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KMS-symmetric Markov semigroups*

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0 Introduction

A theory of symmetric Markov semigroups on W*-algebras was initiated by Albeverio and Høegh-Krohn in the seventies ([AH-K]), and has recently been extensively developed by Davies and Lindsay ([DL1,2]). The setting for this theory is a semi-finite algebra $\mathcal{M}$, and symmetry refers to a trace τ on the algebra. The construction and analysis take place on the Segal space $L^2(\mathcal{M},\tau)$ which, together with each of the interpolating spaces $L^p(\mathcal{M},\tau)$ $(1 \leq p \leq \infty)$, is a subspace of the topological *-algebra of τ-measurable operators acting (as closed densely defined operators, by the strong sense product) on $L^2(\mathcal{M},\tau)$. *Tracially symmetric* Markov semigroups on the algebra $\mathcal{M}$ correspond to closed Dirichlet forms on the Hilbert space $L^2(\mathcal{M},\tau)$, permitting the application of quadratic form techniques to the analysis of dynamical semigroups on von Neumann algebras. For an interesting application of this theory to heat kernel bounds on graphs, through ideas from non-commutative differential geometry, see [Da2] where Schrödinger operators are viewed, in various ways, as restrictions of non-commutative Dirichlet forms. Each such representation suggests a metric on the graph which is then used to define a gaussian-like kernel for comparison with the given kernel. Further connections with non-commmutative geometry have been revealed by Sauvageot ([Sa1,2]) who constructs the transverse heat semigroup on a Riemannian foliation W*-algebra and demonstrates a Feller-type property, namely invariance of the foliation C*-algebra.

In the present paper we extend this theory to the context of states on a W*-algebra to obtain a fully non-commutative theory. As remarked in [DL2] this extension is required for applications to the quantum theory of irreversible dynamics, where tracial symmetry corresponds to an infinite temperature approximation. We work with the Haagerup spaces $L^p(\mathcal{M})$ and Kosaki's interpolating

* Dedicated to the memory of Alberto Frigerio

spaces ([Ha3], [Kos]). Symmetry is defined relative to the *symmetric embedding* of $\mathscr{A}$ into $L^2(\mathscr{A})$ (with respect to the state φ), and this is crucial for our analysis. *φ-symmetry* or *KMS-symmetry*, for a semigroup (P_t) on $\mathscr{A}$ may be expressed formally by

$$\varphi(P_t a\, \sigma^{\varphi}_{-i/2}(b)) = \varphi(\sigma^{\varphi}_{i/2}(a)\, P_t b)$$

where σ^{φ} is the modular automorphism group of φ. It is closely related to the physical condition of *detailed balance*, however detailed balance is usually defined (up to a reversible part) as symmetry with respect to the *GNS embedding* of $\mathscr{A}$ into $\mathscr{H}_{\varphi}$ (see e.g. [KFGV]).

The paper is organised as follows. Section 1 contains our basic definitions and notation, and properties of Haagerup spaces and embeddings that are needed in later sections. Operators induced from $\mathscr{A}$ to $L^p(\mathscr{A})$ are reviewed in Section 2, together with KMS-symmetry and the Markov property, both on $\mathscr{A}$ and on $L^p(\mathscr{A})$. The Hille-Yosida relations between resolvent family, generator and contraction semigroup on a Hilbert space are collected for convenience in Section 3, and in Section 4 the generators of L^2-Markov semigroups are characterised. The Dirichlet property is introduced in Section 5 where it is shown to characterise those quadratic forms which generate symmetric L^2-Markov semigroups. These in turn correspond one to one with KMS-symmetric Markov semigroups on $\mathscr{A}$. In the final section some useful results on form and generator cores, and quadratic form characterisations of further positivity, are collected.

The main results (Theorems 5.7, 5.10 and 6.5) were announced in [GL]. Cipriani, in his recent PhD thesis, has also considered non-commutative Dirichlet forms, and associated Markov semigroups, in the context of von Neumann algebras in standard form ([Cip]). In his approach geometrical properties of the counterpart to our map $x \to x_{\wedge}$ (see Section 4) are exploited. He has developed a Perron-Frobenius theory for positivity preserving operators, and has shown that *hypercontractivity* is a sufficient condition on a symmetric Markov semigroup for the generator to have a (strictly) positive eigenvector with eigenvalue 0. In [DL1] unbounded derivations, in particular generators of *integrable* automorphism groups, are used to construct Markov semigroups. We hope to return to the consideration of the role of such groups and derivations in the non-tracial context in a future article.

Finally a few words about our notational conventions. We use $a, b, ..$ for elements of an algebra $\mathscr{A}$; $x, y, ..$ for measurable operators; S, T, P for linear operators on $\mathscr{A}$ and $L^p(\mathscr{A})$; $\operatorname{Dom} T$ for the domain of T; $L^p_h(\mathscr{A}), \operatorname{Dom}_h T$ etc. for the self-adjoint parts and $L^p_+(\mathscr{A})$ etc. for the non-negative parts.

1 Definitions and embeddings

Let $\mathscr{A}$ be a σ-finite von Neumann algebra acting in a Hilbert space H and ω_0 a faithful normal semifinite weight on $\mathscr{A}$. We denote by $\overset{\bullet}{\mathscr{A}}$ the crossed product $\mathscr{A} \rtimes_{\sigma} \mathbb{R}$, where $\sigma = \{\sigma_t\}_{t \in \mathbb{R}}$ is the modular automorphism group of $\mathscr{A}$ with respect to ω_0. The von Neumann algebra $\dot{\mathcal{M}}$ acts in $L^2(\mathbb{R};H)$ and is generated by the operators $\pi(a)$, $a \in \mathcal{M}$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by

$$\begin{aligned}(\pi(a)\xi)(t) &= \sigma_{-t}(a)\xi(t),\\ (\lambda(s)\xi)(t) &= \xi(t-s),\end{aligned}$$

where $\xi \in L^2(\mathbb{R};H)$ and $t \in \mathbb{R}$. Let $\{\theta_s\}_{s\in\mathbb{R}}$ be the dual action (of $\mathbb{R}$ on $\dot{\mathcal{M}}$) and τ the relatively invariant trace on $\dot{\mathcal{M}}$ (thus, $\tau \circ \theta_s = e^{-s}\tau$ for $s \in \mathbb{R}$). We denote by $\dot{\mathcal{M}}^{\sim}$ the topological $*$-algebra of all τ-measurable operators affiliated with $\dot{\mathcal{M}}$ (with the operations of strong addition and strong multiplication and with the measure topology). The Haagerup L^p-spaces ($1 \le p \le \infty$) are defined as

$$L^p(\mathcal{M}) = \{x \in \dot{\mathcal{M}}^{\sim} : \theta_s(x) = e^{-s/p}x,\ \forall s \in \mathbb{R}\}.$$

In particular, $L^\infty(\mathcal{M}) = \pi(\mathcal{M})$. Define now, for any $\omega \in \mathcal{M}_{*,+}$, an operator $k_\omega \in L^1_+(\mathcal{M})$ by $k_\omega = \frac{d\widetilde{\omega}}{d\tau}$, where $\widetilde{\omega}$ is the dual weight of ω. The mapping $\omega \mapsto k_\omega$ extends to a linear bijection κ of $\mathcal{M}_*$ onto $L^1(\mathcal{M})$. If $k \in L^1(\mathcal{M})$, we write ω_k for $\kappa^{-1}(k)$. A trace-like functional on $L^1(\mathcal{M})$, given by $\mathrm{tr}(k) = \omega_k(1)$, is used to define the p-th norm on $L^p(\mathcal{M})$ for $1 \le p < \infty$:

$$\|x\|_p = \mathrm{tr}(|x|^p)^{1/p}.$$

Additionally, $\|a\|_\infty = \|\pi^{-1}(a)\|$ for $a \in L^\infty(\mathcal{M})$. With these norms, $L^p(\mathcal{M})$ are Banach spaces, π is an isometric isomorphism of $\mathcal{M}$ onto $L^\infty(\mathcal{M})$, $\kappa : \omega \mapsto k_\omega$ is an isometry of $\mathcal{M}_*$ onto $L^1(\mathcal{M})$, and $L^2(\mathcal{M})$ is a Hilbert space with the scalar product $\langle x, y\rangle = \mathrm{tr}(x^*y)$.

The spaces $L^p(\mathcal{M})$ and the relations between them are independent of the choice of ω_0. In particular, we may (and shall) identify $\mathcal{M}$ with $L^\infty(\mathcal{M})$. We can also assume that $\mathcal{M}$ acts in H in a standard way, so that we are given a standard form $(\mathcal{M}, H, J, \mathcal{P})$ of $\mathcal{M}$. In such representations, corresponding to each $\omega \in \mathcal{M}_{*,+}$, there is a unique vector $\xi_\omega \in \mathcal{P}$ such that $\omega(\cdot) = \langle \xi_\omega, \cdot\xi_\omega\rangle_H$. Moreover, if ω is faithful then ξ_ω is a separating and cyclic vector for $\mathcal{M}$.

It is clear from the construction of the crossed product that (with our identification of $\mathcal{M}$ with $\pi(\mathcal{M})$)

$$\sigma_t(a) = \lambda(t)a\lambda(-t), \qquad t \in \mathbb{R},\quad a \in \mathcal{M}.$$

A useful formula for σ_t^ω is given below.

Lemma 1.1 *Let $\omega \in \mathcal{M}_{*,+}$ be faithful. Then*

$$\sigma_t^\omega(a) = k_\omega^{it} a k_\omega^{-it}.$$

Proof. We calculate ([Ha1], Theorem 4.7)

$$\begin{aligned}\sigma_t^{\omega}(a) &= (D\omega : D\omega_0)_t \sigma_t(a)(D\omega : D\omega_0)_{-t}\\ &= (D\widetilde{\omega} : D\widetilde{\omega}_0)_t (D\widetilde{\omega}_0 : D\tau)_t a (D\widetilde{\omega}_0 : D\tau)_{-t}(D\widetilde{\omega} : D\widetilde{\omega}_0)_{-t}\\ &= \left(\frac{d\widetilde{\omega}}{d\tau}\right)^{it} a \left(\frac{d\widetilde{\omega}}{d\tau}\right)^{-it} = k_\omega^{it} a k_\omega^{-it}.\end{aligned}$$

□

There is a particular standard form of $\mathcal{M}$, which is very well suited to our purposes, namely $(\mathcal{M}, L^2(\mathcal{M}), *, L^2_+(\mathcal{M}))$, with $\mathcal{M}$ acting on $L^2(\mathcal{M})$ by left strong multiplication ([Ter], Theorem 36). It is clear that the unique vector in $L^2_+(\mathcal{M})$ corresponding to ω is $k_\omega^{1/2}$. The following important lemma describes the way the modular operator Δ_ω acts in $L^2(\mathcal{M})$, for a faithful $\omega \in \mathcal{M}_{*,+}$.

Lemma 1.2 *For any* $\alpha \in [0, \frac{1}{2}]$ *and* $a \in \mathcal{M}$,

$$\Delta_\omega^{\alpha} a k_\omega^{1/2} = k_\omega^{\alpha} a k_\omega^{1/2-\alpha}.$$

Proof. Put $S_\beta^0 = \{z \in \mathbb{C} : 0 < \Re z < \beta\}$, $S_\beta = \overline{S_\beta^0}$. Since the function $z \mapsto k_\omega^z$ is analytic on $\{z \in \mathbb{C} : \Re z > 0\}$ (with measure topology on $\widetilde{\mathcal{M}}$, see [Ter], Lemma 18), the function $f_1 : z \mapsto \mathrm{tr}(b^* k_\omega^z a k_\omega^{1-z})$ is analytic on the strip S_1^0 for any $a, b \in \mathcal{M}$. On the other hand, the function $f_2 : z \mapsto \langle b k_\omega^{1/2}, \Delta_\omega^z a k_\omega^{1/2}\rangle$ is analytic on the strip $S_{1/2}^0$ and continuous on $S_{1/2}$ ([KaR], Lemma 9.2.12), for any $a, b \in \mathcal{M}$. We have (by Lemma 1.1, noting that $J_\omega = J = *$)

$$\begin{aligned}f_1(\frac{1}{2} + it) &= \mathrm{tr}(b^* k_\omega^{it+1/2} a k_\omega^{-it+1/2}) = \mathrm{tr}(b^* k_\omega^{1/2} \sigma_t^{\omega}(a) k_\omega^{1/2})\\ &= \langle b k_\omega^{1/2}, k_\omega^{1/2}\sigma_t^{\omega}(a)\rangle = \langle b k_\omega^{1/2}, J_\omega \sigma_t^{\omega}(a^*) k_\omega^{1/2}\rangle\\ &= \langle b k_\omega^{1/2}, \Delta_\omega^{1/2}\sigma_t^{\omega}(a) k_\omega^{1/2}\rangle = \langle b k_\omega^{1/2}, \Delta_\omega^{it+1/2} a k_\omega^{1/2}\rangle\\ &= f_2(\frac{1}{2} + it).\end{aligned}$$

Since both functions are analytic on $S_{1/2}^0$ and continuous on $\{z \in \mathbb{C} : \Re z = \frac{1}{2}\}$, we have $f_1(z) = f_2(z)$ for any $z \in S_{1/2}^0$. Since f_1 and f_2 obviously coincide at $t = 0$ the result follows. □

In fact it is easy to see that

$$\begin{aligned}f_1(it) &= \mathrm{tr}(b^* h_\omega^{it} a h_\omega^{-it} k_\omega) = \mathrm{tr}(b^* \sigma_t^{\omega}(a) k_\omega)\\ &= \langle b k_\omega^{1/2}, \sigma_t^{\omega}(a) k_\omega^{1/2}\rangle = \langle b k_\omega^{1/2}, \Delta_\omega^{it} a k_\omega^{1/2}\rangle\\ &= f_2(it),\end{aligned}$$

so that f_1 and f_2 coincide on $S_{1/2}$.

We record here a couple of properties of measurable operators and Haagerup L^p-spaces which will be useful later.

Lemma 1.3 *Let $\mathcal{M}$ be a semifinite von Neumann algebra with faithful normal semifinite trace τ, and let $x, y, z \in \widetilde{\mathcal{M}}$, with x injective and z having dense range. If $x \cdot y \cdot z = 0$ (strong multiplication), then $y = 0$.*

Proof. We have $x(y \cdot z) = 0$ on the τ-dense domain of xyz (see [Ter], Proposition 24), which implies $y \cdot z = 0$. Thus $z^* \cdot y^* = 0$ and $y^* = 0$, as above. □

Lemma 1.4 *Let $x, y \in L_h^{2p}(\mathcal{M})$, $p \geq 1$. Then*

$$\|x^2 - y^2\|_p \leq \|x - y\|_{2p} \|x + y\|_{2p}.$$

Proof. Combine $x^2 - y^2 = \frac{1}{2}\{(x - y)(x + y) + (x + y)(x - y)\}$ with Hölder's inequality. □

Bounded measurable functions on a finite measure space are automatically integrable. In the non-commutative context there is a choice of different ways in which $\mathcal{M}$ may be embedded into $\mathcal{M}_*$. For our purposes the symmetric embedding, with respect to a faithful normal state ω, is the most appropriate:

$$\iota = \iota^{(\omega)} : a \mapsto \langle \xi_\omega, \cdot\, \Delta_\omega^{1/2} a \xi_\omega \rangle_H ,$$

in view of its positivity. By Lemma 1.2, we get

$$\iota : a \mapsto \operatorname{tr}(k_\omega^{1/2} a k_\omega^{1/2} \cdot\ ).$$

We shall also need embeddings of $\mathcal{M}$ into $L^p(\mathcal{M})$ and of $L^p(\mathcal{M})$ into $L^1(\mathcal{M})$. We put

$$\iota_p = \iota_p^{(\omega)} : a \in \mathcal{M} \mapsto k_\omega^{1/2p} a k_\omega^{1/2p}$$

and

$$\kappa_p = \kappa_p^{(\omega)} : x \in L^p(\mathcal{M}) \mapsto k_\omega^{1/2p'} x k_\omega^{1/2p'}, \qquad \text{where } p' = 1 + (p-1)^{-1}$$

The following lemmas yield some information on the above families of embeddings.

Lemma 1.5 *(**Radon-Nikodym type theorem**) Let $h \in L_+^1(\mathcal{M})$ be non-singular. If $0 \leq x \leq h^{1/p}$ with $x \in \widetilde{\mathcal{M}}$, then $x = h^{1/2p} a h^{1/2p}$ for some $a \in \mathcal{M}_+$ with $\|a\| \leq 1$.*

Proof. ([Sch], Lemma 2.2c) □

Lemma 1.6 *For any $p \in [1, \infty]$, $\iota_p(\mathcal{M}_+)$ is dense in $L_+^p(\mathcal{M})$, $\iota_p(\mathcal{M}_h)$ in $L_h^p(\mathcal{M})$, and $\iota_p(\mathcal{M})$ in $L^p(\mathcal{M})$.*

Proof. It is enough to show that $k_\omega^{1/2p} \mathcal{M}_+ k_\omega^{1/2p}$ is dense in $L_+^p(\mathcal{M})$. For $p = 2$ it follows immediately from Lemma 1.2 and the uniqueness of the standard form ([Ha2]):

$$L_+^2(\mathcal{M}) = \mathcal{P} = \{\Delta_\omega^{1/4} a k_\omega^{1/2} : a \in \mathcal{M}_+\}^- = \{k_\omega^{1/4} a k_\omega^{1/4} : a \in \mathcal{M}_+\}^-.$$

Now take $x \in L^1_+(\mathcal{A})$ and fix $\varepsilon > 0$. There is $b \in \mathcal{A}_+$ such that $\|k_\omega^{1/4} b k_\omega^{1/4} - x^{1/2}\|_2 < \varepsilon$. Let $a \in \mathcal{A}_+$ be such that $\|k_\omega^{1/4} a k_\omega^{1/4} - b k_\omega^{1/2} b\| < \varepsilon$. Then, by Lemma 1.4 and Hölder's inequality,

$$\begin{aligned}\|k_\omega^{1/2} a k_\omega^{1/2} - x\|_1 &\leq \|k_\omega^{1/2} a k_\omega^{1/2} - k_\omega^{1/4} b k_\omega^{1/2} b k_\omega^{1/4}\|_1 + \|k_\omega^{1/4} b k_\omega^{1/2} b k_\omega^{1/4} - x\|_1 \\ &\leq \|k_\omega^{1/4}\|_4 \|k_\omega^{1/4} a k_\omega^{1/4} - b k_\omega^{1/2} b\|_2 \|k_\omega^{1/4}\|_4 \\ &\quad + \|k_\omega^{1/4} b k_\omega^{1/4} - x^{1/2}\|_2 \|k_\omega^{1/4} b k_\omega^{1/4} + x^{1/2}\|_2 \\ &\leq \varepsilon + \varepsilon(2\|x^{1/2}\|_2 + \varepsilon).\end{aligned}$$

Thus, $\iota_1(\mathcal{A}_+)$ is dense in $L^1_+(\mathcal{A})$. Now fix an arbitrary $p \in [1, \infty[$ and let $x \in L^p_+(\mathcal{A})$. Then $x^p \in L^1_+(\mathcal{A})$ so that there is a sequence (b_n) in $\mathcal{A}_+$ such that $k_\omega^{1/2} b_n k_\omega^{1/2} \to x^p$ in L^1, and hence also in measure ([Ter], Proposition 26). It follows by ([FaK], Lemmas 2.5iv and 3.1) or ([Tih], Proposition 2.3) that $(k_\omega^{1/2} b_n k_\omega^{1/2})^{1/p} \to x$ as $n \to \infty$ in measure and, consequently, in $\|\cdot\|_p$ ([Ter], Proposition 26). Since $k_\omega^{1/2} b_n k_\omega^{1/2} \leq \|b_n\| k_\omega$, we have $(k_\omega^{1/2} b_n k_\omega^{1/2})^{1/p} \leq \|b_n\|^{1/p} k_\omega^{1/p}$ and by Lemma 1.5 there are $a_n \in \mathcal{A}_+$ such that $k_\omega^{1/2} a_n k_\omega^{1/2} \to x$ as $n \to \infty$. □

Remark The above proof is not the simplest one. The same result follows even more easily from the density of $\mathcal{A} k_\omega^{1/p}$ in $L^p(\mathcal{A})$ (see [Wat]). Nevertheless, our method illustrates many important notions of the theory.

Let us describe now the interpolation L^p-spaces of Kosaki ([Kos]). Put

$$L^p(\mathcal{A}, \omega) := \kappa_p^{(\omega)}(L^p(\mathcal{A})) \quad \text{and} \quad \|x\|_p^{(\omega)} := \|(\kappa_p^{(\omega)})^{-1}(x)\|_p.$$

In particular, $L^1(\mathcal{A}, \omega) = L^1(\mathcal{A})$ and $\|x\|_1^{(\omega)} = \|x\|_1$. As proved by Kosaki

$$L^p(\mathcal{A}, \omega) = C_{1/p}(L^\infty(\mathcal{A}, \omega), L^1(\mathcal{A})),$$

where $C_\alpha(X, Y)$, $\alpha \in [0, 1]$, are the α-interpolation functors between the Banach spaces X and Y given by Calderon's complex interpolation method. Note that, for $1 \leq p_1 \leq p_2 < \infty$,

$$L^\infty(\mathcal{A}, \omega) \subset L^{p_2}(\mathcal{A}, \omega) \subset L^{p_1}(\mathcal{A}, \omega) \subset L^1(\mathcal{A})$$

and, by ([BeL], Theorem 4.2.1),

$$\|x\|_{p_1}^{(\omega)} \leq \|x\|_{p_2}^{(\omega)} \qquad \text{for } x \in L^{p_2}(\mathcal{A}, \omega).$$

As a simple consequence, we get continuity of the maps $\iota_p^{(\omega)}$ and $\kappa_p^{(\omega)}$.

The following theorem sumarises the sallient features of these embeddings.

Theorem 1.7 *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A} = \pi(\mathcal{A}) & \xrightarrow{\iota_p^{(\omega)}} & L^p(\mathcal{A}) \\ {\scriptstyle \iota^{(\omega)}} \downarrow & & \downarrow {\scriptstyle \kappa_p^{(\omega)}} \\ \mathcal{A}_* & \xrightarrow[\kappa]{} & L^1(\mathcal{A}) \end{array}$$

Moreover, the mappings $\iota^{(\omega)}$, $\iota_p^{(\omega)}$ and $\kappa_p^{(\omega)}$ ($1 \leq p \leq \infty$) are injective contractions with dense ranges, while κ is an isometric isomorphism.

Proof. The result follows immediately from Lemmas 1.3 and 1.7 and the above discussion of interpolation spaces. □

2 Linear operators on L^p-spaces

From now on we fix a faithful, normal state φ on $\mathcal{A}$. ι, ι_p and κ_p will denote embeddings with respect to φ and k_φ will be abbreviated to h. Let S be a linear mapping of a subspace $\mathcal{E}$ of $L^p(\mathcal{A})$ into $L^p(\mathcal{A})$, $p \in [1, \infty]$. We say that S is *real* if $\mathcal{E}$ is $*$-invariant and $Sx^* = (Sx)^*$ for $x \in \mathcal{E}$; we say that S is *positivity-preserving* if $\mathcal{E}_+ := \mathcal{E} \cap \dot{\mathcal{A}}^{\sim}_+$ linearly generates $\mathcal{E}$ and $Sx \geq 0$ for $x \in \mathcal{E}_+$.

Now let T be a linear operator on $\mathcal{A}$. Define $T^{(p)}$ on $\iota_p(\operatorname{Dom} T)$ by

$$T^{(p)}(\iota_p(a)) = \iota_p(Ta) \qquad \text{for } a \in \operatorname{Dom} T.$$

If T is real, then all the $T^{(p)}$'s are real; if T is positivity-preserving, then so are all the $T^{(p)}$'s.

If the domain of T is all of $\mathcal{A}$ then $T^{(p)}$ is densely defined by Lemma 1.6. We say that T is *p-integrable* (with respect to φ) if the induced operator $T^{(p)}$ is L^p-bounded, in which case we denote its unique continuous extension to $L^p(\mathcal{A})$ by $T^{(p)}$ also.

Lemma 2.1 *If $T : \mathcal{A} \to \mathcal{A}$ is p-integrable and also bounded, then T is r-integrable for any $r > p$.*

Proof. Let $T^{[p]}$ be the (unique) bounded operator on $L^p(\mathcal{A}, \varphi)$ such that $T^{[p]} \circ \kappa_p = \kappa_p \circ T^{(p)}$. It is clear that $T^{[p]}|L^\infty(\mathcal{A}, \varphi) = T^{[\infty]}$ (we identify T with $T^{(\infty)}$). By the reiteration property of the complex interpolation method ([BeL], Theorem 4.6.1), $T^{[r]} = T^{[p]}|L^r(\mathcal{A}, \varphi)$ is bounded, which implies boundedness of $T^{(r)}$. □

If $T : \mathcal{A} \to \mathcal{A}$ is σ-weakly continuous (i.e. $\sigma(\mathcal{A}, \mathcal{A}_*) - \sigma(\mathcal{A}, \mathcal{A}_*)$-continuous) then one can define the predual map $T_* : \mathcal{A}_* \to \mathcal{A}_*$ by $T_*\omega = \omega \circ T$. Note that T_* is automatically bounded, T is also bounded and $\|T_*\| = \|T\|$ ([Ped], 2.4.12 and 2.3.10). If T is real (positivity-preserving), then T_* is real (positivity-preserving), i.e. $T_*(\mathcal{A}_{*,h}) \subset \mathcal{A}_{*,h}$ ($T_*(\mathcal{A}_{*,+}) \subset \mathcal{A}_{*,+}$).

For such maps the integrability condition can be expressed in terms of the predual map.

Proposition 2.2 *Let $T : \mathcal{A} \to \mathcal{A}$ be σ-weakly continuous and positivity preserving. Then T is integrable (i.e. 1-integrable with respect to φ) if and only if $T_*\varphi \leq \gamma\varphi$ for some $\gamma > 0$.*

Proof. Assume T integrable. In view of the tracial property, $\varphi(a) = \mathrm{tr}(h^{1/2}ah^{1/2}) = \mathrm{tr}(\iota_1(a))$. Thus, for $a \geq 0$,

$$\|\iota_1(Ta)\|_1 = \|T^{(1)}\iota_1(a)\|_1 \leq \|T^{(1)}\|\|\iota_1(a)\|_1 = \|T^{(1)}\|\varphi(a).$$

Suppose now that $T_*\varphi \leq \gamma\varphi$ for some $\gamma > 0$. We have $\|\iota_1(Ta)\|_1 \leq \gamma\|\iota_1(a)\|_1$ for every positive a. Thus, $T^{(1)}$ (defined on $\iota_1(\mathcal{A})$) can be extended, first, by continuity, to $L^1_+(\mathcal{A})$, and then, by linearity, to the whole of $L^1(\mathcal{A})$. Since $T^{(1)}$ is bounded on $L^1_+(\mathcal{A})$ and any element x of $L^1(\mathcal{A})$ is expressible in the form $\sum i^k x_k$ where $x_1, \cdots, x_4$ are non-negative and have norm at most $\|x\|$, T is integrable. □

Assume again that T is σ-weakly continuous. The predual map T_* can be easily transported to $L^1(\mathcal{A})$, and the map obtained is denoted by $T_{(1)}$. Thus, $T_{(1)} = \kappa \circ T_* \circ \kappa^{-1}$. It is not hard to check that $L^\infty(\mathcal{A}, \omega)$ is $T_{(1)}$-invariant if and only if T is integrable. Moreover, in that case $T_{(1)}|L^\infty(\mathcal{A}, \omega)$ is bounded with norm at most $\|T^{(1)}\|$. Since $T_{(1)}$ is bounded (with norm equal to $\|T\|$), we get, by interpolation, the $T_{(1)}$-invariance of all the spaces $L^p(\mathcal{A}, \omega)$. Putting $T_{(p)} = \kappa_p^{-1} \circ T_{(1)}|L^p(\mathcal{A}, \omega) \circ \kappa_p$ we get a family of operators on the spaces $L^p(\mathcal{A})$, similar to $T^{(p)}$ in that $T_{(p)} = (T_{(\infty)})^{(p)}$. In fact they stand in adjoint relationship

$$(T^{(p)})^* = T_{(p')} \quad \text{for } p < \infty \quad \text{and } (T^{(p)})_* = T_{(p')} \quad \text{for } p > 1$$

A linear operator $T : \mathcal{A} \to \mathcal{A}$ is called KMS-*symmetric* (with respect to φ), or φ-*symmetric*, if, for any $a, b \in \mathcal{A}$,

$$\mathrm{tr}(h^{1/2}(Ta)h^{1/2}b) = \mathrm{tr}(ah^{1/2}(Tb)h^{1/2})$$

where, as always now, $h = k_\varphi$.

Proposition 2.3 *Let $T : \mathcal{A} \to \mathcal{A}$ be KMS-symmetric. The following conditions are equivalent:*

(a) T is σ-weakly continuous;
(b) T is bounded;
(c) T is integrable.

In which case $T_{(p)} = T^{(p)}$ for each $p \in [1, \infty]$ and $\|T^{(1)}\| = \|T\|$.

Proof. (a) $\Rightarrow$ (b) As noted earlier, this implication is true even without the symmetry assumption.

(b) $\Rightarrow$ (c) Let u be the partial isometry from the polar decomposition of $h^{1/2}(Ta)h^{1/2}$. Then,

$$\begin{aligned}\|h^{1/2}(Ta)h^{1/2}\|_1 &= \mathrm{tr}(h^{1/2}(Ta)h^{1/2}u^*) = \mathrm{tr}(h^{1/2}ah^{1/2}(Tu^*)) \\ &\leq \|h^{1/2}ah^{1/2}\|_1\|T\|,\end{aligned}$$

which implies integrability of T.

(c) $\Rightarrow$ (a) We have, for $a, b \in \mathcal{A}$,

$$\operatorname{tr}((Ta)h^{1/2}bh^{1/2}) = \operatorname{tr}(ah^{1/2}(Tb)h^{1/2}) = \operatorname{tr}(aT^{(1)}(h^{1/2}bh^{1/2})).$$

Since $T^{(1)}$ is continuous, Lemma 1.6 gives

$$\operatorname{tr}((Ta)k) = \operatorname{tr}(a(T^{(1)}k))$$

for any $k \in L^1(\mathcal{A})$. If $a_\lambda \to a$ σ-weakly in $\mathcal{A}$, then

$$\operatorname{tr}((Ta_\lambda)k) = \operatorname{tr}(a_\lambda(T^{(1)}k)) \to \operatorname{tr}(a(T^{(1)}k)) = \operatorname{tr}((Ta)k),$$

which implies σ-weak continuity of T.

Now, for all $a, b \in \mathcal{A}$,

$$\begin{aligned}\operatorname{tr}(T_{(1)}(h^{1/2}ah^{1/2})b) &= \operatorname{tr}(h^{1/2}ah^{1/2}(Tb)) = \operatorname{tr}(h^{1/2}(Ta)h^{1/2}b) \\ &= \operatorname{tr}(T^{(1)}(h^{1/2}ah^{1/2})b).\end{aligned}$$

Since both $T_{(1)}$ and $T^{(1)}$ are continuous, we have $T_{(1)} = T^{(1)}$, which implies $T_{(p)} = T^{(p)}$ for each $p \in [1, \infty]$. Moreover, $\|T\| = \|T_*\| = \|T_{(1)}\| = \|T^{(1)}\|$, which ends the proof. □

Proposition 2.4 *Let $T : \mathcal{A} \to \mathcal{A}$ be bounded and integrable. If T is KMS-symmetric, then $(T^{(p)})^* = T^{(p')}$ for each $p \in [1, \infty[$, with $p' = 1 + (p-1)^{-1}$. In particular, $T^{(2)}$ is self-adjoint. Conversely, if, for some $p \in [1, \infty[$, $(T^{(p)})^* = T^{(p')}$, then T is KMS-symmetric.*

Proof. It is enough to observe that the condition

$$\operatorname{tr}(h^{1/2}(Ta)h^{1/2}b) = \operatorname{tr}(ah^{1/2}(Tb)h^{1/2})$$

can be written as

$$\operatorname{tr}(T^{(p)}(h^{1/2p}ah^{1/2p})h^{1/2p'}bh^{1/2p'}) = \operatorname{tr}(h^{1/2p}ah^{1/2p}T^{(p')}(h^{1/2p'}bh^{1/2p'})).$$

□

We now turn our attention to Markov operators. We say that a linear operator $T : \mathcal{A} \to \mathcal{A}$ is *Markov* if $0 \le a \le 1$ implies $0 \le Ta \le 1$, and an operator S on $L^p(\mathcal{A})$ is *L^p-Markov* if $0 \le x \le h^{1/p}$ implies $0 \le Sx \le h^{1/p}$. It follows from Kadison's inequality for positive operators on normal elements of a C*-algebra, together with the Russo-Dye theorem, ([KaR], 10.5.10) that T is Markov if and only if it is a positivity preserving contraction. On the other hand, *identity preserving* contractions on $\mathcal{A}$ are automatically positivity preserving ([BrR], 3.2.6), and therefore Markov. Markov usually includes the condition of identity preserving - our condition being merely *sub-Markov*. However we prefer to reserve the term *conservative* for the (extra) condition $T1 = 1$, following mathematical physics terminology.

Proposition 2.5 *If $T : \mathcal{A} \to \mathcal{A}$ is Markov and p-integrable, then $T^{(r)} : L^r(\mathcal{A}) \to L^r(\mathcal{A})$ is L^r-Markov for each $r > p$. Conversely, if $S : L^p(\mathcal{A}) \to L^p(\mathcal{A})$ is L^p-Markov, then there exists a unique Markov operator $T : \mathcal{A} \to \mathcal{A}$ such that $S = T^{(p)}$. If S is bounded, then T is p-integrable. In any case, S is positivity-preserving.*

Proof. Assume T Markov and p-integrable and let $0 \leq x \leq h^{1/p}$ for some $x \in L^p(\mathcal{M})$. By Lemma 1.5, there is an $a \in \mathcal{M}$ such that $x = h^{1/2p}ah^{1/2p}$, so that $0 \leq T^{(p)}x = h^{1/2p}(Ta)h^{1/2p} \leq h^{1/p}$. Now use Lemma 2.1 to end the proof of the first part of the proposition. Conversely, let S be L^p-Markov. If $a \in \mathcal{M}$, then $0 \leq S(h^{1/2p}ah^{1/2p}) \leq h^{1/p}$, so that $S(h^{1/2p}ah^{1/2p}) = h^{1/2p}bh^{1/2p}$ for some $b \in \mathcal{M}$ again by Lemma 1.5. Put $Ta = b$. By Lemma 1.3, T is well-defined and linear. If $0 \leq a \leq 1$, then $0 \leq h^{1/2p}bh^{1/2p} \leq h^{1/p}$. By ([Sch], Lemma 2.2e]), $0 \leq b \leq 1$. Thus, T is Markov and, evidently, $S = T^{(p)}$. The uniqueness of T follows directly from Lemma 1.3. The p-integrability of T for a bounded S is obvious. Since T is positivity-preserving, so is S. □

To sum up, we have the following

Theorem 2.6 *Let T be a KMS-symmetric Markov operator on $\mathcal{M}$. It follows that T is a σ-weakly continuous, integrable, positivity-preserving contraction, and, for each $p \in [1, \infty[$, $T^{(p)}$ is an L^p-Markov, positivity-preserving contraction on $L^p(\mathcal{M})$. Moreover, $(T^{(p)})^* = T^{(p')}$ for $p \in [1, \infty[$ and $T^{(2)}$ is self-adjoint. Conversely, if S is a self-adjoint Markov operator on $L^2(\mathcal{M})$, then $S = T^{(2)}$ for a unique KMS-symmetric Markov operator T on $\mathcal{M}$. In particular self-adjoint L^2-Markov operators are contractive.*

By common abuse of language, we speak about a symmetric L^2-Markov operator instead of a self-adjoint L^2-Markov operator.

3 The Hilbert space theory

In the sequel, we shall consider non-negative quadratic forms Q defined on a dense subspace $\operatorname{Dom} Q$ of a Hilbert space $\mathcal{H}$. Such a form Q is *closed* if $\operatorname{Dom} Q$ equipped with the norm $\|\cdot\|_Q$ given by $\|x\|_Q^2 = \|x\|^2 + Q(x)$ is a Hilbert space. It is *closable* if there exists a closed form $\widehat{Q}$ extending Q, i.e. such that $\operatorname{Dom} \widehat{Q} \supset \operatorname{Dom} Q$ and $\widehat{Q}(x) = Q(x)$ for $x \in \operatorname{Dom} Q$. There exists in such a case a closed form $\overline{Q}$, called the closure of Q, which is the smallest closed extension of Q. Moreover, $\operatorname{Dom} Q$ is dense in $\operatorname{Dom} \overline{Q}$ endowed with its Hilbertian structure. A form Q is closable if and only if, for any sequence (x_n) in $\operatorname{Dom} Q$, $x_n \to 0$ and $Q(x_n - x_m) \to 0$ imply $Q(x_n) \to 0$.

We consider the folowing objects:

(i) a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ on $\mathcal{H}$;
(ii) a strongly continuous contraction resolvent $(R_\lambda)_{\lambda>0}$ on $\mathcal{H}$;
(iii) a densely defined closed linear operator G on $\mathcal{H}$ such that $]0, \infty[$ is contained in its resolvent set and $\|\lambda(\lambda I - G)^{-1}\| \leq 1$ for $\lambda > 0$.

By a *strongly continuous contraction resolvent* we mean a family of (everywhere defined) linear operators on $\mathcal{H}$ satisfying:

(a) $\lim_{\lambda\to\infty} \lambda R_\lambda x = x$ for $x \in \mathcal{H}$;
(b) λR_λ is a contraction on $\mathcal{H}$ for all $\lambda > 0$;

(c) $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$ for all $\lambda, \mu > 0$.

We have the following, classical theorem, most of which can be attributed to Hille and Yosida (see e.g. [Da1]):

Theorem 3.1 *Each of the objects (i), (ii), (iii) uniquely determines the others through the relations:*

$$G = \text{st.}\lim_{t\downarrow 0} \frac{1}{t}(P_t - I); \qquad G = \lambda I - R_\lambda^{-1};$$

$$R_\lambda = \text{st.}\int_0^\infty e^{-\lambda t} P_t dt; \qquad R_\lambda = (\lambda I - G)^{-1},$$

$$P_t = \text{st.}\lim_{n\to\infty} (n/tR_{n/t})^n.$$

The operator G is called the *generator* of (P_t) and of (R_λ). It determines the semigroup via the resolvent family. The correspondence is enriched by symmetry as follows.

Proposition 3.2 *The correspondences in Theorem 3.1 remain true if we add the assumption of self-adjointness of all P_t, $t \geq 0$ in (i), of R_λ, $\lambda > 0$ in (ii), and exchange (iii) for*

(iii′) a self-adjoint, non-negative operator $H = -G$ on $\mathcal{H}$.

In this case, one has $P_t = e^{-tH}$ for $t > 0$ (in the sense of the functional calculus for self-adjoint operators), and a fourth ingredient can be added, namely,

(iv) a closed, densely defined, non-negative quadratic form Q,

given uniquely by $\|H^{1/2}\cdot\|^2_{\mathcal{H}}$ (and, of course, determining the other three objects).

4 Markov semigroups and Dirichlet operators

We briefly mention the correspondence between Markov properties of a contraction semigroup and its resolvent family, and a Dirichlet property for its generator, in the absense of symmetry assumptions. Quadratic forms are useful under weaker constraints than symmetry, such as sector conditions on the resolvent family. Such forms are called coercive — the anti-symmetric part being controlled by the symmetric part. For a recent account of such (Dirichlet) forms and their associated processes we recommend [MaR].

Some further terminology is needed. An *L^p-Markov semigroup* ($1 \leq p < \infty$) is a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ consisting of L^p-Markov operators. An *L^p-Markov resolvent family* is a strongly continuous contraction resolvent $(R_\lambda)_{\lambda>0}$ such that each λR_λ is L^p-Markov. A closed densely defined operator G on $L^2(\mathcal{A})$ is called *Dirichlet* if G is real and $\langle Gx, (x - h^{1/2})_+\rangle \leq 0$ for all $x \in \text{Dom}_h G$. Finally a *Markov semigroup on $\mathcal{A}$* is a weak*-continuous semigroup consisting of Markov operators.

With any $x \in L^2_h(\mathcal{A})$ we associate an element $x_\wedge$ by the formula

$$x_\wedge := x - (x - h^{1/2})_+ = h^{1/2} - (x - h^{1/2})_- .$$

Note that $x_\wedge \leq x$ and $x_\wedge \leq h^{1/2}$.

We have the following

Theorem 4.1 *Let $(R_\lambda)_{\lambda>0}$ be a strongly continous contraction resolvent on $L^2(\mathcal{A})$ with generator G and semigroup $(P_t)_{t\geq 0}$. The following conditions are equivalent:*

(i) each λR_λ is L^2-Markov;
(ii) each P_t is L^2-Markov;
(iii) G is Dirichlet.

Proof. (i) $\Rightarrow$ (ii) The relations given in Theorem 3.1 imply the equivalence of (i) and (ii).

(ii) $\Rightarrow$ (iii) Let P be a single L^2-Markov contraction. Then, for $x \in L^2_h(\mathcal{A})$,

$$\begin{aligned}\langle Px, (x - h^{1/2})_+\rangle &= \langle P(x - h^{1/2})_+, (x - h^{1/2})_+\rangle + \langle Px_\wedge, (x - h^{1/2})_+\rangle \\ &\leq \langle x - h^{1/2}, (x - h^{1/2})_+\rangle + \langle h^{1/2}, (x - h^{1/2})_+\rangle \\ &= \langle x, (x - h^{1/2})_+\rangle .\end{aligned}$$

Hence, for all $x \in \mathrm{Dom}_h G$,

$$\langle Gx, (x - h^{1/2})_+\rangle = \lim_{t\downarrow 0} \frac{1}{t}\langle P_t x - x, (x - h^{1/2})_+\rangle \leq 0.$$

(iii) $\Rightarrow$ (i) Let $x \in L^2_h(\mathcal{A})$ and $y = \lambda R_\lambda x$. If $x \leq h^{1/2}$, then

$$\begin{aligned}\lambda\langle y, (y - h^{1/2})_+\rangle &= \langle \lambda y - Gy, (y - h^{1/2})_+\rangle + \langle Gy, (y - h^{1/2})_+\rangle \\ &\leq \lambda\langle x, (y - h^{1/2})_+\rangle \leq \lambda\langle h^{1/2}, (y - h^{1/2})_+\rangle .\end{aligned}$$

Thus, $\|(y - h^{1/2})_+\| \leq 0$, which means that $y \leq h^{1/2}$. If $x \geq 0$, then $-nx \leq h^{1/2}$, which implies $-ny \leq h^{1/2}$ for all $n \in \mathbb{N}$. Consequently, $y \geq 0$. □

Thus, in the Hille-Yosida correspondence of Theorem 3.1, the generator is Dirichlet if and only if the semigroup (and resolvent family) are Markov.

5 Dirichlet forms and symmetry

A non-negative quadratic form Q on $L^2(\mathcal{A})$ with dense domain $\mathrm{Dom}\, Q$ is called $\mathbb{R}$-*Dirichlet* (or simply *Dirichlet*) if:

(a) Q is real - in other words $\mathrm{Dom}\, Q$ is *-invariant and $Qx^* = Qx$ for $x \in \mathrm{Dom}\, Q$;
(b) $x_+, x_\wedge \in \mathrm{Dom}_h Q$ for $x \in \mathrm{Dom}_h Q$;
(c) $Qx_+ \leq Qx$ and $Qx_\wedge \leq Qx$ for $x \in \mathrm{Dom}_h Q$.

Lemma 5.1 *Let S be symmetric L^2-Markov. It follows that $Q_S : x \in L^2(\mathscr{A}) \mapsto \langle x, (I - S)x\rangle$ is a Dirichlet form.*

Proof. By assumption, S is a self-adjoint, positivity-preserving contraction. Hence, $Q := Q_S$ is a non-negative, real quadratic form on $L^2(\mathscr{A})$. We have, for $x \in L^2_h(\mathscr{A})$,

$$Qx - Qx_+ = Qx_- - 2\langle x_+, (I - S)x_-\rangle = Qx_- + 2\langle x_+, Sx_-\rangle \geq 0$$

using once more the fact that the inner product of non-negative operators is necessarily non-negative. Further, since $x = (x - h^{1/2})_+ - (x - h^{1/2})_- + h^{1/2}$,

$$\begin{aligned} Qx_\wedge &= Qx + Q(x - h^{1/2})_+ - 2\langle (x - h^{1/2})_+, (I - S)x\rangle \\ &= Qx - Q(x - h^{1/2})_+ - 2\langle (x - h^{1/2})_+, S(x - h^{1/2})_-\rangle \\ &\quad -2\langle (x - h^{1/2})_+, (I - S)h^{1/2}\rangle. \end{aligned}$$

Thus, $Sh^{1/2} \leq h^{1/2}$ yields $Qx_\wedge \leq Qx$. □

This easily implies that the form generator of a symmetric Markov semigroup is Dirichlet. The converse result that Dirichlet forms generate L^2-Markov semigroups is proved by a series of Lemmas as in [DL1].

Lemma 5.2 *If $x \in L^2_+(\mathscr{A})$ and $y \in L^2_h(\mathscr{A})$, then*

$$\|y_+\|_2^2 - 2\langle x, y_+\rangle \leq \|y\|_2^2 - 2\langle x, y\rangle.$$

Proof. We have

$$\begin{aligned} & \|y\|_2^2 - 2\langle x, y\rangle - \|y_+\|_2^2 + 2\langle x, y_+\rangle \\ = \ & \|y_-\|_2^2 + 2\langle x, y_-\rangle \\ \geq \ & 0. \end{aligned}$$

□

Lemma 5.3 *If $x, y \in L^2_h(\mathscr{A})$ and $x \leq h^{1/2}$, then*

$$\|y_\wedge\|_2^2 - 2\langle x, y_\wedge\rangle \leq \|y\|_2^2 - 2\langle x, y\rangle.$$

Proof. We have

$$\begin{aligned} & \|y\|_2^2 - 2\langle x, y\rangle - \|y - (y - h^{1/2})_+\|_2^2 + 2\langle x, y - (y - h^{1/2})_+\rangle \\ = \ & \|(y - h^{1/2})_+\|_2^2 + 2\langle y - (y - h^{1/2})_+, (y - h^{1/2})_+\rangle - 2\langle x, (y - h^{1/2})_+\rangle \\ = \ & \langle 2y - (y - h^{1/2})_+ - 2(x - h^{1/2}) - 2h^{1/2}, (y - h^{1/2})_+\rangle \\ = \ & \langle 2(h^{1/2} - x) + (y - h^{1/2})_+, (y - h^{1/2})_+\rangle \\ \geq \ & 0. \end{aligned}$$

Lemma 5.4 *Let Q be a non-negative, closed, real, densely defined quadratic form on $L^2(\mathscr{A})$ and H the self-adjoint, non-negative real operator on $L^2(\mathscr{A})$ such that $Q = \|H^{1/2} \cdot \|_2^2$. Let $x \in L_h^2(\mathscr{A})$, $\lambda > 0$, $y = \lambda(\lambda I + H)^{-1}x$ and $z \in Dom_h Q$ with $\|z\|_2^2 - 2\langle x, z\rangle \leq \|y\|_2^2 - 2\langle x, y\rangle$. It follows that $Qz \leq Qy$ implies $z = y$.*

Proof. Note that $\lambda x = \lambda y + Hy$, so that $\langle x, y\rangle = \|y\|_2^2 + \lambda^{-1}Qy$. Thus, if $Qz \leq Qy$,

$$\begin{aligned} \|(I + \lambda^{-1}H)^{1/2}(y - z)\|_2^2 &= \langle x, y\rangle - 2\langle x, y\rangle + \|z\|_2^2 + \lambda^{-1}Qz \\ &\leq \langle x, y\rangle - 2\langle x, y\rangle + \|y\|_2^2 + \lambda^{-1}Qy \\ &= 0, \end{aligned}$$

which implies $z = y$. □

Proposition 5.5 *Let Q be a closed Dirichlet form on $L^2(\mathscr{A})$ and H the corresponding non-negative operator. It follows that, for all $\lambda > 0$, $\lambda R_\lambda = (I + \lambda H)^{-1}$ is L^2-Markov.*

Proof. This follows immediately from 5.2, 5.3 and 5.4. □

Lemma 5.6 *A self-adjoint Dirichlet operator G on $L^2(\mathscr{A})$ is necessarily non-positive.*

Proof. It is enough to show that $\langle x, Gx\rangle \leq 0$ for all $x \in \mathrm{Dom}_h G$. Put nx in place of x in the definition of a Dirichlet operator to obtain

$$\langle Gx, (x - \frac{1}{n}h^{1/2})_+\rangle \leq 0 \qquad \text{for } x \in \mathrm{Dom}_h G, \quad n \in \mathbb{N}.$$

Since $\|x - y\|^2 - \|x_+ - y_+\|^2 = \|x_- - y_-\|^2 + 2\mathrm{tr}(x_+y_- + x_-y_+) \geq 0$, if we let $n \to \infty$ we have $\langle Gx, x_+\rangle \leq 0$ for any $x \in \mathrm{Dom}_h G$. If we exchange x for $-x$ in the definition of a Dirichlet operator, we obtain

$$\langle Gx, (x + \frac{1}{n}h^{1/2})_-\rangle \geq 0 \qquad \text{for } x \in \mathrm{Dom}_h G, \quad n \in \mathbb{N}.$$

Hence $\langle Gx, x_-\rangle \geq 0$ for $x \in \mathrm{Dom}_h G$, and the result follows. □

Theorem 5.7 *The following conditions are equivalent:*

(i) $(P_t)_{t\geq 0}$ *is a symmetric L^2-Markov semigroup;*
(ii) $(R_\lambda)_\lambda > 0$ *is a symmetric L^2-Markov resolvent family;*
(iii) $-H$ *is a self-adjoint Dirichlet operator;*
(iv) Q *is a closed Dirichlet form.*

Here, P_t, R_λ, H and Q are interconnected as in Theorem 3.1 and Proposition 3.2.

Proof. Since symmetric Markov operators are contractions, the equivalence of (i), (ii) and (iii) is given by Theorem 4.1. Lemma 5.5 yields (iv) $\Rightarrow$ (ii). Assume (i). By Lemma 5.1, $Q_t : x \mapsto t^{-1}\langle x, (I - P_t)x\rangle$ is Dirichlet. Put $\widehat{Q}x = \limsup_{t\downarrow 0} Q_t x$ for $x \in L^2(\mathscr{A})$. Then $Q = \widehat{Q} \mid \mathrm{Dom}\, Q$, hence Q is Dirichlet. □

This settles the L^2 situation. In order to relate back to the algebra we need to tie up the respective topologies.

Proposition 5.8 *Let (P_t) be a weak*-continuous, uniformly p-integrable semigroup on $\mathcal{M}$. Then $(P_t^{(p)})$ is a strongly continuous semigroup on $L^p(\mathcal{M})$. Conversely, if (S_t) is a (strongly continuous) L^p-Markov semigroup, then the semigroup (P_t) on $\mathcal{M}$ such that $S_t = (P_t^{(p)})$ for each t (given by Proposition 2.5) is weak*-continuous.*

Proof. For all $a \in \mathcal{M}$ and $y \in L^{p'}(\mathcal{M})$

$$\langle (P_t^{(p)} - I)h^{1/2p}ah^{1/2p}, y\rangle = \langle (P_t - I)a, h^{1/2p}yh^{1/2p}\rangle \to 0 \qquad \text{as } t \downarrow 0,$$

by weak*-continuity of (P_t). Weak continuity of $(P_t^{(p)})$ follows now from Theorem 1.7 and $\sup_t \|P_t^{(p)}\| < \infty$. Hence ([Dal], Proposition 1.23), $(P_t^{(p)})$ is strongly continuous. Conversely, if (S_t) is L^p-Markov, then, by Proposition 2.5, the corresponding semigroup P_t is Markov, hence uniformly bounded. Note that, for any $a \in \mathcal{M}$ and $y \in L^{p'}(\mathcal{M})$,

$$\langle P_t a - a, h^{1/2p}yh^{1/2p}\rangle = \langle S_t(h^{1/2p}ah^{1/2p}) - h^{1/2p}ah^{1/2p}, y\rangle \to 0 \qquad \text{as } t \downarrow 0.$$

As before, use Theorem 1.7 to end the proof of the proposition. □

Corollary 5.9 *Let (P_t) be a semigroup of KMS-symmetric Markov operators and (S_t) the corresponding semigroup of symmetric L^2-Markov operators. Then (P_t) is weak*-continuous if and only if (S_t) is strongly continuous.*

To sum up, we have the following

Theorem 5.10 *Let $\mathcal{M}$ be a von Neumann algebra with faithful, normal state φ. There is a bijective correspondence between KMS-symmetric Markov semigroups on $\mathcal{M}$ and closed Dirichlet forms on $L^2(\mathcal{M})$.*

6 More on Dirichlet forms and positivity

In this final section we verify that the Dirichlet property refines to characterise stronger forms of positivity for KMS-symmetric Markov semigroups, and give some useful results on cores of Dirichlet forms.

Proposition 6.1 *Let Q be a non-negative closed real densely defined quadratic form on $L^2(\mathcal{M})$ such that $Qx_\wedge \le Qx$ for $x \in L^2_+(\mathcal{M})$. It follows that Q is Dirichlet.*

Proof. By Lemmas 5.3 and 5.4, $\lambda R_\lambda x \le h^{1/2}$ for all $\lambda > 0$ and $x \le h^{1/2}$. The same kind of reasoning as that used in the proof of Theorem 4.1 (iii) ⇒ (i) shows that $x \ge 0$ implies $\lambda R_\lambda x \ge 0$. Thus, λR_λ is Markov and one gets (ii) in Theorem 5.7, so Q is Dirichlet. □

Lemma 6.2 *Let $x, y \in L^2_h(\mathcal{M})$, then*

$$\|x_\wedge - y_\wedge\|_2 \le \|x - y\|_2.$$

Proof. We have

$$\begin{aligned}
& \|x-y\|_2^2 - \|x_\wedge - y_\wedge\|_2^2 \\
= \; & \langle 2(x-y), (x-h^{1/2})_+ - (y-h^{1/2})_+\rangle - \|(x-h^{1/2})_+ - (y-h^{1/2})_+\|_2^2 \\
= \; & \langle 2\{(x-h^{1/2}) - (y-h^{1/2})\} - \{(x-h^{1/2})_+ - (y-h^{1/2})_+\}, \\
& (x-h^{1/2})_+ - (y-h^{1/2})_+\rangle \\
= \; & \langle (x-h^{1/2})_+ - (y-h^{1/2})_+ + 2\{(y-h^{1/2})_- - (x-h^{1/2})_-\}, \\
& (x-h^{1/2})_+ - (y-h^{1/2})_+\rangle \\
= \; & \|(x-h^{1/2})_+ - (y-h^{1/2})_+\|_2^2 + 2\{\langle (y-h^{1/2})_-, (x-h^{1/2})_+\rangle \\
& + \langle (x-h^{1/2})_-, (y-h^{1/2})_+\rangle\} \\
\geq \; & 0.
\end{aligned}$$

□

As in the tracial case, the algebraic condition (on a quadratic form) of being Dirichlet may be de-coupled from the analytic requirement of being closed, as the next result demonstrates.

Theorem 6.3 *If Q is Dirichlet and closable, then $\overline{Q}$ is Dirichlet.*

Proof. For $x \in \operatorname{Dom}\overline{Q}$, let (x_n) be a sequence in $\operatorname{Dom} Q$, converging to x in the quadratic form norm $\|\cdot\|_{\overline{Q}}$. By Lemma 6.2, $(x_n)_\wedge \to x_\wedge$, so by lower semicontinuity of $\overline{Q}$,

$$\overline{Q}x_\wedge \leq \liminf_{n\to\infty} Q(x_n)_\wedge \leq \liminf_{n\to\infty} Qx_n = \overline{Q}x.$$

Apply Proposition 6.1 to end the proof (or observe that the above argument applies also to x_+ since $\|x_+ - y_+\|_2 \leq \|x-y\|_2$). □

The next result provides defining domains for (form) generators of symmetric Markov semigroups.

Proposition 6.4 *Let $G^{(p)}$ denote the L^p-generators of a KMS-symmetric Markov semigroup (P_t) (weak*-generator in the case $p = \infty$). If C is any subset of $\operatorname{Dom} G^{(\infty)}$ which is weak*-dense, and invariant under each P_t, then $h^{1/2p}Ch^{1/2p}$ is a core for $G^{(p)}$, for $1 \leq p < \infty$. Moreover $h^{1/4}Ch^{1/4}$ is a core for the corresponding Dirichlet form on $L^2(\mathcal{A})$.*

Proof. This follows from [Da1], 1.24, 4.15 and [BrR], 3.1.7 . □

Now let M_n denote the algebra of $n \times n$ matrices acting on $\mathbb{C}^n$, and τ_n the usual trace on M_n. By the uniqueness of the modular automorphism group, $\sigma_t^{\omega^{(n)}} = \mathrm{id}_n \otimes \sigma_t^\omega$, where id_n is the identitity mapping on M_n, $\omega \in \mathcal{A}_{*,+}$ is faithful and $\omega^{(n)} := \tau_n \otimes \omega$ (see, e.g. [Str], section 3.9, formula (2)). Let $\pi^{(n)}$ and $\lambda^{(n)}$ correspond to π and λ, but with $M_n \otimes \mathcal{A}$ and $\omega^{(n)}$ in place of $\mathcal{A}$ and ω, respectively. If we identify $L^2(\mathbb{R};\mathbb{C}^n \otimes H)$ with $\mathbb{C}^n \otimes L^2(\mathbb{R};H)$, then $\pi^{(n)} = \mathrm{id}_n \otimes \pi$ and $\lambda^{(n)}(s) = 1_n \otimes \lambda(s)$, $s \in \mathbb{R}$, where 1_n is the unit of M_n. As an easy consequence, we get $M_n \otimes \mathcal{A}^{\bullet} = (M_n \otimes \mathcal{A})^{\bullet}$. Moreover, the dual action is given by $\theta_s^{(n)} = \mathrm{id}_n \otimes \theta_s$, $L^p(M_n \otimes \mathcal{A}) = M_n \otimes L^p(\mathcal{A})$ for $p \in [1, \infty]$, and the dual weight of $\omega^{(n)}$ is $\tilde{\omega}^{(n)}$. Thus, $k_{\omega^{(n)}} = 1_n \otimes k_\omega$ for each $\omega \in \mathcal{A}_*$, which implies that $\mathrm{tr}^{(n)} = \tau_n \otimes \mathrm{tr}$. For a quadratic form Q on $L^2(\mathcal{A})$ let $Q^{(n)}$ denote the quadratic form on $L^2(M_n \otimes \mathcal{A}) = M_n \otimes L^2(\mathcal{A}) = M_n(L^2(\mathcal{A}))$ given by

$$\mathrm{Dom}\, Q^{(n)} = M_n \otimes \mathrm{Dom}\, Q, \qquad Q^{(n)}((x_{ij})) = \sum_{i,j} Q x_{ij}.$$

If S is a bounded operator on $L^2(\mathcal{A})$, then, in the notation of Lemma 5.1,

$$(Q_S)^{(n)} = Q_{S^{(n)}}, \tag{*}$$

where $S^{(n)} = \mathrm{id}_n \otimes S$.

We say that a non-negative, densely defined quadratic form on $L^2(\mathcal{A})$ is

(i) *n-Dirichlet*, if $Q^{(n)}$ is Dirichlet;
(ii) *completely Dirichlet*, if it is n-Dirichlet for all n.

We say that an $\mathbb{R}$-Dirichlet form is *$\mathbb{C}$-Dirichlet*, if it satisfies

(i) $|x| \in \mathrm{Dom}\, Q$ for $x \in \mathrm{Dom}\, Q$;
(ii) $Q|x| \leq Qx$ for $x \in \mathrm{Dom}\, Q$.

$(n+1)$-Dirichlet forms are n-Dirichlet, but there are elementary examples showing that the converse is false for non-commutative algebras. The importance of this notion is demonstrated by the following result.

Theorem 6.5 *Let Q be the Dirichlet form on $L^2(\mathcal{A})$ corresponding to a symmetric Markov semigroup (P_t) on $\mathcal{A}$. Then Q is n-Dirichlet if and only if each P_t is n-positive.*

In view of the relation (*) the proof of the corresponding result in [DL1] carries over to the non-tracial context without change. We may also view $\mathbb{C}$-Dirichlet as a condition of $1\frac{1}{2}$-Dirichlet — the proof of the tracial case again carrying over to the present context.

Theorem 6.6 *Let Q be a quadratic form on $L^2(\mathcal{A})$. If Q is 2-Dirichlet, then Q is $\mathbb{C}$-Dirichlet.*

We have therefore characterised completely positive KMS-symmetric Markov semigroups by their form generators.

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