqrt{-1}\,e_2$ and $\langle e_1,e_1\rangle=-\langle e_2,e_2\rangle=-1$, $\langle e_1,e_2\rangle=0$. By letting $\lambda_i=a_i+\sqrt{-1}\,b_i$ we get (3.7). Since $V=\operatorname{span}_{\mathbb{R}}\{e_1,e_2\}$ is an invariant subspace of $\{A_i\}$ and $V\cap V^\perp=\{0\}$, by the irreducibility of V_0 we know that $V_0=V$. Q.E.D.

In the rest of this section we assume that $\{A_i\}$ is of Type III. Let V_0 be as in (3.6) and λ_i as in Definition 3.8 we have

Proposition 3.10 *Let* γ *be the smallest positive integer such that* $(A_i-\lambda_i I)^\gamma=0$ *in* V_0 *for all* $i=1,2,\ldots,m$, *then* $\gamma\leqq 3$.

Proof. Let $x\in V_0\otimes\mathbb{C}$ be a common eigenvector of $\{A_i\}$, then $A_i x=\lambda_i x$, $i=1,2,\ldots,m$. Since $\{A_i\}$ and $\{\lambda_i\}$ are real, we may assume that $x\in V_0$. By the definition of γ we can find j and $u\in V_0$ such that $(A_j-\lambda_j I)^{\gamma-1}u\neq 0$. We define $y=(A_j-\lambda_j I)^{\gamma-2}u$. If $\gamma\geqq 4$, then $2\gamma-4\geqq\gamma$, we have $\langle y,y\rangle=\langle u,(A_j-\lambda_j I)^{2\gamma-4}u\rangle=0$ and $\langle y,x\rangle=0$. But Proposition 3.6 implies $\langle x,x\rangle=0$, we conclude from Proposition 3.4 that $y=kx$ for some $k\in\mathbb{R}$. Thus $(A_j-\lambda_1 I)y=(A_j-\lambda_j I)^{\gamma-1}u=0$, we get a contradiction. Q.E.D.

Proposition 3.11 *Suppose that* $\gamma=2$, *then* $\dim V_0=2$. *Let* j *satisfy* $(A_j-\lambda_j I)\neq 0$ *in* V_0, *then there is a basis* $\{x,y\}$ *for* V_0 *with* $\langle x,y\rangle=1$ *and* $\langle x,x\rangle=\langle y,y\rangle=0$ *such that*

$$A_i(x,y)=(x,y)\begin{pmatrix}\lambda_i & a_i\\ 0 & \lambda_i\end{pmatrix}, \quad i=1,2,\ldots,m;\quad a_i\in\mathbb{R};\quad a_j=\pm 1. \tag{3.8}$$

Proof. Let $x\in V_0$ be a common eigenvector of $\{A_i\}$. Then $A_i x=\lambda_i x$, $i=1,2,\ldots,m$, and $\langle x,x\rangle=0$. Let $u\in V_0$ with $\langle u,x\rangle\neq 0$, then u and x are linearly independent. Since $\langle(A_i-\lambda_i I)u,x\rangle=0$ and $\langle(A_i-\lambda_i I)u,(A_i-\lambda_i I)u\rangle=0$, we have by Proposition 3.4 that $(A_i-\lambda_i I)u=b_i x$, $i=1,2,\ldots,m$; $b_i\in\mathbb{R}$. Thus $V=\operatorname{span}_{\mathbb{R}}\{u,x\}$ is an invariant subspace of $\{A_i\}$ with $V\cap V^\perp=\{0\}$. By the irreducibility of V_0 we have $V_0=V$, so $\dim V_0=2$. Since $(A_j-\lambda_j I)\neq 0$ in V_0, we have $(A_j-\lambda_j I)u=b_j x\neq 0$. By modifying u we may assume $\langle(A_j-\lambda_j I)u,u\rangle=\varepsilon\in\{+1,-1\}$. Then by modifying x we have $\langle u,x\rangle=1$. Now let $y=u-\frac{\langle u,u\rangle}{2}x$, we get $\langle x,x\rangle=\langle y,y\rangle=0$

and $\langle x, y\rangle = 1$. Moreover, $(A_i - \lambda_i I)y = a_i x$ for some $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, and $a_j = \varepsilon$. Q.E.D.

Proposition 3.12 *Suppose that $\gamma = 3$ and j satisfies $(A_j - \lambda_j I)^2 \neq 0$ in V_0. Then there is a basis $\{x, y, e_3, \ldots, e_n\}$ for V_0 with $\langle x, x\rangle = \langle y, y\rangle = \langle x, e_\alpha\rangle = \langle y, e_\alpha\rangle = 0$, $\langle x, y\rangle = 1$ and $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}$, $3 \leq \alpha, \beta \leq r$, such that we have the matrix representations*

$$(3.9) \qquad A_i = \begin{pmatrix} \lambda_i & a_{i2} & a_{i3} & a_{i4} & \ldots & a_{iv} \\ 0 & \lambda_i & 0 & 0 & \ldots & 0 \\ 0 & a_{i3} & \lambda_i & & & \\ 0 & a_{i4} & & \lambda_i & & \\ \vdots & \vdots & & & & \\ 0 & a_{iv} & & 0 & & \lambda_i \end{pmatrix}, \quad i = 1, 2, \ldots, m,$$

and $a_{ik} \in \mathbb{R}$ with $a_{j3} = 1$, $a_{j2} = a_{j4} = \ldots = a_{jr} = 0$.

Proof. Let $x \in V_0$ be a common eigenvector for $\{A_i\}$, then $A_i x = \lambda_i x$ for all i and $\langle x, x\rangle = 0$. Let $u \in V_0$ with $(A_j - \lambda_j I)^2 u \neq 0$. Since $\langle (A_j - \lambda_j I)u, x\rangle = 0$, we know by Proposition 3.4 that $v = (A_j - \lambda_j I)u$ satisfies $\langle v, v\rangle \neq 0$. We claim that $\langle v, v\rangle > 0$. Otherwise we have $\langle v, v\rangle < 0$, then we have the orthonogal decomposition $V_0 = \mathbb{R}v \oplus v^\perp$. Since $\langle\ ,\ \rangle$ is negative definite in $\mathbb{R}v$, it must be positive definite in $v^\perp$. But $x \in v^\perp$ and $\langle x, x\rangle = 0$, we get a contradiction. Thus by modifying u we may assume that $\langle v, v\rangle = 1$. By Proposition 3.4 and the fact that $\gamma = 3$ we know that $(A_j - \lambda_j I)^2 u = kx \neq 0$ for some $k \in \mathbb{R}$, so by modifying x we may assume that $k = 1$, i.e., $\langle u, x\rangle = \langle v, v\rangle = 1$. Now let $y = u - \frac{1}{2}\langle u, v\rangle v - [\frac{1}{2}\langle u, u\rangle - \frac{3}{8}\langle u, v\rangle^2] x$ and $e_3 = v - \frac{1}{2}\langle u, v\rangle x$, we have

$$(3.10) \qquad \langle x, x\rangle = \langle y, y\rangle = \langle x, e_3\rangle = \langle y, e_3\rangle = 0, \qquad \langle x, y\rangle = \langle e_3, e_3\rangle = 1.$$

We define $W = \mathrm{span}_{\mathbb{R}}\{x, y, e_3\}$, then $W \cap W^\perp = \{0\}$ and $V_0 = W \oplus W^\perp$. Since $\langle\ ,\ \rangle$ is indefinite in W, it must be positive definite in $W^\perp$. So $A_j\colon W^\perp \to W^\perp$ is diagonalizable. We can choose an orthonormal basis $\{e_4, e_5, \ldots, e_v\}$ for $W^\perp$ such that $A_j e_\alpha = \lambda_j e_\alpha$, $\alpha = 4, 5, \ldots, v$. Thus $\{x, y, e_3, e_4, \ldots, e_v\}$ is a basis for V_0. By Proposition 3.4 we know that $(A_i - \lambda_i I)^2 e_\alpha = k_{i\alpha} x$ for some $k_{i\alpha} \in \mathbb{R}$, $i = 1, 2, \ldots, m$; so $\langle (A_i - \lambda_i I)e_\alpha, (A_i - \lambda_i I)e_\alpha\rangle = \langle e_\alpha, k_{i\alpha} x\rangle = 0$. Again by Proposition 3.4 we get

$$(3.11) \qquad (A_i - \lambda_i I)e_\alpha = a_{i\alpha} x, \quad a_{i\alpha} \in \mathbb{R}, \quad i = 1, 2, \ldots, m; \quad \alpha = 3, 4, \ldots, v.$$

Using the formula $A_i y = \langle A_i y, y\rangle x + \langle A_i y, x\rangle y + \sum_{\alpha=3}^{v} \langle A_i y, e_\alpha\rangle e_\alpha$ and (3.11) we get

$$(3.12) \qquad A_i y = a_{i2} x + \lambda_i y + \sum_{\alpha=3}^{v} a_{i\alpha} e_\alpha, \quad i = 1, 2, \ldots, m, \quad a_{i2} = \langle A_i y, y\rangle.$$

Thus (3.11) and (3.12) imply (3.9). Furthermore, we have $(A_j - \lambda_j I)y = v - \frac{1}{2}\langle u, v\rangle x = e_3$, so we get $a_{j3} = 1$ and $a_{j2} = a_{j4} = \ldots = a_{jv} = 0$. Q.E.D.

Thus the algebraic lemma follows from the orthonogal decomposition (3.6), Proposition 3.9, 3.11 and 3.12.

4 The proof of the main theorem

Let $f\colon M\to\mathbb{R}^{n+1}$ be a proper canonical affine hypersphere and G its Blaschke metric. We assume that the maximal negative definite subspaces of G have dimension 1. Then as in § 2 we know that there exist n mutually commutative linear operators $\{A_i\}$ in $\mathbb{R}^{n+1}$ which are self-adjoint with respect to the Lorentz inner product $\langle\,,\rangle$,

$$(4.1)\qquad \langle x,x\rangle=-x_1^2+x_2^2+\dots+x_{n+1}^2,\qquad x={}^t(x_1,x_2,\dots,x_{n+1})\in\mathbb{R}^{n+1}.$$

Moreover, $\{A_i\}$ satisfy also the properties (i): $\operatorname{tr}(A_i)=0$, $i=1,2,\dots,n$; and (ii): $\exists\xi\in\mathbb{R}^{n+1}$ such that $\{A_1\xi,A_2\xi,\dots,A_n\xi,\xi\}$ is an orthonormal basis for $\mathbb{R}^{n+1}$ with respect to $\langle\,,\rangle$.

We know from the algebraic lemma in § 1 that $\{A_i\}$ have three possible types, namely the Type I, II and III.

If $\{A_i\}$ is of Type I, then there is an orthonormal basis $\{e_1,e_2,\dots,e_{n+1}\}$ for $\mathbb{R}^{n+1}$ such that we have the matrix representations

$$(4.2)\qquad A_i=(a_{i1})\oplus(a_{i2})\oplus\dots\oplus(a_{in+1}),\qquad i=1,2,\dots,n.$$

By the property (ii) we know that $A_1,A_2,\dots,A_n$ are linearly independent, so the matrix

$$(4.3)\qquad \mathbb{T}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n+1}\\ a_{21} & a_{22} & \dots & a_{2n+1}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn+1}\end{pmatrix}$$

has rank n, i.e., the kernal of $\mathbb{T}\colon\mathbb{R}^{n+1}\to\mathbb{R}^n$ has dimension 1. Now let $\xi=\sum_{\alpha=1}^{n+1}\xi_\alpha e_\alpha$. By the property (ii) we have

$$\langle A_i\xi,\xi\rangle=a_{i1}(-\xi_1^2)+\sum_{\alpha=2}^{n+1}a_{i\alpha}(\xi_\alpha)^2=0,\qquad i=1,2,\dots,n.$$

Thus ${}^t(-\xi_1^2,\xi_2^2,\dots,\xi_{n+1}^2)\in\ker\mathbb{T}$. But the property (i) implies that ${}^t(1,1,\dots,1)\in\ker\mathbb{T}$, so there is $k\in\mathbb{R}$ such that $(-\xi_1^2,\xi_2^2,\dots,\xi_{n+1}^2)=k(1,1,\dots,1)$. It is impossible for $\langle\xi,\xi\rangle=1$. Thus $\{A_i\}$ cannot be of Type I.

Let $\{A_i\}$ be of Type II. Then there is an orthonormal basis $\{e_1,e_2,\dots,e_{n+1}\}$ for $\mathbb{R}^{n+1}$ such that we have the matrix representations

$$(4.4)\qquad A_i=\begin{pmatrix} a_{i1} & a_{i2}\\ -a_{i2} & a_{i1}\end{pmatrix}\oplus(a_{i3})\oplus\dots\oplus(a_{in+1}),\qquad i=1,2,\dots,n.$$

Let $\xi=\sum_{\alpha=1}^{n+1}\xi_\alpha e_\alpha$. By the property (ii) we have

$$\langle A_i\xi,\xi\rangle=a_{i1}(\xi_2^2-\xi_1^2)+a_{i2}(-2\xi_1\xi_2)+\sum_{\alpha=3}^{n+1}a_{i\alpha}(\xi_\alpha)^2=0,\qquad i=1,2,\dots,n.$$

We define $\mathbb{T}$ by (4.3), then ${}^t(\xi_2^2-\xi_1^2, -2\xi_1\xi_2, \xi_3^2, \dots, \xi_{n+1}^2) \in \ker \mathbb{T}$. But the property (i) implies that ${}^t(2, 0, 1, \dots, 1) \in \ker \mathbb{T}$, so there is $k \in \mathbb{R}$ such that $(\xi_2^2-\xi_1^2, -2\xi_1\xi_2, \xi_3^2, \dots, \xi_{n+1}^2) = k(2, 0, 1, \dots, 1)$. Thus $\xi_1 = 0$. From $\langle \xi, \xi \rangle = 1$ we get $k = (n+1)^{-1}$. By changing e_α to $-e_\alpha$ if necessary we may assume that $\xi = \frac{1}{\sqrt{n+1}}(\sqrt{2}e_2 + e_3 + \dots + e_{n+1})$. As we know in Example 2.2 that $\{A_i\}$ determine the proper canonical hypersphere

$$(x_1^2+x_2^2)x_3 \dots x_{n+1} = 2(n+1)^{-\frac{n+1}{2}}. \tag{4.5}$$

Finally we assume that $\{A_i\}$ is of Type III. Then by the algebraic lemma we have a basis $\{x, y, e_3, \dots, e_{n+1}\}$ such that we have the matrix representations as in the lemma in § 1. We write

$$A_i = a_{i1} T_1 + a_{i2} T_2 + \dots + a_{in+1} T_{n+1}, \qquad i = 1, 2, \dots, n, \tag{4.6}$$

and $\xi = \xi_1 x + \xi_2 y + \sum_{\alpha=3}^{n+1} \xi_\alpha e_\alpha$. By the property (ii) we have

$$\langle A_i \xi, \xi \rangle = \sum_{\alpha=1}^{n+1} a_{i\alpha} \langle T_\alpha \xi, \xi \rangle = 0, \qquad i = 1, 2, \dots, n.$$

Thus ${}^t(\langle T_1 \xi, \xi \rangle, \dots, \langle T_{n+1} \xi, \xi \rangle) \in \ker \mathbb{T}$, where $\mathbb{T}$ is defined by (4.3). But the property (i) implies that ${}^t(v, 0, \dots, 0, 1, \dots, 1) \in \ker \mathbb{T}$, and we know that $\dim(\ker \mathbb{T}) = 1$, so there is $k \in \mathbb{R}$ such that $(\langle T_1 \xi, \xi \rangle, \dots, \langle T_{n+1} \xi, \xi \rangle) = k(v, 0, \dots, 0, 1, \dots, 1)$, where we have $v-1$ copies of 0. Since $v \geqq 2$, we get $0 = \langle T_2 \xi, \xi \rangle = \langle \xi_2 x, \xi \rangle = \xi_2^2 = 0$, i.e., $\xi_2 = 0$. So we have $\langle \xi, x \rangle = \xi_2 = 0$ and $\langle A_i \xi, x \rangle = \langle \xi, a_{i1} x \rangle = 0$, $i = 1, 2, \dots, n$. It is impossible because of the property (ii).

Thus we complete the proof of the main theorem.

Acknowledgements. I would like to thank Professor U. Simon and Professor U. Pinkall for their direction and Professor A.M. Li for helpful discussions.

References

1. Blaschke, W.: Vorlesungen über Differentialgeometrie. II. Affine Differentialgeometrie. Berlin Heidelberg New York: Springer 1923
2. Bokan, N., Nomizu, K., Simon, U.: Affine hypersurfaces with parallel cubic forms. Tôhoku Math. J., II. Ser. **42**, 101–108 (1990)
3. Calabi, E.: Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jögens. Mich. Math. J **5**, 105–126 (1958)
4. Calabi, E.: Complete affine hypersurfaces. I. Symp. Math. **10**, 19–38 (1972)
5. Cheng, S.Y., Yau, S.T.: Complete affine hypersurfaces, part I, the completeness of affine metrics. Commun. Pure Appl. Math. **36**, 839–866 (1986)
6. Li, A.M.: Calabi conjecture on hyperbolic affine hyperspheres. Math. Z. **203**, 483–491 (1990)
7. Li, A.M.: Some theorems in affine differential geometry. Acta Math. Sin. New Ser. **5** (no. 4), 345–354 (1989)
8. Li, A.M., Penn, G.: Uniqueness theorems in affine differential geometry, part II. Results Math. **13**, 308–317 (1988)
9. Magid, A., Ryan, P.: Flat affine spheres in $\mathbb{R}^3$. Geom. Dedicata **33** (no. 3), 277–288 (1990)

10. Magid, A., Ryan, P.: Affine 3-spheres with constant affine curvature. Trans. Am. Math. Soc. (to appear)
11. Nomizu, K.: Introduction to affine differential geometry, part I. Lecture Notes. Preprint 88-37. MPI (1988)
12. Nomizu, K.: A survey of recent results in affine differential geometry. In: Verstraelen, L., West, A. (eds.) Geometry and Topology of Submanifolds, III. Leeds-Conference 1990, pp. 227–256. Singapore: World Scientific 1991
13. Nomizu, K., Pinkall, U.: Cayley surfaces in affine differential geometry. Tôhoku Math. J., II. Ser. **41** (no. 4), 589–596 (1989)
14. Radon, J.: Zur Affingeometrie der Regelflächen. Leip. Ber. **70**, 147–155 (1918)
15. Sasaki, T.: Hyperbolic affine hyperspheres. Nagoya Math. J. **77**, 107–123 (1980)
16. Schneider, R.: Zur affinen Differentialgeometrie im Großen. II. Über eine Abschätzung der Pickschen Invariante auf Affinesphären. Math. Z. **102**, 1–8 (1967)
17. Simon, U.: Local classification of two dimensional affine spheres with constant curvature metric. Differ. Geom. Appl. **1**, 123–132 (1991)
18. Simon, U.: Recent developments in affine differential geometry. In: Bokan, N. et al. (eds.) Proceedings of the Conference Differential Geometry and its Applications, 1988, pp. 327–347. Dubrovnik, Yugoslavie: Novi Sad 1989
19. Simon, U.: Zur Entwicklung der affinen Differentialgeometrie nach Blaschke. In: Buran, W. et al. (eds.) W. Blaschke, Gesammelte Werke. Essen: Bd. IV, pp. 35–88. Thales 1985
20. Simon, U., Wang, C.P.: Local theory of affine 2-spheres. In: Greene, R.E., Yau, S.T. (eds.) Proceedings 1990 Summer Institute of Differential Geometry. (to appear)
21. Vrancken, L., Li, A.M., Simon, U.: Affine spheres with constant sectional curvature. Math. Z. **206**, 651–658 (1991)
22. Wang, C.P.: Some examples of complete hyperbolic affine 2-spheres in $\mathbb{R}^3$. In: Ferus, D. et al. (eds.) Proceedings Conf. Global Analysis Global Differential Geometry, Berlin 1990. (Lect. Notes Math., vol. 1481, pp. 272–280) Berlin Heidelberg New York: Springer 1991
23. Yu, J.H.: Affine spheres with constant sectional curvature in A^4. J. Sichuan Univ., Nat. Sci. Ed. **27**, no. 4 (1990)