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Symplectic classification of quadratic forms, and general Mehler formulas

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1 Introduction

In [4, Sect. 21.5] we discussed the symplectic classification of quadratic forms which occur in the study of hyperbolic or hypoelliptic differential equations. However, no general classification was given. In particular, the results in [4] do not answer a question raised in a recent preprint by Derezinski in connection with the Weyl calculus. (In a revised version [3] of the preprint an answer given in an early version of this paper has been added.) The question raised by Derezinski has spurred us to write down a complete discussion supplementing [4], along similar lines as there, for the complex case in Sect. 2 and for the real case in Sect. 3. The result is not new, for a complete classification was already given by Williamson [6] for arbitrary fields of characteristic 0. However, because of the generality, the results in [6] are less explicit and the proofs are less elementary than those in Sects. 2 and 3. Explicit canonical forms were given by Laub and Meyer [8],[1] but in the case of purely imaginary eigenvalues they listed one decomposable case and two other cases can be simplified to one. A closely related classification of symplectic linear maps was given by Cushman and Duistermaat [2]. They listed a complete set of invariants but no explicit normal forms.[2]

In [3] Derezinski also determined the quadratic forms Q such that the Weyl symbol of the exponential of the corresponding Weyl operator Q^w is a function of the quadratic form. In Sect. 4 we shall discuss the symbol of the exponential quite generally, at first in a formal sense. When Re $Q \leq 0$ so that $\exp Q^w$ is defined in the sense of operator theory we prove that the Weyl symbol of $\exp Q^w$ is a continuous function of Q with values in $\mathscr{S}'$ and give it explicitly first when Re Q is negative definite or more generally the Hamilton map F of Q (see Sect. 2) has no eigenvalue λ with $\cos\lambda = 0$. Mehler's formula is a very special

[1] We owe this reference to the referee. Further references are given in [8].

[2] We are grateful to J. J. Duistermaat for calling the references [2] and [6] to our attention.

case. Finally we determine the Weyl symbol of $\exp Q^w$ for a general Q with $\operatorname{Re} Q \le 0$; it is given by a Gaussian on the subspace symplectically orthogonal to the kernel of $\cos F$. The normal forms in Sect. 3 are essential in the proof and give very explicit formulas when $\operatorname{Re} Q = 0$.

In Sect. 5 we reinterpret the results in terms of the infinitesimal version of Fourier integral operators, that is, the calculus based on Gaussian kernels. We give an essentially selfcontained exposition of this technique. The well known connection between the metaplectic group and the exponentials of Weyl operators iQ^w where Q is a real quadratic form is extended to the semigroup corresponding to forms with $\operatorname{Im} Q \ge 0$.

2 The complex case

Let S be a finite dimensional complex symplectic vector space with symplectic form σ, and let Q be a quadratic form in S. Denote by F the Hamilton map of Q defined by

$$\sigma(Y, FX) = Q(Y, X), \quad X, Y \in S,$$

where the right-hand side contains the polarized form of Q. Then F is skew symmetric with respect to σ. If V_λ denotes the space of generalized eigenvectors of F belonging to the eigenvalue $\lambda \in \mathbf{C}$, then ([4, Lemma 21.5.2])

$$\sigma(V_\lambda, V_\mu) = 0, \quad \text{if } \lambda + \mu \ne 0.$$

Hence V_λ and $V_{-\lambda}$ are isotropic spaces dual with respect to the symplectic form and $V_\lambda \oplus V_{-\lambda}$ is symplectic if $\lambda \ne 0$, while V_0 is a symplectic vector space. Thus we obtain a decomposition of S in a direct sum of symplectic subspaces which are mutually σ orthogonal and also Q orthogonal, since they are F invariant. To determine the structure of Q it suffices therefore to consider two cases:

i) $S = V_\lambda \oplus V_{-\lambda}$ where $\lambda \ne 0$.

ii) $S = V_0$.

In case ii) F is any skew symmetric nilpotent map in the symplectic vector space S. In case i) F restricts to a map T in $W = V_\lambda$ such that $T - \lambda I$ is nilpotent; $V_{-\lambda}$ is isomorphic to W', and if the duality is denoted $-\langle x, \xi\rangle$ for $x \in W$, $\xi \in W'$, then S is identified with $W \oplus W' = T^*W$ with the symplectic form

$$\sigma((x,\xi),(y,\eta)) = \langle y, \xi\rangle - \langle x \eta\rangle, \quad \text{if } x, y \in W, \ \xi, \eta \in W'. \tag{2.1}$$

A Jordan decomposition of W with respect to T yields a decomposition of S, so we may assume that T with suitable coordinates $x = (x_1, \dots, x_n)$ in W has the form

$$Tx = \lambda x + (x_2, \dots, x_n, 0).$$

By the skew symmetry we obtain $F(x,\xi) = (\lambda x + (x_2, \dots, x_n, 0), -\lambda\xi - (0, \xi_1, \dots, \xi_{n-1}))$, hence $Q(x,\xi) = 2\lambda \sum_1^n x_j\xi_j + 2\sum_1^{n-1} x_{j+1}\xi_j$.

Case ii) is harder. Before examining it we state the general result:

Theorem 2.1 *Let S be a complex symplectic vector space with symplectic form σ, and let Q be a quadratic form in S. Then S is a direct sum of subspaces orthogonal with respect to Q and with respect to σ of one of the following types:*

a) $S = T^*\mathbf{C}^n$ and with $\lambda \neq 0$

$$Q(x,\xi) = 2\lambda \sum_1^n x_j\xi_j + 2\sum_1^{n-1} x_{j+1}\xi_j.$$

Then the Jordan decomposition has one $n \times n$ box for each of the eigenvalues λ and $-\lambda$.

b) $S = T^*\mathbf{C}$ and $Q = 0$.

c) $S = T^*\mathbf{C}$ with coordinates (x,ξ), and $Q(\xi) = \xi^2$; then $F^2 = 0$.

d) $S = T^*\mathbf{C}^n$ with $n \geq 2$ and

$$Q(x,\xi) = 2\sum_1^{n-1} x_j\xi_{j+1} + (-1)^n x_n^2;$$

then $F^{2n} = 0$ but $F^{2n-1} \neq 0$, so the Jordan decomposition of F has just one $2n \times 2n$ box.

e) $S = T^*\mathbf{C}^n$ with n odd ≥ 3 and

$$Q(x,\xi) = 2\sum_1^{n-1} x_j\xi_{j+1};$$

then $F^n = 0$ but F^{n-1} has rank 2, so there are two $n \times n$ boxes in the Jordan decomposition of F.

Proof. What remains is to study the nilpotent case, so assume that $F^N = 0$ but that $F^{N-1} \neq 0$. If $N = 1$, then $Q = 0$ and we have case b). If $N = 2$ then Im F is isotropic and the σ orthogonal space Ker F, the radical of Q, is involutive. Hence we can choose symplectic coordinates so that Q is a quadratic form in ξ only, so we have just cases b) and c). From now on we assume that $N \geq 3$.

Set

$$B(X,Y) = \sigma(F^{N-1}X, Y), \quad X, Y \in S.$$

We have

$$B(X,Y) = (-1)^{N-1}\sigma(X, F^{N-1}Y) = (-1)^N\sigma(F^{N-1}Y, X) = (-1)^N B(Y,X),$$

so B is symmetric (skew symmetric) if N is even (odd). The bilinear form B induces a non-degenerate bilinear form $\widetilde{B}$ on $S/\text{Ker } F^{N-1}$, which is mapped bijectively on Im F^{N-1} by F^{N-1}.

1) Assume first that N is even, and choose $X \in S$ with $B(X,X) = 1$. Then

$$X, FX, \ldots, F^{N-1}X \tag{2.2}$$

span a symplectic F invariant space of dimension N. In fact, if $\sum_{j<N} a_j F^j X$ is σ orthogonal to all the vectors (2.2), we first obtain $a_0 = 0$ by orthogonality

to $F^{N-1}X$, then $a_1 = 0$ by orthogonality to $F^{N-2}X, \dots$, and finally $a_{N-1} = 0$ by orthogonality to X. To obtain a symplectic basis where Q is given as in d) we must consider all the scalar products

$$\sigma(F^jX, F^kX) = (-1)^k \sigma(F^{j+k}X, X);$$

they vanish if $j + k \geq N$ or if $j + k$ is even, and they are equal to $(-1)^k$ when $j + k = N - 1$. Without changing the space spanned by the vectors F^jX we can add to X any linear combination of F^jX with $j \neq 0$, and we shall use this to make $\sigma(F^jX, X) = 0$ for odd $j < N - 1$. First set $\widetilde{X} = X + \alpha F^2X$, and note that

$$\sigma(F^j\widetilde{X}, \widetilde{X}) = \sigma(F^jX, X) + 2\alpha\sigma(F^{j+2}X, X) + \alpha^2\sigma(F^{j+4}X, X).$$

The last term vanishes if $j + 2 = N - 1$, and we can then choose α so that $\sigma(F^j\widetilde{X}, \widetilde{X}) = 0$. Replacing X by $\widetilde{X}$ we have achieved that $\sigma(F^{N-3}X, X) = 0$. Suppose that we have already achieved that

$$\sigma(F^{N-3}X, X) = \dots = \sigma(F^{N+1-2k}X, X) = 0 \tag{2.3}$$

for some $k \geq 2$ where $2k + 1 < N$. Set $\widetilde{X} = X + \alpha F^{2k}X$. Then the conditions (2.3) remain valid with X replaced by $\widetilde{X}$, and

$$\sigma(F^{N-1-2k}\widetilde{X}, \widetilde{X}) = \sigma(F^{N-1-2k}X, X) + 2\alpha\sigma(F^{N-1}X, X)$$

is also 0 for a suitable choice of α. Replacing X by $\widetilde{X}$ we have then increased k by 1 in (2.3) and conclude by induction that we may choose X so that

$$\sigma(F^jX, X) = 0, \ j < N - 1; \quad \sigma(F^{N-1}X, X) = 1.$$

Now a symplectic basis in the space spanned by the vectors (2.2) is given by

$$e_j = F^{j-1}X, \varepsilon_j = (-1)^{j-1}F^{N-j}X, \quad 1 \leq j \leq n = N/2.$$

If $Z = \sum_1^n x_je_j + \sum_1^n \xi_j\varepsilon_j$ then

$$FZ = \sum_1^{n-1} x_je_{j+1} + (-1)^{n+1}x_n\varepsilon_n - \sum_2^n \xi_j\varepsilon_{j-1},$$

and since $Q(Z) = \sigma(Z, FZ)$, it follows that

$$Q(Z) = \sum_1^{n-1} x_j\xi_{j+1} + (-1)^n x_n^2 + \sum_2^n \xi_jx_{j-1} = 2\sum_1^{n-1} x_j\xi_{j+1} + (-1)^n x_n^2$$

as claimed in d). Note that for this form $F(x, \xi) = (x', \xi')$ where

$$x' = (0, x_1, \dots, x_{n-1}), \quad \xi' = (-\xi_2, \dots, -\xi_n, (-1)^{n+1}x_n),$$

so application of the powers of F starting with $x = (1, 0, \dots, 0)$, $\xi = 0$, gives all the basis vectors. — We can continue splitting off such spaces until we are left with a space where $F^{N-1} = 0$.

2) Assume now that N is odd. Then $\widetilde{B}$ is skew symmetric and non-degenerate, so we can choose two vectors X and Y such that $B(X,Y) = \sigma(F^{N-1}X, Y) = 1$. We claim that the $2N$ vectors

$$F^jX, F^jY, \quad 0 \le j < N \tag{2.4}$$

span a symplectic F invariant space of dimension $2N$. (In this space the Jordan form of F has two $N \times N$ boxes.) As before we prove that if

$$\sum_{j<N} a_j F^j X + \sum_{j<N} b_j F^j Y$$

is σ orthogonal to all vectors (2.4), it follows successively for increasing j that $a_j = b_j = 0$ by taking the σ scalar product with $F^{N-1-j}X$ and $F^{N-1-j}Y$. To obtain a suitable symplectic basis we would like to know that

$$\sigma(F^jX,X)=0,\ \sigma(F^jY,Y)=0,\ \sigma(F^jX,Y)=0, \quad 0 \le j < N-1. \tag{2.5}$$

The first two conditions are automatically fulfilled when j is even. To achieve this is slightly more complicated than in case 1). We begin by making (2.5) valid for $j = N-2$. To do so we set

$$\widetilde{X} = X + \alpha FY + \gamma FX, \quad \widetilde{Y} = Y + \beta FX.$$

Then

$$\begin{aligned}
\sigma(F^{N-2}\widetilde{X},\widetilde{X}) &= \sigma(F^{N-2}X,X) - 2\alpha\sigma(F^{N-1}X,Y),\\
\sigma(F^{N-2}\widetilde{Y},\widetilde{Y}) &= \sigma(F^{N-2}Y,Y) + 2\beta\sigma(F^{N-1}X,Y),\\
\sigma(F^{N-2}\widetilde{X},\widetilde{Y}) &= \sigma(F^{N-2}X,Y) + \gamma\sigma(F^{N-1}X,Y).
\end{aligned}$$

We choose α, β, γ so that these scalar products vanish and replace X, Y by $\widetilde{X}, \widetilde{Y}$. Then we have achieved that (2.5) holds when $j = N-2$. To make this true also when $j = N-3$ we consider $\widetilde{X} = X + \delta F^2X$. Since $F^NX = 0$ this does not affect the part of (2.5) already attained, and we get

$$\sigma(F^{N-3}\widetilde{X}, Y) = \sigma(F^{N-3}X, Y) + \delta\sigma(F^{N-1}X, Y).$$

We choose δ so that this is equal to 0. Next we make (2.5) valid for $j = N-4$ by arguing as for $j = N-2$ but with F replaced by F^3, then we deal with the case $j = N-5$ as the case $j = N-3$ replacing F^2X by F^4X. Proceeding in this way we make (2.5) valid without restriction.

Now we define a symplectic basis by

$$e_j = F^{j-1}X,\ \varepsilon_j = (-1)^{N-j-1}F^{N-j}Y, \quad j = 1, \dots, N.$$

With $Z = \sum_1^N x_je_j + \sum_1^N \xi_j\varepsilon_j$ we obtain

$$FZ = \sum_1^{N-1} x_je_{j+1} - \sum_2^N \xi_j\varepsilon_{j-1},$$

hence $Q(Z) = \sigma(Z, FZ) = \sum_1^{N-1} x_j\xi_{j+1} + \sum_2^N \xi_j x_{j-1} = 2\sum_1^{N-1} x_j\xi_{j+1}$. This agrees with e). Note that for this form $F(x,\xi) = (x',\xi')$ where

$$x' = (0, x_1, \dots, x_{n-1}), \quad \xi' = (\xi_2, \dots, \xi_n, 0).$$

We split off such spaces until in the remaining symplectic subspace we have $F^{N-1} = 0$, and then the claim follows by induction with respect to N.

3 The real case

Now assume that Q is a real quadratic form in a real symplectic vector space S. We can then apply the results of Sect. 2 to the complexification $S_{\mathbf{C}}$ of S. We keep the notation V_λ for the generalized eigenspaces of the complexification $F_{\mathbf{C}}$ of F in $S_{\mathbf{C}}$.

The space V_0 is invariant under conjugation, hence generated by its real part $V_0^{\mathbf{R}}$, and we can examine F there by the arguments used in the nilpotent case in Sect. 2. The only difference is that we must now distinguish two kinds of non-zero values of a quadratic form, positive and negative ones. This means that in addition to cases c) and d) there appear two other cases with a change of sign. These are not equivalent for the signatures are different.

If $0 \neq \lambda \in \mathbf{R}$ then V_λ is also generated by its real elements $V_\lambda^{\mathbf{R}}$, so the discussion in Sect. 2 requires no further modification. However, when $\lambda \in \mathbf{C} \setminus \mathbf{R}$ a new situation is encountered. Then there is no real element $\neq 0$ in V_λ, for if $x \in V_\lambda$ then $\bar{x} \in V_{\bar\lambda}$, and $V_\lambda \cap V_{\bar\lambda} = \{0\}$. Thus the projection $V_\lambda \ni X \mapsto \operatorname{Re} X \in \operatorname{Re} V_\lambda$ is a bijection defining an analytic structure in $\operatorname{Re} V_\lambda$. If $\operatorname{Re}\lambda = 0$, this is a symplectic vector space but if $\operatorname{Re}\lambda \neq 0$ it is isotropic, so we are led to distinguish two different cases.

a) First assume that $\lambda = \lambda_1 + i\lambda_2$ where $\lambda_1\lambda_2 \neq 0$. Then $\operatorname{Re} V_\lambda$ and $\operatorname{Re} V_{-\lambda}$ are isotropic F invariant spaces which are dual with respect to the symplectic form and σ orthogonal to all spaces V_μ with $\mu \neq \pm\lambda, \pm\bar\lambda$. It is therefore enough to study the case where $F_{\mathbf{C}}$ has just the four eigenvalues $\pm\lambda$ and $\pm\bar\lambda$. Thus $S \sim W \oplus W'$ where $W = \operatorname{Re} V_\lambda$ is F invariant and the complexification T of the restriction of F to W has the two complex eigenvalues $\lambda, \bar\lambda$. By a Jordan decomposition we can reduce further to the case where there is just one Jordan box for each eigenvalue. Choose $e \in W_{\mathbf{C}}$ so that $e, \dots, (T-\lambda)^{n-1}e$ is a complex basis for the generalized eigenvectors belonging to the eigenvalue λ, and introduce real coordinates in W with the basis vectors $\operatorname{Re} e$, $\operatorname{Im} e$, …, $\operatorname{Im}(T-\lambda)^{n-1}e$. Then we have with the corresponding complex coordinates in $W_{\mathbf{C}}$

$$\sum_1^n x_j(T-\lambda)^{j-1}e = (x_1, ix_1, x_2, ix_2, \dots, x_n, ix_n).$$

Applying $T - \lambda$ just shifts the coordinates to the right two steps, so

$$(T-\lambda_1)(x_1, ix_1, x_2, ix_2, \dots, ix_n) = (0, 0, x_1, ix_1, \dots, ix_{n-1}) + i\lambda_2(x_1, ix_1, \dots, ix_n).$$

Separating real and imaginary parts and changing notation we conclude that

$$(T-\lambda_1)(x_1,x_2,\dots,x_{2n})=(0,0,x_1,x_2,\dots,x_{2n-2})+\lambda_2(x_2,-x_1,\dots,x_{2n},-x_{2n-1}).$$

This allows us to compute $2\langle\xi,Tx\rangle$, and we conclude that S is symplectically equivalent to $T^*\mathbf{R}^{2n}$ with the usual symplectic coordinates and

$$Q(x,\xi)=2\Big(\sum_1^{2n-2}\xi_{j+2}x_j+\lambda_1\sum_1^{2n}x_j\xi_j+\lambda_2\sum_1^{n}(x_{2j}\xi_{2j-1}-x_{2j-1}\xi_{2j})\Big). \quad (3.1)$$

The form is non-degenerate since $V_0=\{0\}$, and the signature is $2n,2n$ since it vanishes in a space of dimension $2n$. For this form $F(x,\xi)=(x',\xi')$ where

$$\begin{aligned} x' &= (0,0,x_1,\dots,x_{2n-2})+\lambda_1 x+\lambda_2(x_2,-x_1,\dots,x_{2n},-x_{2n-1}),\\ \xi' &= -(\xi_3,\xi_4,\dots,\xi_{2n},0,0)-\lambda_1\xi+\lambda_2(\xi_2,-\xi_1,\dots,\xi_{2n},-\xi_{2n-1}).\end{aligned}$$

When $x_{2k}=ix_{2k-1}$ for $k=1,\dots,n$, then $x'-(\lambda_1+i\lambda_2)x=(0,0,x_1,\dots,x_{2n-2})$, and since this shift operator is nilpotent we recognize the generalized eigenspace for the eigenvalue $\lambda_1+i\lambda_2$. Similarly we recognize the other eigenspaces.

b) What remains is the case where $S_{\mathbf{C}}=V_{i\mu}\oplus V_{-i\mu}$ for some $\mu>0$. Choose $N>0$ so that $(F_{\mathbf{C}}-i\mu)^N V_{i\mu}=\{0\}$ but $(F_{\mathbf{C}}-i\mu)^{N-1}V_{i\mu}\neq\{0\}$. The sesquilinear form

$$\sigma((iF_{\mathbf{C}}+\mu)^jX,\overline{Y})/i,\quad X,Y\in V_{i\mu},$$

is Hermitian symmetric for every j. If $j<N$ it cannot vanish identically, for then we would have $(iF_{\mathbf{C}}+\mu)^jX=0$ for all $X\in V_{i\mu}$ since $V_{-i\mu}=\overline{V_{i\mu}}$ is dual to $V_{i\mu}$. Thus we can choose X so that

$$\sigma((iF_{\mathbf{C}}+\mu)^{N-1}X,\overline{X})=2\gamma i,\quad \text{where } \gamma=\pm1. \quad (3.2)$$

This implies that $X_j=(iF_{\mathbf{C}}+\mu)^jX$ and $\overline{X}_j$, $0\le j<N$, span a symplectic space. It is F invariant so the real part can be split off from S. Hence it is no restriction to assume that it is equal to $S_{\mathbf{C}}$. Then $V_{i\mu}$ is spanned by the vectors $X_0,\dots,X_{N-1}$. We may replace $X=X_0$ by $X+\sum_1^{N-1}a_jX_j$ for arbitrary a_j, and this can be used to achieve that

$$\sigma(X_k,\overline{X}_0)=0,\quad 0\le k<N-1. \quad (3.3)$$

In fact, $\sigma(X_k,\overline{X}_0)$ is purely imaginary and is not affected by the coefficients a_j with $k+j\ge N$ if we make such a change of X. The coefficient a_{N-1-k} only enters then in the term $2\sigma(X_{N-1},\overline{X}_0)\mathrm{Re}\, a_{N-1-k}$. Thus we can successively achieve the desired goal (3.3) for $k=N-2,\dots,0$. We shall now extract a symplectic basis from the real and imaginary parts of $X_0,\dots,X_{N-1}$.

Set $X_k=X_k'+iX_k''$. Then

$$\sigma(X_k',X_j')=\sigma(X_k'',X_j''),\quad \sigma(X_k',X_j'')+\sigma(X_k'',X_j')=0,\quad j,k=0,\dots,N-1, \quad (3.4)$$

for $V_{i\mu}$ is Lagrangean. Since $\sigma(X_k, \overline{X}_j) = 0$ when $k + j \neq N - 1$ by (3.3), we get additional equations which prove that

$$\sigma(X_k', X_j') = \sigma(X_k'', X_j'') = \sigma(X_k', X_j'') = 0, \quad \text{if } j + k \neq N - 1. \tag{3.5}$$

Using (3.2) we obtain

$$\sigma(X_{N-k-1}', X_k') + \sigma(X_{N-k-1}'', X_k'') = 0, \quad -\sigma(X_{N-k-1}', X_k'') + \sigma(X_{N-k-1}'', X_k') = 2\gamma,$$

which in view of (3.4) implies

$$\sigma(X_{N-k-1}', X_k') = \sigma(X_{N-k-1}'', X_k'') = 0, \quad \sigma(X_{N-k-1}'', X_k') = \gamma. \tag{3.6}$$

By (3.5) and (3.6) we obtain a symplectic basis by setting

$$\varepsilon_j = X_{j-1}'', \; e_j = X_{N-j}'/\gamma, \quad j = 1, \dots, N.$$

To calculate Q in the corresponding coordinates we just have to find $F\varepsilon_j$ and Fe_j. By definition

$$(iF_{\mathbf{C}} + \mu)(X_j' + iX_j'') = X_{j+1}' + iX_{j+1}'', \quad j = 0, \dots, N - 1$$

where we define $X_N' = X_N'' = 0$. Thus

$$FX_j' = -\mu X_j'' + X_{j+1}'', \quad FX_j'' = \mu X_j' - X_{j+1}',$$

which means that

$$F\varepsilon_j = \gamma(\mu e_{N+1-j} - e_{N-j}), \quad Fe_j = \gamma^{-1}(-\mu\varepsilon_{N+1-j} + \varepsilon_{N+2-j}).$$

(We interpret e_0 and ε_{N+1} as 0.) The polynomial Q is given by

$$\sigma\left(\sum x_k e_k + \sum \xi_k \varepsilon_k, \sum x_j Fe_j + \sum \xi_j F\varepsilon_j\right) = \sum x_j x_k \sigma(e_k, Fe_j) + \sum \xi_j \xi_k \sigma(\varepsilon_k, F\varepsilon_j).$$

Thus we obtain

$$Q(x,\xi) = \gamma^{-1}\left(\mu \sum_1^N x_j x_{N+1-j} - \sum_2^N x_j x_{N+2-j}\right) + \gamma\left(\mu \sum_1^N \xi_j \xi_{N+1-j} - \sum_1^{N-1} \xi_j \xi_{N-j}\right).$$

Recall that $\gamma = \pm 1$, so the two alternatives differ just by the sign. For this form $F(x,\xi) = (x', \xi')$ where

$$x' = \gamma(\mu(\xi_N, \dots, \xi_1) - (\xi_{N-1}, \dots, \xi_1, 0)), \; \xi' = -\gamma(\mu(x_N, \dots, x_1) - (0, x_N, \dots, x_2)).$$

When $x = (0, \dots, 0, \gamma)$ and $\xi = (i, 0, \dots, 0)$ then

$$x' - i\mu x = -(0, \dots, 0, \gamma i, 0), \quad \xi' - i\mu\xi = (0, 1, 0, \dots, 0)$$

which apart from a factor $-i$ means a left or right shift of the coordinates. Repetition confirms the nilpotent structure.

To determine the signature when $\gamma = 1$ we note that it must be independent of μ, for Q is always non-degenerate since $V_0 = \{0\}$. For large μ we have essentially two copies of the form

$$\sum_1^N x_j x_{N+1-j} = x_1 x_N + x_2 x_{N-1} + \dots .$$

If N is even the signature is $N/2, N/2$ but when N is odd we have a middle term $x_{(N+1)/2}^2$ which makes the signature equal to $(N+1)/2, (N-1)/2$. For the full form Q the signature is therefore N, N and $N+1, N-1$ in the two cases when $\gamma = 1$. Thus the signature distinguishes between the cases $\gamma = \pm 1$ when N is odd, but when N is even the signatures are the same although we know from (3.2) that the sign is determined.

We are now ready to sum up the results as a complete classification theorem.

Theorem 3.1 *Let S be a real symplectic vector space with symplectic form σ, and let Q be a real quadratic form in S. Then S is a direct sum of subspaces orthogonal with respect to Q and σ of one of the following types, and the number of spaces of each type is uniquely determined:*

a) $S = T^*\mathbf{R}^n$ and with $\lambda > 0$

$$Q(x,\xi) = 2\lambda \sum_1^n x_j \xi_j + 2\sum_1^{n-1} x_{j+1}\xi_j .$$

Then the Jordan decomposition of F has one $n \times n$ box for each of the eigenvalues λ and $-\lambda$. The signature of Q is n, n.

b) $S = T^*\mathbf{R}^{2n}$ and with $\lambda_1 > 0$, $\lambda_2 > 0$

$$Q(x,\xi) = 2\Big(\sum_0^{2n-2} \xi_{j+2} x_j + \lambda_1 \sum_1^{2n} x_j \xi_j + \lambda_2 \sum_1^n (x_{2j}\xi_{2j-1} - x_{2j-1}\xi_{2j})\Big).$$

The Jordan decomposition has one $n \times n$ box for each of the eigenvalues $\pm\lambda_1 \pm i\lambda_2$. The signature of Q is $2n, 2n$.

c) $S = T^*\mathbf{R}^n$ and with $\mu > 0$, $\gamma = \pm 1$,

$$Q(x,\xi) = \gamma\Big(\mu \sum_1^n x_j x_{n+1-j} - \sum_2^n x_j x_{n+2-j} + \mu \sum_1^n \xi_j \xi_{n+1-j} - \sum_1^{n-1} \xi_j \xi_{n-j}\Big).$$

The Jordan decomposition of F has one $n \times n$ box for each of the eigenvalues $\pm i\mu$. The signature of Q is n, n if n is even and $n+\gamma, n-\gamma$ when n is odd.

d) $S = T^*\mathbf{R}$ and $Q = 0$, thus $F = 0$.

e) $S = T^*\mathbf{R}$ and $Q = \pm\xi^2$; then $F^2 = 0$.

f) $S = T^*\mathbf{R}^n$ with $n \geq 2$, and

$$Q(x,\xi) = \pm\Big(2\sum_1^{n-1} x_j\xi_{j+1} + (-1)^n x_n^2\Big);$$

then $F^{2n} = 0$ but $F^{2n-1} \neq 0$, so the Jordan decomposition of F has just one $2n \times 2n$ box. The signature of Q is $n, n-1$ or $n-1, n$.

g) $S = T^*\mathbf{R}^n$ with n odd ≥ 3 and

$$Q(x,\xi) = 2\sum_1^{n-1} x_j\xi_{j+1};$$

then $F^n = 0$ but F^{n-1} has rank 2 so there are two $n \times n$ boxes in the Jordan decomposition. The signature of Q is $n-1, n-1$.

Williamson [6] lists the possibilities which can occur when S is of dimension 4 and Q is non-degenerate.We may then have case b) with $n = 1$, cases a) or c) with $n = 2$ or a direct sum of cases a) and/or c) with $n = 1$, altogether 6 possibilities. No explicit conclusions are given otherwise but Theorem 3.1 is contained in principle in the results of [6]. The explicit normal forms given in [8] agree with Theorem 3.1 apart from case c) where in [8] there appear two cases depending on the parity of n, due to another choice of symplectic bases, and one case which is decomposable into two spaces of type c).

4 General Mehler formulas

Let Q be a quadratic form in $T^*\mathbf{R}^n$, and let Q^w be the corresponding Weyl operator (see [4, Sect. 18.5]). Assuming that e^{tQ^w} makes sense we want to determine the Weyl symbol which we denote by e^{q_t}, so that $(\exp q_t)^w = \exp(tQ^w)$. Differentiation with respect to t should give

$$(\partial q_t/\partial t \exp q_t)^w = Q^w \exp(tQ^w) = Q^w(\exp q_t)^w,$$

which by the calculus (see [4, (18.5.6)]) means that

$$\begin{aligned}\partial q_t/\partial t \exp q_t &= e^{\frac{i}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}Q(x,\xi)\exp q_t(y,\eta)|_{(x,\xi)=(y,\eta)}\\ &= Q\exp q_t + \tfrac{1}{2i}\{Q,\exp q_t\} - \tfrac{1}{8}\Big(\sum_{j,k=1}^n \partial^2 Q/\partial\xi_j\partial\xi_k\partial^2/\partial x_j\partial x_k\\ &+ \sum_{j,k=1}^n \partial^2 Q/\partial x_j\partial x_k\partial^2/\partial\xi_j\partial\xi_k - 2\sum_{j,k=1}^n \partial^2 Q/\partial x_j\partial\xi_k\partial^2/\partial\xi_j\partial x_k\Big)\exp q_t\end{aligned}$$

or if the differentiation is carried out,

$$\partial q_t/\partial t = Q + \tfrac{1}{2i}\{Q, q_t\} - \tfrac{1}{8}\Big(\sum_{j,k=1}^{n} \partial^2 Q/\partial\xi_j\partial\xi_k \partial^2 q_t/\partial x_j\partial x_k$$
$$+ \sum_{j,k=1}^{n} \partial^2 Q/\partial x_j\partial x_k \partial^2 q_t/\partial\xi_j\partial\xi_k - 2\sum_{j,k=1}^{n} \partial^2 Q/\partial x_j\partial\xi_k \partial^2 q_t/\partial\xi_j\partial x_k\Big)$$
$$-\tfrac{1}{8}\Big(\sum_{j,k=1}^{n} \partial^2 Q/\partial\xi_j\partial\xi_k \partial q_t/\partial x_j \partial q_t/\partial x_k$$
$$+ \sum_{j,k=1}^{n} \partial^2 Q/\partial x_j\partial x_k \partial q_t/\partial\xi_j \partial q_t/\partial\xi_k - 2\sum_{j,k=1}^{n} \partial^2 Q/\partial x_j\partial\xi_k \partial q_t/\partial\xi_j \partial q_t/\partial x_k\Big).$$

We claim that there is a solution of the form

$$q_t = g_t + h(t)$$

where g_t is a quadratic form in (x, ξ). Separating terms of degree 2 and 0 with respect to (x, ξ) gives the equations

$$\partial g_t/\partial t = Q + \tfrac{1}{2i}\{Q, g_t\} - \tfrac{1}{8}\Big(\textstyle\sum_{j,k=1}^{n} \partial^2 Q/\partial\xi_j\partial\xi_k \partial g_t/\partial x_j \partial g_t/\partial x_k$$
$$+\textstyle\sum_{j,k=1}^{n} \partial^2 Q/\partial x_j\partial x_k \partial g_t/\partial\xi_j \partial g_t/\partial\xi_k - 2\sum_{j,k=1}^{n} \partial^2 Q/\partial x_j\partial\xi_k \partial g_t/\partial\xi_j \partial g_t/\partial x_k\Big),$$
$$h'(t) = -\tfrac{1}{8}\Big(\textstyle\sum_{j,k=1}^{n} \partial^2 Q/\partial\xi_j\partial\xi_k \partial^2 g_t/\partial x_j\partial x_k$$
$$+\textstyle\sum_{j,k=1}^{n} \partial^2 Q/\partial x_j\partial x_k \partial^2 g_t/\partial\xi_j\partial\xi_k - 2\sum_{j,k=1}^{n} \partial^2 Q/\partial x_j\partial\xi_k \partial^2 g_t/\partial\xi_j\partial x_k\Big). \tag{4.1}$$

This can be regarded as a system of non-linear ordinary differential equations for h and the $n(2n+1)$ coefficients of g_t, so for small t there is a unique solution with $q_0 = 0$, that is, $g_0 = 0$ and $h(0) = 0$. We want to examine if there is a global solution. In doing so we can clearly work over the complex numbers since the preceding equations are valid over $\mathbf{C}$, and we look for g_t and $h(t)$ as analytic functions of t.

First consider the non-degenerate case with one degree of freedom. It is then enough to study the case $Q(x,\xi) = 2\lambda x\xi$ for some $\lambda \in \mathbf{C}\setminus\{0\}$. The equation for g_t is

$$\partial g_t/\partial t = 2\lambda(x\xi + \tfrac{1}{2i}\{x\xi, g_t\} + \tfrac{1}{4}\partial g_t/\partial\xi \partial g_t/\partial x),$$

which is solved by $g_t = \gamma(t)x\xi$ where

$$\gamma'(t) = 2\lambda(1 + \tfrac{1}{4}\gamma^2), \quad \gamma(0) = 0; \quad \text{thus } \gamma(t) = 2\tan(\lambda t).$$

The equation $h'(t) = \lambda\gamma(t)/2 = \lambda\tan(\lambda t)$ has the solution $h(t) = -\log\cos(\lambda t)$ so

$$\exp q_t(x,\xi) = \frac{e^{2x\xi\tan(\lambda t)}}{\cos(\lambda t)},$$

which is analytic except at the zeros of $\cos(\lambda t)$.

Suppose now that we have n degrees of freedom and that

$$Q(x,\xi)=\sum_1^n 2\lambda_j x_j\xi_j.$$

A generic form is symplectically equivalent to such a form. Then we have

$$g_t(x,\xi)=2\sum_1^n x_j\xi_j\tan(\lambda_j t),\quad h'(t)=\sum_1^n \lambda_j\tan(\lambda_j t),$$

$$\text{thus } \exp(-h(t))=\prod_1^n\cos(\lambda_j t).$$

The Hamilton map of Q is

$$F=\begin{pmatrix}\Lambda & 0\\ 0 & -\Lambda\end{pmatrix}\quad \Lambda=\operatorname{diag}(\lambda_1,\dots,\lambda_n).$$

If $X=(x,\xi)$ then $\sigma(X,\tan(tF)X)=2\langle\xi,\tan(t\Lambda)\rangle=g_t(X)$, so we obtain

$$g_t(X)=\sigma(X,\tan(tF)X),\quad \exp(-h(t))=\prod_1^n\cos(\lambda_j t),\tag{4.2}$$

except at the zeros of the product.

For an arbitrary quadratic form Q in $T^*\mathbf{C}^n$, with Hamilton map F, it is clear that $\sin F$ and $\cos F$ are entire analytic functions of (the coefficients of) F, and $\det(\cos F)=\prod_1^{2n}\cos\lambda_j$ where λ_j are all the $2n$ eigenvalues of F with multiple ones repeated. Now the secular equation $\det(F-\lambda I)=0$ is even in λ, for in the generic special case above it is equal to the product $\prod(\lambda^2-\lambda_i^2)$, taken over one half of the zeros. Let $\mu_1,\dots,\mu_n$ be the zeros of $\det(F-\lambda I)$ as a polynomial in $\mu=\lambda^2$. Then $\det(\cos F)=\prod_1^n(\cos\sqrt{\mu_j})^2$, and the square root $\sqrt{\det(\cos F)}=\prod_1^n\cos\sqrt{\mu_j}$ is an analytic function of $\mu_1,\dots,\mu_n$ which is symmetric under permutations, hence an entire function of the elementary symmetric functions which are polynomials in F. Thus $\sqrt{\det(\cos F)}$ is an entire analytic function of F. Outside the zeros it is clear that $\tan F=\sin F(\cos F)^{-1}$ is analytic.

Theorem 4.1 *For every quadratic form Q in $T^*\mathbf{C}^n$ the system of differential equations* (4.1) *for a quadratic form $g_t(x,\xi)$ and a scalar function $h(t)$ has a unique solution with vanishing initial values such that g_t and $\exp h(t)$ are meromorphic functions,*

$$\exp(q_t(X))=\exp(g_t(X)+h(t))=\exp(\sigma(X,\tan(tF)X))/\sqrt{\det(\cos tF)}\tag{4.3}$$

where F is the Hamilton map of Q. It is analytic in t and F except at the zeros of the denominator, that is, where $t\lambda_j=\frac12\pi+k\pi$ for some eigenvalue λ_j of F and some $k\in\mathbf{Z}$.

If we have a symplectic decomposition of the whole space such that Q has a corresponding splitting, then we can compute q_t separately for the different factors. This makes it easy to make explicit computations by using the normal forms for F.

Derezinski [3, Theorem 3.2] proved that the symbol of every operator of the form $f(Q^w)$ is a function of Q precisely when $F^3 = cF$ for some c. This is an immediate consequence of Theorem 4.1, for if $\tan(tF)$ is a multiple of F for every t then F^3 must be a multiple cF of F. Conversely, if $F^3 = cF$ then $\tan(tF) = F\tan(t\sqrt{c})/\sqrt{c}$, interpreted as Ft when $c = 0$.

As an example consider the quadratic form $Q(x,\xi) = -x^2 - \xi^2$ for $(x,\xi) \in T^*\mathbf{R}$. Then $F(x,\xi) = (-\xi, x)$ so $F^3 = -F$, $\lambda = \pm i$, and $\tan(tF) = F\tan(ti)/i = F\tanh t$. Thus

$$e^{q_t(x,\xi)} = e^{-(x^2+\xi^2)\tanh t}/\cosh t. \tag{4.4}$$

It is well known (and will be proved more generally below) that $Q^w = -x^2 - D^2$ defines a self-adjoint operator ≤ 0 in $L^2(\mathbf{R})$, so e^{tQ^w} is a well defined self-adjoint semigroup of contractions in $L^2(\mathbf{R})$ when $t \geq 0$. We claim that the Weyl symbol is given by (4.4) when $t > 0$; this is a function in $\mathscr{S}(\mathbf{R}^2)$. To prove this we recall that if $\mathscr{A}_t$ is the Weyl operator corresponding to (4.4) we have by the derivation of (4.1)

$$Q^w \mathscr{A}_t u = \frac{\partial}{\partial t}\mathscr{A}_t u, \quad u \in \mathscr{S}(\mathbf{R}), \quad \text{hence}$$
$$\frac{\partial}{\partial s}(e^{(t-s)Q^w}\mathscr{A}_s u) = e^{(t-s)Q^w}(-Q^w + Q^w)\mathscr{A}_s u = 0, \quad u \in \mathscr{S}(\mathbf{R}),$$

for $0 < s < t$, which proves that $\mathscr{A}_t u = e^{tQ^w}u$ as claimed.

The kernel of the operator $\mathscr{A}_t$ is (cf. [4, (18.5.4)])

$$A_t(x,y) = \frac{1}{2\pi}\int \exp\big(-((x+y)^2/4 + \xi^2)\tanh t + i(x-y)\xi\big)\, d\xi/\cosh t,$$

and evaluation of the Gaussian integral gives the classical Mehler formula:

Corollary 4.2 *The Weyl symbol of the self-adjoint contraction* $\exp(-t(x^2+D^2))$ *in* $L^2(\mathbf{R})$, $t > 0$, *is given by* (4.4), *and the kernel is given by Mehler's formula*

$$(x,y) \mapsto \exp\big(-\tfrac{1}{2}((x^2+y^2)\cosh(2t) - 2xy)/\sinh(2t)\big)/\sqrt{2\pi\sinh(2t)}. \tag{4.5}$$

The formula (4.4) for the Weyl symbol is so much simpler since it respects the orthogonal invariance of $x^2 + \xi^2$.

For any quadratic form Q in $T^*\mathbf{R}^n$ we denote by M_Q the maximal operator defined by Q^w in $L^2(\mathbf{R}^n)$; the domain consists of all $u \in L^2(\mathbf{R}^n)$ such that $M_Q u = Q^w u$, defined in the distribution sense, is in $L^2(\mathbf{R}^n)$. The continuity of Q^w in $\mathscr{D}'(\mathbf{R}^n)$ proves that M_Q is a closed operator. It is the closure of the restriction to $\mathscr{S}(\mathbf{R}^n)$. In fact, if $\chi \in C_0^\infty(\mathbf{R}^{2n})$ is equal to 1 in a neighborhood of 0, and $\chi_\varepsilon(\xi,\eta) = \chi(\varepsilon\xi, \varepsilon\eta)$, then

$$Q^w\chi_\varepsilon^w - \chi_\varepsilon^w Q^w = -i\{Q,\chi_\varepsilon\}^w + \varepsilon^2\psi_\varepsilon^w$$

where ψ_ε is defined as χ_ε with another function $\psi \in C_0^\infty$. Since

$$\{Q, \chi_\varepsilon\}(x,\xi) = \sum_1^n (\partial Q(x,\xi)/\partial\xi_j \partial\chi(\varepsilon x, \varepsilon\xi_j)/\partial x_j \\ -\partial Q(x,\xi)/\partial x_j \partial\chi(\varepsilon x, \varepsilon\xi)/\partial\xi_j)$$

is uniformly bounded in the symbol space $S(1,(dx^2 + d\xi^2)/(1 + |x|^2 + |\xi|^2))$ as $0 < \varepsilon < 1$ and vanishes on any compact set for small ε, it follows that $(Q^w\chi_\varepsilon^w - \chi_\varepsilon^w Q^w)u \to 0$ in $L^2(\mathbf{R}^n)$ as $\varepsilon \to 0$, if $u \in L^2(\mathbf{R}^n)$. If u is in the domain of M_Q it follows that $\chi_\varepsilon u \in \mathcal{S}$ converges to u in the graph norm of M_Q. (Since translation of Q yields a unitarily equivalent Weyl operator it is also obvious that Re $Q \le 0$ if $M_{\text{Re}\,Q}$ is bounded above.) Thus M_Q is the closure of the restriction to $\mathcal{S}$ or C_0^∞, which proves that the adjoint is equal to $M_{\overline{Q}}$.

If Re $Q \le 0$ then

$$\text{Re}\,(M_Q u, u) = (M_{\text{Re}\,Q} u, u) \le 0$$

when u is in the domain of M_Q. This follows from the metaplectic invariance of the Weyl calculus (cf. [4, Theorem 18.5.9]) since Re Q is symplectically equivalent to a sum of (possibly degenerate) harmonic oscillators ([3, Theorem 21.5.3] or Theorem 3.1, cases c) with $n = 1$, d), e)). Thus M_Q and its adjoint are both dissipative, so M_Q generates a contraction semigroup (cf. Yosida [7, p. 251]) which we shall denote by $\exp(tQ^w)$, $t \ge 0$.

Theorem 4.2 *If* Re Q *is negative definite, then the Weyl symbol* A_t *of* $\exp(tQ^w)$ *is for* $t > 0$ *a function in* $\mathcal{S}(T^*\mathbf{R}^n)$ *given by* (4.3) *where the quadratic form* g_t *and* $\exp(h(t))$ *are always finite, and* g_t *has negative definite real part. When* Re Q *is just negative semidefinite, the map* $Q \mapsto \exp(Q^w)u$ *is a continuous function (resp.* C^∞ *function) of* Q *with values in* $L^2(\mathbf{R}^n)$ *(resp.* $\mathcal{S}(\mathbf{R}^n)$*) if* $u \in L^2(\mathbf{R}^n)$ *(resp.* $u \in \mathcal{S}(\mathbf{R}^n)$*), and the Weyl symbol of* $\exp(Q^w)$ *is a continuous function of* Q *with values in* $\mathcal{S}'(\mathbf{R}^{2n})$ *given by* (4.3) *when* $\det(\cos F) \ne 0$.

Proof. If Re Q is negative definite, then no eigenvalue λ_j of F is real (cf. [4, Theorem 21.5.4]). Hence $\prod \cos(t\lambda_j) \ne 0$ when $t \ge 0$, so A_t is well defined and analytic in t then. When $t > 0$ is small enough then Re $g_t = t$Re $Q + O(t^3)(|x|^2 + |\xi|^2)$ is negative definite. Thus A_t is then in $\mathcal{S}$, and we can prove just as for the harmonic oscillator above that A_t^w is equal to $\exp(tQ^w)$ when $0 < t < t_0$, say. By the semigroup property it follows that if $t = t_1 + \ldots + t_N$ where $0 < t_j < t_0$ for $j = 1, \ldots, N$, then $\exp(tQ^w)$ is the composition of the operators $A_{t_j}^w$. They have kernels in $\mathcal{S}$ which are real analytic in t_j, so it follows that $\exp(tQ^w)$ for every $t > 0$ has a Weyl symbol in $\mathcal{S}$ and that it is analytic in t. Hence it must always be equal to A_t, and A_t must be in $\mathcal{S}$, which proves that Re g_t is negative definite. If $u \in \mathcal{S}$ it follows that $u(t) = \exp(tQ^w)u$ is a C^∞ function of t and Q with values in $\mathcal{S}$ when $t \ge 0$ and Re Q is negative definite. We can estimate the L^2 norms of the derivatives $u_{\alpha\beta} = D^\alpha x^\beta u$ by noting that for any $N \ge 0$

$$\partial u_{\alpha\beta}/\partial t = D^\alpha x^\beta Q^w u = Q^w u_{\alpha\beta} + \sum_{|\alpha'+\beta'|\le N} c_{\alpha\beta,\alpha'\beta'} u_{\alpha'\beta'}, \quad |\alpha + \beta| \le N$$

where $c_{\alpha\beta,\alpha'\beta'}$ are linear forms in the coefficients of Q. Hence

$$\frac{d}{dt}\sum_{|\alpha+\beta|\le N}\|u_{\alpha\beta}\|^2 = 2\sum_{|\alpha+\beta|\le N}\operatorname{Re}\,(\partial u_{\alpha\beta}/\partial t, u_{\alpha\beta}) \le C_{N,Q}\sum_{|\alpha+\beta|\le N}\|u_{\alpha\beta}\|^2.$$

Integration gives a bound for $\sum_{|\alpha+\beta|\le N}\|u_{\alpha\beta}\|^2$ when $t\ge 0$ depending only on upper bounds for t, the coefficients of Q and the values when $t=0$. When Re Q is negative semidefinite, and u is a C^∞ function of $t\ge 0$ with values in $\mathscr{S}$ such that $\partial u/\partial t = Q^w u$ for $t\ge 0$, then $u(t)=\exp(tQ^w)u(0)$, $t\ge 0$, so the bounds just obtained in the negative definite case extend to the negative semidefinite case. Since $\mathscr{S}$ is dense in L^2 the continuity with values in L^2 for $u\in L^2$ follows in view of the contraction property.

By the Schwartz kernel theorem and the continuity just proved it follows that the kernel of $\exp(Q^w)$ is a continuous function of Q with values in $\mathscr{S}'$, so this is also true for the Weyl symbol (see [4, (18.5.4)$''$]).

When Re Q is negative semidefinite, it follows from Theorem 4.3 that the Weyl symbol of $\exp(Q^w)$ is given by (4.3) with $t=1$ provided that the denominator is not 0, that is, if F has no eigenvalue of the form $\frac{1}{2}\pi + k\pi$ with integer k. To prepare for the discussion of the remaining case it is instructive to examine the harmonic oscillator in detail. We know already that the Weyl symbol of the operator $\exp(-t(x^2+D^2))$ in $L^2(\mathbf{R})$ is given by

$$(x,\xi)\mapsto \exp(-(x^2+\xi^2)\tanh t)/\cosh t, \tag{4.6}$$

when Re $t\ge 0$ and $\cosh t\ne 0$. In particular, if $t=is$ with $s\in\mathbf{R}$ and $\cos s\ne 0$ it is given by

$$(x,\xi)\mapsto \exp(-i(x^2+\xi^2)\tan s)/\cos s. \tag{4.7}$$

If $s=\pi/2+k\pi$ for some integer k, then the Weyl symbol is the limit in $\mathscr{S}'(\mathbf{R}^2)$ as $\varepsilon\to +0$ of the symbol for $t=is+\varepsilon$,

$$(x,\xi)\mapsto \exp(-(x^2+\xi^2)/\tanh\varepsilon)i(-1)^{k+1}/\sinh\varepsilon,$$

that is, $i(-1)^{k+1}\pi\delta_0$. The corresponding Weyl operator has the kernel $i(-1)^{k+1}\times\delta_0(x+y)$, which defines a reflection operator. When $\sin s=0$ then the Weyl symbol (4.7) is ± 1, and the corresponding kernel is $\pm\delta_0(x-y)$, so we have $\pm$ the identity operator. When $2\sin s\cos s=\sin(2s)\ne 0$ the kernel of $e^{-is(x^2+D^2)}$ is easily obtained from (4.5); it is

$$(x,y)\mapsto \exp\big(\frac{i}{2}((x^2+y^2)\cos(2s)-2xy)/\sin(2s)\big)/\sqrt{2\pi i\sin(2s)}. \tag{4.8}$$

Since $\sqrt{2\pi\sinh(2t)}=\cosh t\sqrt{4\pi\tanh t}$ and Re $\tanh t>0$ when Re $t>0$, the square root in (4.8) should be taken in the right (left) half plane when $\cos s>0$ ($\cos s<0$). When $\cos(2s)=0$, hence $\sin(2s)=\pm 1$, the exponential reduces to $\exp(\mp ixy)$, so $e^{-is(x^2+D^2)}$ becomes the (inverse) Fourier transformation apart from a factor $\pm e^{\pm\pi i/4}$. In particular, when $s=\pi/4$ then the operator is $e^{-\pi i/4}$ times the Fourier transformation, and from the group property and Fourier's inversion formula we obtain the values at $s = \nu\pi/4$ for any integer ν which we have already given. (Anders Melin has pointed out that this follows from the expansion in Hermite functions which are eigenfunctions of the Fourier transformation.)

We turn now to the case of a general quadratic form Q in $T^*\mathbf{R}^n$ with Re $Q \leq 0$. First we shall rewrite the result already proved when $\det\cos F \neq 0$. Then $\sec F = (\cos F)^{-1}$ is defined, and

$$\sigma_F(X,Y) = \sigma(X,(\sec F)Y) \tag{4.9}$$

is also a symplectic form in $T^*(\mathbf{C}^n)$. The corresponding measure vol_{σ_F}, defined by the nth power of σ_F and the standard orientation, is equal to $\mathrm{vol}_\sigma/\sqrt{\det\cos F}$ where vol_σ is defined by the standard symplectic form. That σ_F is skew symmetric is obvious since $\cos F$ is even in F, and it is non-degenerate since $\sec F$ is bijective. When $F = \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}$ with $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ then

$$\sigma_F = \sum_1^n d\xi_j \wedge dx_j / \cos\lambda_j$$

which proves that $\mathrm{vol}_{\sigma_F} = \mathrm{vol}_\sigma/\prod_1^n \cos\lambda_j$ as claimed. Since this situation is symplectically equivalent to the generic case, the statement follows. The result of Theorem 4.3 is now that

$$A_t(X)\mathrm{vol}_\sigma = \exp(\sigma(X,(\tan F)X))\mathrm{vol}_{\sigma_F}. \tag{4.10}$$

Our next goal is to give a sense to the two factors in the right-hand side when $\det\cos F = 0$. As suggested by the example of the harmonic oscillator vol_{σ_F} will be replaced by a translation invariant measure on the range of $\cos F$, and the quadratic form in the exponential will only be defined there. We need some preliminaries:

Proposition 4.4 *If* $Q = Q_1 + iQ_2$ *where* $Q_1 \leq 0$, *then* Ker $(F - \lambda)$ *is the complex conjugate of* Ker $(F+\lambda)$ *for every* $\lambda \in \mathbf{R}$, *and if* $F = F_1 + iF_2$ *then* F_1Ker $(F \pm \lambda) = 0$. *Thus* Ker $(F - \lambda) \oplus$ Ker $(F + \lambda)$, $0 \neq \lambda \in \mathbf{R}$, *is the complexification of its intersection with* $T^*\mathbf{R}^n$, *and so is* Ker F.

Proof. Assume that $X \in T^*\mathbf{C}^n$ and that $(F - \lambda)X = 0$. This means that

$$Q(Y,X) = \sigma(Y,FX) = \lambda\sigma(Y,X)$$

for every Y, thus

$$Q(\overline{X},X) = \lambda\sigma(\overline{X},X) = 2\lambda i\sigma(\mathrm{Re}\,X,\mathrm{Im}\,X),$$

so $Q_1(\overline{X},X) = 0$. Since Q_1 is semidefinite it follows that $Q_1(Y,X) = 0$ for arbitrary Y, that is, $F_1X = 0$. Thus $(iF_2 - \lambda)X = 0$ so $(-iF_2 - \lambda)\overline{X} = 0$, and $(F + \lambda)\overline{X} = 0$. The proof is complete.

Remark. The generalized eigenspaces need not be complex conjugates. For example, if $Q = -\xi_1^2 - \xi_2^2 - x_2^2 - 2i\xi_1 x_2$ then $F(x,\xi) = (x',\xi')$ where $x' = (-\xi_1 - ix_2, -\xi_2)$ and $\xi' = (0, x_2 + i\xi_1)$. The kernel of F is the x_1 axis and the kernel of F^2 is defined by $\xi_2 = 0$ and $x_2 = -i\xi_1$. The kernel of $\overline{F}^2$ is defined by $\xi_2 = 0$ and $x_2 = i\xi_1$, so the intersection of the kernels of F^2 and $\overline{F}^2$ is the kernel of F.

Proposition 4.5 *If* $\operatorname{Re} Q \leq 0$ *then the kernel (resp. range) of* $\cos F$ *is the complexification of its intersection* $K(F)$ *(resp.* $W(F)$*) with* $T^*\mathbf{R}^n$. *The restriction of* $(\sin F)/i$ *to* $K(F)$ *is a bijection with square equal to minus the identity, so it defines a complex vector space structure in* $K(F)$ *and the determinant is equal to* 1. *The corresponding orientation induces an orientation in* $W(F)$.

Proof. The kernel of $\cos F$ is the direct sum of the kernels of $F - \lambda$ for all λ with $\cos\lambda = 0$. In fact, the restriction of F to V_λ is equal to $\lambda + T$ where T is nilpotent. Hence $\cos F - \cos\lambda = \cos(\lambda + T) - \cos\lambda$ is nilpotent and $\cos F$ is invertible if $\cos\lambda \neq 0$. If $\cos\lambda = 0$ then $\cos F = \cos(\lambda + T) = -\sin\lambda \sin T$. Since $(\sin T)/T - 1$ is nilpotent the operator $(\sin T)/T$ is invertible so the kernel of $\sin T$ is equal to that of T. From Proposition 4.4 it follows now that the kernel is the complexification of its intersection with $T^*\mathbf{R}^n$, and since the range is the σ orthogonal space it has the same property.

The restriction of $\sin F$ to Ker $(F - \lambda) \oplus$ Ker $(F + \lambda)$ is equal to $\sin\lambda$ and $-\sin\lambda$ in the two factors. Hence $(\sin F)^2$ is equal to the identity in Ker $\cos F$. If $X \in$ Ker $(F - \lambda)$ and $\lambda \in \mathbf{R}$, it follows from Proposition 4.4 that $F_1 X = 0$ and $iF_2 X = \lambda X$, hence $(\sin F)X = (\sin(iF_2))X = i(\sinh F_2)X$, so $(\sin F)/i$ defines a bijection in $K(F)$, with square equal to minus the identity. It makes $K(F)$ a complex vector space, so it defines a natural orientation which gives rise to an orientation in $T^*\mathbf{R}^n/W(F)$, which is dual with respect to the symplectic form. Since $T^*\mathbf{R}^n$ has a natural orientation as a real symplectic vector space and the real dimension of $K(F)$ is even, this defines an orientation in $W(F)$.

We can define a symplectic form in the complex vector space $W(F)_\mathbf{C}$ by

$$\sigma_F((\cos F)X, (\cos F)Y) = \sigma((\cos F)X, Y), \quad X, Y \in T^*\mathbf{C}^n. \tag{4.9$'$}$$

This agrees with (4.9) when $\det\cos F \neq 0$. The definition (4.9)$'$ is unique, for $\cos F$ is even in F so the right-hand side is equal to $\sigma(X, (\cos F)Y)$, hence equal to 0 if $(\cos F)X = 0$ or $(\cos F)Y = 0$. It is obvious that σ_F is skew symmetric. If the right-hand side of (4.9)$'$ vanishes for every Y then $(\cos F)X = 0$ which proves that the form is non-degenerate. Thus we have a symplectic form defined in the range $W(F)_\mathbf{C}$ of $\cos F$, and the corresponding volume form defines a translation invariant measure $\text{vol}\sigma_F \neq 0$ in $W(F)$ when combined with the orientation defined in Proposition 4.5.

We define the quadratic form similarly. The quadratic form in (4.10) corresponds to the symmetric bilinear form

$$E_F(X, Y) = \sigma(X, (\tan F)Y),$$

and we extend the definition to the case where $\det\cos F = 0$ by

$$E_F((\cos F)X, (\cos F)Y) = \sigma((\cos F)X, (\sin F)Y), \quad X, Y \in T^*\mathbf{C}^n. \tag{4.11}$$

Since $\cos F$ is symmetric and $\sin F$ is skew symmetric with respect to σ the right-hand side is equal to $\sigma((\cos F)Y, (\sin F)X)$, which proves that the form is uniquely defined and symmetric. We can now state the main result of this section:

Theorem 4.6 *If* Re $Q \leq 0$ *then* (4.9)′ *defines a symplectic form in the range* $W(F)_{\mathbf{C}}$ *of* $\cos F$ *where* F *is the Hamilton map of* Q, *and* (4.11) *defines a symmetric bilinear form with*

$$\operatorname{Re} E_F(X, X) \leq 0 \quad \textit{when } X \in W(F).$$

The product of the Weyl symbol of $\exp(Q^w)$ *and* vol_σ *is equal to* $(\pi i)^\nu \exp(E_F)\mathrm{vol}_{\sigma_F}$ *where* 2ν *is the dimension of the kernel* $K(F)$ *of* $\cos F$. *Here* vol_σ *is the positive measure defined by the symplectic form, and* vol_{σ_F} *is the measure in* $W(F)$ *defined by the form* σ_F *and the orientation of* $W(F)$ *in Proposition 4.5.*

Proof. If $u_\nu \in \mathscr{S}'$ and $u_\nu \to u$ in $\mathscr{S}'$ then $\langle u_\nu, e^G\rangle \to \langle u, e^G\rangle$ for every quadratic form G with negative definite real part. If u_ν is even then u is even and this determines u. In fact, if $v \in \mathscr{S}'$ and $\langle v, e^G\rangle = 0$ for all such forms G, then differentiation with respect to the coefficients of G shows that $\langle v, X^\alpha G\rangle = 0$ for every monomial X^α of even degree, hence for every α if v is even. The Fourier-Laplace transform of ve^G is then an entire function with all derivatives equal to 0 at the origin so $v = 0$. We can therefore prove the theorem by examining only scalar products with Gaussians.

Let us first recall some well known facts on integrals of Gaussians. If A is a symmetric $N \times N$ matrix with Re A positive definite then

$$\int_{\mathbf{R}^N} e^{-\langle AX, X\rangle}\, dX = \pi^{N/2}/\sqrt{\det A}. \tag{4.12}$$

Here dX is the Lebesgue measure in $\mathbf{R}^N$ and $\sqrt{\det A}$ is defined so that it is an analytic function of A in the convex set of matrices with Re A positive definite, and equal to 1 at the identity matrix. If Q is a complex valued quadratic form in a real symplectic vector space S of dimension $2n$ and Re Q is negative definite, we write as before $Q(X, Y) = \sigma(X, FY)$ where F is the Hamilton map and claim that

$$\int_S e^Q \mathrm{vol}_\sigma = \pi^n/\sqrt{\det F} \tag{4.13}$$

with $\sqrt{\det F} = \sqrt{\prod_1^{2n} \lambda_j}$ defined analytically so that it is positive when Q is real, thus the eigenvalues λ_j occur in complex conjugate pairs. To prove (4.13) we may assume that $S = T^*\mathbf{R}^n$ with standard coordinates $X = (x, \xi)$ and we write

$$\sigma(X, Y) = \langle EX, Y\rangle, \quad E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbf{R}^{2n} \cong T^*\mathbf{R}^n$. Then

$$Q(X,Y) = \sigma(X,FY) = \langle EX,FY\rangle = -\langle X,EFY\rangle,$$

so EF is symmetric and we can apply (4.12) with $A = EF$. Since $\det E = 1$ this proves (4.13).

If $M : S_{\mathbf{C}} \to S_{\mathbf{C}}$ is an invertible complex linear map which is symmetric with respect to σ, then $\tilde{\sigma}(X,Y) = \sigma(X,M^{-1}Y)$ is a complex symplectic form in $S_{\mathbf{C}}$, and every complex symplectic form there has such a representation. Since $\sigma(X,FY) = \tilde{\sigma}(X,MFY)$ the Hamilton map $\tilde{F}$ of Q with respect to $\tilde{\sigma}$ is equal to MF. Since $\mathrm{vol}_\sigma = \sqrt{\det M}\,\mathrm{vol}_{\tilde{\sigma}}$ if we keep the orientation in S defined by σ, it follows from (4.13) that

$$\int_S e^Q \mathrm{vol}_{\tilde{\sigma}} = \pi^n/(\sqrt{\det F}\sqrt{\det M}) = \pi^n/\sqrt{\det \tilde{F}}$$

for a suitable choice of the square root. Thus (4.13) remains valid with such an uncertainty of the sign if σ is any complex symplectic form in $S_{\mathbf{C}}$.

If F is the Hamilton map of the quadratic form Q in the theorem, then the zeros $t \in \mathbf{R}$ of the entire analytic function $\det\cos(tF)$ are isolated. For $t \neq 1$ in some neighborhood of 1 it follows from Theorem 4.3 that the Weyl symbol A_t of $\exp(tQ^w)$ is equal to

$$\exp(\sigma(X,\tan(tF)X)/\sqrt{\det\cos tF},$$

and that $A_t \to A_1$ in $\mathscr{S}'$ as $t \to 1$. Since $\mathrm{Re}\ \sigma(X,\tan(tF)\overline{X}) \le 0$ when $\det\cos(tF) \neq 0$, by Theorem 4.3, we have

$$0 \ge \mathrm{Re}\ \sigma(\cos(tF)X,\tan(tF)\cos(t\overline{F})\overline{X}) = \mathrm{Re}\ \sigma(\cos(t\overline{F})\overline{X},\sin(tF)X)$$
$$\to \mathrm{Re}\ \sigma((\cos\overline{F})\overline{X},(\sin F)X) \text{ as } t \to 1.$$

On the other hand, since the range of $\cos F$ is invariant under conjugation by Proposition 4.5, we may replace $(\cos F)X$ by $\overline{(\cos F)Y}$ in (4.11) which gives

$$\mathrm{Re}\ E_F(\overline{(\cos F)Y},(\cos F)Y) = \mathrm{Re}\ \sigma((\cos\overline{F})\overline{Y},(\sin F)Y) \le 0,$$

and proves that $\mathrm{Re}\ E_F(X,X) \le 0$ when X is real and in the range of $\cos F$.

Let G be a quadratic form with negative definite real part and denote its Hamilton map by Φ. Then for $t \neq 1$ real and close to 1

$$\begin{aligned}\langle A_t, e^G\rangle &= \int \exp(\sigma(X,(\tan(tF)+\Phi)X))\mathrm{vol}_\sigma/\sqrt{\det\cos(tF)}\\ &= \pi^n/\sqrt{\det\cos(tF)\det(\tan(tF)+\Phi)}\\ &= \pi^n/\sqrt{\det(\sin(tF)+\cos(tF)\Phi)}\\ &\to \pi^n/\sqrt{\det(\sin F+(\cos F)\Phi)}, \quad \text{when } t \to 1.\end{aligned}$$

(We postpone discussing the choice of the square root.) To calculate the determinant we note that $\sin F + (\cos F)\Phi$ maps the range $W(F)_{\mathbf{C}}$ of $\cos F$ to itself since $\sin F$ commutes with $\cos F$. The determinant is therefore equal to the product of the determinant of the restriction to $W(F)_{\mathbf{C}}$ and the determinant of the map induced in the quotient of $T^*\mathbf{C}^n$ and $W(F)_{\mathbf{C}}$. Now the σ orthogonal space Ker $\cos F$ of $W(F)_{\mathbf{C}}$ is dual to this quotient space and the adjoint of the map induced in the quotient is the restriction of the adjoint $-\sin F - \Phi\cos F$ to Ker $\cos F$, hence equal to $-\sin F$. With $2\nu = \dim \operatorname{Ker} \cos F$ it follows from Proposition 4.5 that the determinant of $\sin F + (\cos F)\Phi$ is $(-1)^\nu$ times the determinant D of the restriction to $W(F)_{\mathbf{C}}$. Thus for some choice of the square root

$$\langle A_1, e^G\rangle = \pi^n i^\nu/\sqrt{D}. \tag{4.14}$$

The theorem will be proved if we show that the right-hand side is equal to

$$\pi^\nu i^\nu \int_{W(F)} \exp(E_F(X,X) + G(X))\mathrm{vol}\sigma_F. \tag{4.15}$$

By the remarks made on (4.13) we can evaluate the integral in (4.15) using (4.13) in spite of the fact that σ_F is a complex valued symplectic form. By (4.9)$'$ we have

$$\sigma((\cos F)X, \Phi(\cos F)Y) = \sigma_F((\cos F)X, (\cos F)\Phi(\cos F)Y),$$

which proves that with respect to σ_F the Hamilton map of the restriction of G to W is equal to $(\cos F)\Phi$. By (4.11) and (4.9)$'$

$$E_F((\cos F)X, (\cos F)Y) = \sigma((\cos F)X, (\sin F)Y)$$
$$= \sigma_F((\cos F)X, (\cos F \sin F)Y) = \sigma_F((\cos F)X, (\sin F)((\cos F)Y)),$$

which proves that with respect to σ_F the Hamilton map of E_F is defined by $\sin F$. Hence the integral in (4.15) is equal to $\pi^{n-\nu}/\sqrt{D}$, and we have proved that

$$\langle A_1, e^G\rangle = \pm(\pi i)^\nu \int_{W(F)} \exp(E_F(X,X) + G(X))\mathrm{vol}\sigma_F.$$

The sign must be independent of G since both sides are continuous functions of G, so we have proved the statement except for the uncertainty of the sign, which we have not kept track of in the choice of square roots above.

To verify that the sign is correct in Theorem 4.6 we shall use a continuity method connecting to the case where Q is purely imaginary so that we can use the classification in Theorem 3.1. This argument will be postponed until we have made an explicit calculation in that case.

Thus assume now that Q/i is a real valued symplectic form in $T^*\mathbf{R}^n$. By the metaplectic invariance of the Weyl calculus we may assume that Q/i is a sum of polynomials as listed in Theorem 3.1 in different groups of variables. The formula for the Weyl symbol of $\exp(Q^w)$ is immediately given by Theorem 4.3 in all cases except case c) in Theorem 3.1 when $\cos\mu = 0$ so we assume now that

$$Q(x,\xi) = \gamma i\left(\mu\sum_1^n x_j x_{n+1-j} - \sum_2^n x_j x_{n+2-j} + \mu\sum_1^n \xi_j\xi_{n+1-j} - \sum_1^{n-1}\xi_j\xi_{n-j}\right), \tag{4.16}$$

where $\gamma = \pm 1$ and $\cos\mu = 0$. Then $F(x,\xi) = (x',\xi')$ where

$$x' = \gamma i\big(\mu(\xi_n,\dots,\xi_1) - (\xi_{n-1},\dots,\xi_1,0)\big),$$
$$\xi' = -\gamma i\big(\mu(x_n,\dots,x_1) - (0,x_n,\dots,x_2)\big).$$

If $\check{\xi} = (\xi_n,\dots,\xi_1) = -ix/\gamma$ it follows that

$$x' - \mu x = -Lx, \quad \xi' - \mu\xi = -R\xi$$

where $Lx = (x_2,\dots,x_n,0)$ is a left shift and $R\xi = (0,\xi_1,\dots,\xi_{n-1})$ is a right shift. This proves again that this is the space of generalized eigenvectors belonging to the eigenvalue μ. Similarly, when $\check{\xi} = ix/\gamma$ then

$$x' + \mu x = Lx, \quad \xi' + \mu\xi = R\xi$$

and we obtain the other generalized eigenspace. $K(F)$ is the $x_1\xi_n$ plane and $W(F)$ is defined by $\xi_1 = x_n = 0$. In the kernel of $F \mp \mu$ we have $\sin F = \pm\sin\mu$. Writing

$$X = (x_1,0,\dots,0,\xi_n) = (y_1,0,\dots,0,-iy_1/\gamma) + (z_1,0,\dots,0,iz_1/\gamma)$$

we obtain

$$(\sin F)X = \sin\mu(y_1 - z_1,0,\dots,0,-i(y_1+z_1)/\gamma) = i\gamma\sin\mu(\xi_n,0,\dots,0,-x_1).$$

Thus the analytic structure defined in $K(F)$ by $(\sin F)/i$ makes $\xi_n \pm ix_1$ an analytic coordinate if $\gamma\sin\mu = \pm 1$, so $K(F)$ is oriented by the form $\gamma\sin\mu d\xi_n\wedge dx_1$. The quotient space $T^*\mathbf{R}^n/W(F)$ is parametrized by x_n,ξ_1, and the duality with $K(F)$ is given by

$$\sigma((0,\dots,0,x_n,\xi_1,0,\dots,0),(y_1,0,\dots,0,\eta_n)) = \xi_1 y_1 - x_n\eta_n$$

so the orientation as a dual space of $K(F)$ is given by the orientation form $\gamma\sin\mu d\xi_1\wedge dx_n$. Since $T^*\mathbf{R}^n$ is oriented as a symplectic vector space by $d\xi_1\wedge dx_1\wedge\ldots\wedge d\xi_n\wedge dx_n$ where we can move dx_n to the right of $d\xi_1$ without changing the sign, it follows that $W(F)$, with coordinates $x_1,\dots,x_{n-1},\xi_2,\dots,\xi_n$, is oriented by the form

$$\gamma\sin\mu dx_1\wedge d\xi_2\wedge\ldots\wedge dx_{n-1}\wedge d\xi_n. \tag{4.17}$$

Next we must calculate σ_F using (4.9)′. To find $(\cos F)X$ we decompose X into its components in the two generalized eigenspaces

$$X = (x,\xi) = (y,-i\check{y}/\gamma) + (z,i\check{z}/\gamma).$$

In the space of generalized eigenvectors belonging to the eigenvalue μ we write $\cos F = \cos(F - \mu + \mu) = -\sin\mu\sin(F - \mu)$, which acts on the x coordinates as $(\sin\mu)\sin L$. For the other eigenvalue we find that $\cos F = \cos(F + \mu - \mu) = \sin\mu\sin(F + \mu)$ also acts as $(\sin\mu)\sin L$ in the x coordinates. Hence

$$(\cos F)X = \sin\mu((\sin L)x, (\sin R)\xi),$$

so

$$\sigma_F((\cos F)X, (\cos F)Y) = (\sin\mu)\sigma(((\sin L)x, (\sin R)\xi), (y, \eta)).$$

This means that for $X, Y \in W(F)$ and $Y = (y, \eta)$

$$\sigma_F(X, Y) = (\sin\mu)\sigma(X, ((\sin L)^{-1}y, (\sin R)^{-1}\eta)).$$

The right-hand side is defined since y is in the range of L and η is in the range of R, for $L/\sin L$ and $R/\sin R$ are equal to the identity plus nilpotent maps, so the right argument in the scalar product is defined modulo an element in $K(F)$ which is σ orthogonal to X. When we take the n − 1st power of σ_F regarded as a differential form we can replace $(\sin L)^{-1}$ and $(\sin R)^{-1}$ by L^{-1} and R^{-1} for the determinant of the identity plus a nilpotent matrix is equal to 1. Now the n − 1st power of the differential form

$$\sin\mu(d\xi_2 \wedge dx_1 + \ldots + d\xi_n \wedge dx_{n-1})$$

is equal to $(\sin\mu)^{n-1}d\xi_2 \wedge dx_1 \wedge \ldots \wedge d\xi_n \wedge dx_{n-1}$ which differs from (4.17) by the factor $\gamma(\sin\mu)^n(-1)^{n-1}$. Thus vol_{σ_F} is equal to $\gamma(\sin\mu)^n(-1)^{n-1}$ times the Lebesgue measure in the coordinates in $W(F)$.

To check Theorem 4.6 for this case we must also calculate the Weyl symbol of $\exp(Q^w)$ explicitly as the limit of that of $\exp(tQ^w)$ as $t = 1 + s/\mu \to 1$. Then $\sqrt{\det\cos(tF)} = (\cos(\mu + s))^n = (-\sin\mu\sin s)^n$. We have

$$\tan(tF) = \tan(t(F \mp \mu) \pm (\mu + s)) = -\cot(t(F \mp \mu) \pm s).$$

Recalling the Taylor expansion

$$\cot z = z^{-1} + G(z), \quad G(z) = \sum_1^\infty \frac{b_{2k}}{(2k)!}2^{2k}(-1)^k z^{2k-1}, \ |z| < \tfrac{1}{2}\pi,$$

where b_{2k} are the Bernoulli numbers, we obtain in the eigenspaces where $F \mp \mu$ is nilpotent

$$\tan(tF)X = \mp\Big(\sum_1^n s^{-j}t^{j-1}(\mu \mp F)^{j-1} + G(s - t(\mu \mp F))\Big)X.$$

If $X = (x, \xi)$ then $\mu \mp F$ acts as the left shift L on the x coordinates and we obtain as in the discussion of $\sin F$ above, when $s \to 0$,

$$\sigma(X, \tan(tF)X) = -\gamma i\Big(\sum_1^n s^{-j}t^{j-1}\langle\xi, L^{j-1}\check{\xi}\rangle$$

$$-\sum_{1\le k\le n/2}\frac{b_{2k}}{(2k)!}2^{2k}(-1)^k\langle\xi, L^{2k-1}\check{\xi}\rangle$$

$$+\sum_1^n s^{-j}t^{j-1}\langle x, R^{j-1}\check{x}\rangle - \sum_{1\le k\le n/2}\frac{b_{2k}}{(2k)!}2^{2k}(-1)^k\langle x, R^{2k-1}\check{x}\rangle\Big) + O(s).$$

Here $\langle\xi, L^{j-1}\check{\xi}\rangle = \sum_{\nu+\mu=n+2-j}\xi_\nu\xi_\mu$ so the first sum can be written

$$s^{-n}t^{n-1}\Big(\sum_1^n \xi_\nu(s/t)^{\nu-1}\Big)^2 - \sum_{\nu+\mu\ge 2+n} s^{\nu+\mu-2-n}t^{n+1-\nu-\mu}\xi_\nu\xi_\mu.$$

The second sum here converges to $\sum_{\nu+\mu=2+n}\xi_\nu\xi_\mu$. The function $\mathbf{R}\ni\tau\mapsto e^{-i\gamma s^{-n}\tau^2}$ is asymptotic in $\mathcal{S}'(\mathbf{R})$ to $\delta_0/\sqrt{i\gamma s^{-n}/\pi}$ as $s\to 0$, with a suitable choice of the square root. The same square root appears when we take the limit of the factors involving x, so a change of variables yields the limit

$$\lim_{t\to 1}\exp(\sigma(X, \tan(tF)X)/\sqrt{\det(\cos tF)} = -\pi\gamma i(-\sin\mu)^n\delta_0(x_n)\delta_0(\xi_1)\exp(E),$$

$$E = i\gamma\sum_{0\le k\le n/2}\frac{b_{2k}}{(2k)!}2^{2k}(-1)^k\Big(\sum_{\nu+\mu=n+2-2k}\xi_\nu\xi_\mu + \sum_{\nu+\mu=n+2k}x_\nu x_\mu\Big). \tag{4.18}$$

Here we have used that $b_0 = 1$. Since $-\pi\gamma i(-\sin\mu)^n = \pi i\gamma(\sin\mu)^n(-1)^{n-1}$ this completes the verification of Theorem 4.6 for the quadratic form (4.16) and provides in addition the completely explicit formula (4.18).

End of proof of Theorem 4.6. Let $Q = Q_1 + iQ_2$ be any quadratic form in $T^*\mathbf{R}^n$ with $Q_1 = \operatorname{Re} Q \le 0$, and denote the Hamilton map by $F = F_1 + iF_2$. We can choose a purely imaginary quadratic form iQ_0, with Hamilton map iF_0, such that the kernel of $F - \lambda$ is equal to the kernel of $iF_0 - \lambda$ when $\cos\lambda = 0$. In fact,

$$M_\lambda = T^*\mathbf{R}^n \cap (\operatorname{Ker}(F-\lambda)\oplus\operatorname{Ker}(F-\lambda))$$

is symplectically orthogonal to M_μ if $\mu\ne\pm\lambda$ and $\cos\mu = 0$. We can find a symplectic decomposition $T^*\mathbf{R}^n = S_0\oplus\bigoplus_{\cos\mu=0,\mu>0}S_\mu$ such that $M_\mu\subset S_\mu$ for every $\mu > 0$ with $\cos\mu = 0$. Using the forms c) in Theorem 3.1 with $n = 1$ or $n = 2$ when $\mu > 0$ and the forms a) with $n = 1$ when $\mu = 0$ we obtain a polynomial Q_0 with the required properties as a sum of polynomials in the different spaces in the decomposition.

If $\varepsilon > 0$ is sufficiently small, then the kernel of $F_1 + i((1-\tau)F_2 + \tau F_0) - \lambda$ is independent of τ when $0\le\tau\le\varepsilon$ and $\cos\lambda = 0$, for it contains the kernel of $F-\lambda$ and the dimension is upper semicontinuous. Also $i((1-\tau)F_2 + \tau F_0) - \lambda$ has the same kernel except for finitely many values of τ, for the kernel cannot be smaller than that of $F-\lambda$ and is equal to it except at the zeros of a determinant which is a polynomial in τ and $\neq 0$ when $\tau = 1$. For such a value of $\tau_0 \in (0, \varepsilon]$ we define

$$F^\tau = \begin{cases} F_1 + i((1-\tau)F_2 + \tau F_0), & \text{if } 0 \leq \tau \leq \tau_0, \\ (2 - \tau/\tau_0)F_1 + i((1-\tau_0)F_2 + \tau_0 F_0), & \text{if } \tau_0 \leq \tau \leq 2\tau_0 \end{cases} \tag{4.19}$$

It follows from Proposition 4.4 that the kernel of $\cos(F^\tau)$ is independent of τ when $\tau_0 \leq \tau < 2\tau_0$, and since it contains the kernel when $\tau = 2\tau_0$ the semicontinuity of the dimension of the kernel completes the proof that the kernel is independent of τ.

Now our definitions of σ_F and E_F in (4.9)$'$ and (4.11) are continuous in F when Ker $\cos F$ is fixed. If Q^τ is the quadratic form with Hamilton map F^τ obtained when F_j is replaced by Q_j in (4.19) then the Weyl symbol for $\exp(Q^{\tau w})$ is as stated in Theorem 4.6 for all $\tau \in [0, 2\tau_0]$ since this is true when $\tau = 2\tau_0$. In particular this is true for $\exp(Q^{0w}) = \exp(Q^w)$, which completes the proof.

The literature also contains some formulas for the symbol of $\exp(itQ^w(x,D))$ when Q is an inhomogeneous quadratic polynomial. An example is the formula of Avron and Herbst [1] for $Q(x,\xi) = \xi^2 + x$ in $T^*\mathbf{R}$. We shall now show how such formulas can be derived from the results in this section.

First assume that $Q(X)$, where $X = (x,\xi) \in T^*\mathbf{R}^n$, is a quadratic polynomial with real coefficients, and that the principal part $q(x,\xi)$ is non-singular. We denote the Hamilton map of q by F and write

$$Q(X) = q(X) + 2\sigma(X,a) + b = \sigma(X, FX) + 2\sigma(X,a) + b. \tag{4.20}$$

Then we have

$$Q(X) = q(X+\theta) + c, \quad \text{where } \theta = F^{-1}a,\ c = b - q(\theta) = b - \sigma(\theta, F\theta).$$

There is a unitary operator, the composition of a translation and multiplication by an exponential, such that

$$U^{-1}A^wU = B^w \quad \text{if } B(X) = A(X+\theta).$$

Hence

$$Q^w = U^{-1}q^wU + c, \quad \exp(itQ^w) = U^{-1}\exp(itq^w)U\exp(itc),$$

so it follows from Theorem 4.3 that the Weyl symbol of $\exp(itQ^w)$ is equal to

$$\exp(i\sigma(X+\theta, \tanh(tF)(X+\theta)) + itc)/\sqrt{\det\cosh(tF)}$$

when the denominator is not equal to 0. Here

$$\begin{aligned} &\sigma(X+\theta, \tanh(tF)(X+\theta)) + tc \\ &= \sigma(X, \tanh(tF)X) + 2\sigma(X, F^{-1}\tanh(tF)a) - \sigma(a, F^{-2}(\tanh(tF) - tF))a) + tb. \end{aligned}$$

The right-hand side is well defined even if F is not invertible, provided that $\det\cosh(tF) \neq 0$. For reasons of continuity we conclude that the symbol of $\exp(iQ^w)$ is then equal to the product of the symbol of $\exp(itq^w)$ by

$$\exp\left(i(2\sigma(X, F^{-1}\tanh(tF)a) - \sigma(a, F^{-2}(\tanh(tF) - tF)a) + tb)\right) \tag{4.21}$$

if $\det\cosh(tF) \neq 0$, even if F is not invertible.

The preceding result means that we have solved the equations analogous to (4.1) for an inhomogeneous Q when Q has real coefficients, and the analyticity of the result shows that the solution is also valid when the coefficients are complex. We can therefore apply the proof of Theorem 4.3 again, which gives:

Theorem 4.7 *Let Q be a quadratic polynomial in $T^*\mathbf{R}^n$ such that* $\operatorname{Im} Q$ *is bounded below. Write Q in the form* (4.20) *with q homogeneous, and let F be the Hamilton map of q. Then the Weyl symbol of* $\exp(itQ^w)$ *is equal to that of* $\exp(itq^w)$*, described in Theorem 4.3, multiplied by* (4.21)*, provided that $t > 0$ and that* $\det\cosh(tF) \neq 0$.

We shall not discuss the case where $\det\cosh(tF) = 0$ but content ourselves with the example where $F^3 = cF$ as in Derezinski [3]. As observed above, $\tanh(tF) = F\tanh(t\sqrt{c})/\sqrt{c}$ then, and similarly

$$F^{-2}(\tanh(tF) - tF) = F(\tanh(t\sqrt{c}) - t\sqrt{c})/(c\sqrt{c}),$$

so the symbol of $\exp(itQ^w)$ is equal to

$$\exp\left(i(\tanh(t\sqrt{c})/\sqrt{c}(q(X) + 2\sigma(X,a)))\right)K,$$
$$K = \frac{\exp\left(i\left(q(a)(t\sqrt{c} - \tanh(t\sqrt{c}))/(c\sqrt{c}) + tb\right)\right)}{\sqrt{\det\cosh(tF)}},$$

or equivalently,

$$\exp\left(i(\tanh t\sqrt{c})/\sqrt{c}(q(X) + 2\sigma(X,a) - q(a)/c)\right)\frac{\exp\left(it(q(a)/c^2 + b)\right)}{\sqrt{\det\cosh(tF)}},$$

when $c \neq 0$,

$$\exp\left(i(tQ(X) + q(a)t^3/3)\right),$$

when $c = 0$.

When $Q(X) = \xi^2 + x$, $X = (x,\xi) \in T^*\mathbf{R}$, we obtain the Weyl symbol

$$a(x,\xi) = \exp(i(t\xi^2 + tx + t^3/12)). \tag{4.22}$$

By [4, Theorem 18.5.10] we have $a^w(x,D) = b(x,D)$, where $b(x,D)$ is defined by the standard calculus, if $b(x,\xi) = \exp(\frac{1}{2}iD_x D_\xi)a(x,\xi)$, that is,

$$b(x,\xi) = \frac{1}{\pi}\iint a(x-y, \xi-\eta)\exp(-2iy\eta)\,dy\,d\eta.$$

Evaluation of this Gaussian integral gives

$$b(x,\xi) = \exp(i(t\xi^2 + t^2\xi + tx + t^3/3)), \tag{4.23}$$

which is the formula of Avron and Herbst. Again we notice that the Weyl symbol (4.22) is simpler than the standard symbol (4.23); the lack of commutativity only affects the constant term in (4.22).

5 Gaussian calculus

In this section we shall put the results of Sect. 4 in the framework of general Gaussians studied as infinitesimal Fourier integral distributions (operators). We start with recalling some essentially well known facts on Gaussian distributions. (See [4, Sect. 21.6] for the case of real Lagrangians.)

Let $0 \neq u \in \mathscr{D}'(\mathbf{R}^n)$ and set

$$\mathscr{L}_u = \{L; L(x,D)u = 0\}, \quad \text{where } L(x,D) = \sum_1^n a_j D_j + \sum_1^n b_j x_j. \tag{5.1}$$

One calls u a *Gaussian* if every $v \in \mathscr{D}'(\mathbf{R}^n)$ such that $L(x,D)v = 0$ for all $L \in \mathscr{L}_u$ is a multiple of u. If $L_1, L_2 \in \mathscr{L}_u$ then the commutator $[L_1, L_2]$ is equal to 0, for it is a constant and $[L_1, L_2]u = 0$. Hence

$$\lambda_u = \{(x,\xi) \in T^*\mathbf{C}^n; L(x,\xi) = 0 \ \forall\, L \in \mathscr{L}_u\} \tag{5.2}$$

is an involutive subspace.

Proposition 5.1 *If u is a Gaussian then* (5.1), (5.2) *define a complex Lagrangian plane such that $V = \{\xi; (0,\xi) \in \lambda_u\}$ is invariant under conjugation, hence generated by its intersection with $\mathbf{R}^n$. Conversely, for every such Lagrangian plane λ a distribution u such that $L(x,D)u = 0$ for every L which vanishes on λ is a Gaussian, and $u = ce^q d$ where d is a δ-function on a linear subspace and q is a quadratic form there, both determined by λ, and $c \in \mathbf{C} \setminus \{0\}$.*

Proof. We introduce new coordinates by first taking a basis $e_1, \dots, e_k$ for $W \cap \mathbf{R}^n$ where $W = \{L(0, \partial/\partial x); L \in \mathscr{L}_u\}$, and then extending it to a basis in Re W by taking real and imaginary parts of elements $e_{k+1}, \dots, e_{k+l}$ such that $e_1, \dots, e_k, e_{k+1}, \dots, e_{k+l}$ is a complex basis in W. Then we can write the equations $L(x,D)v = 0$, $L \in \mathscr{L}_u$, in the form

$$(\partial/\partial x_j + a_j(x))v = 0,\ 1 \le j \le k; \quad (\partial/\partial \bar{z}_j + b_j(x))v = 0,\ 1 \le j \le l;$$
$$c_j(x)v = 0,\ 1 \le j \le m.$$

Here $\partial/\partial \bar{z}_j = \frac{1}{2}(\partial/\partial x_{k+2j-1} + i\partial/\partial x_{2k+2j})$. Since the operators commute we have

$$\partial a_j/\partial x_i = \partial a_i/\partial x_j, \quad \partial a_j/\partial \bar{z}_i = \partial b_i/\partial x_j, \quad \partial b_i/\partial \bar{z}_j = \partial b_j/\partial \bar{z}_i,$$

and $c_1, \dots, c_m$ are independent of $x_1, \dots, x_k$ and analytic in $z_1, \dots, z_l$. This means that if

$$Q(x) = \tfrac{1}{2}\left(\sum_1^k a_j(x)x_j + \sum_1^l b_j(x)\bar{z}_j\right)$$

then

$$\alpha_j(x) = a_j(x) - \partial Q(x)/\partial x_j,\ j = 1, \dots, k,$$
$$\beta_j(x) = b_j(x) - \partial Q(x)/\partial \bar{z}_j,\ j = 1, \dots, l,$$

are independent of $x_1, \dots, x_k$ and analytic in $z_1, \dots, z_l$. With $v = e^{-Q}w$ the equations reduce to

$$\partial w/\partial x_j + \alpha_j w = 0,\ j = 1, \dots, k; \quad \partial w/\partial \bar z_j + \beta_j w = 0,\ j = 1, \dots, l;$$
$$c_j(x)w = 0,\ j = 1, \dots, m.$$

Thus $W(x) = w(x)\exp(\sum_1^k \alpha_j(x)x_j + \sum_1^l \bar z_j \beta_j(x))$ is independent of $x_1, \dots, x_k$ and analytic in $z_1, \dots, z_l$ so the support is invariant under translation in the directions of the $x_1, \dots, x_{k+2l}$ plane. The product of a solution by any analytic function of $z_1, \dots, z_l$ is another solution, and since all solutions are supposed to be proportional it follows that $l = 0$. The forms c_j must be independent of $x_1, \dots, x_k$ for otherwise W would have to be equal to 0 by the translation invariance in these variables. As at the beginning of the proof we can change the coordinates $x_{k+1}, \dots, x_n$ so that the linear combinations of the forms c_j are precisely the linear combinations of $x_{k+1} + ix_{k+2}, \dots, x_{k+2\mu-1} + ix_{k+2\mu}, x_{k+2\mu+1}, \dots, x_N$ for some $\mu \geq 0$ and $N \leq n$. Since the solution W is supposed to be unique it is clear that $N = n$. Then we have $x_{k+1} = \dots = x_n = 0$ for every solution W. However, if we apply $\partial/\partial x_{k+1} + i\partial/\partial x_{k+2}$ to a solution we obtain a new solution, so the uniqueness also implies that $\mu = 0$. Thus λ_u is defined by

$$\xi_j - ia_j(x) = 0,\ j = 1, \dots, k, \quad x_j = 0,\ j = k + 1, \dots, n,$$

which is a complex Lagrangian such that V is defined by $\xi_j = 0, j \leq k$, hence invariant under conjugation. We have $u(x) = ce^{-q}\delta(x_{k+1}, \dots, x_n)$ where $q(x) = \frac{1}{2}\sum_1^k x_j a_j(x)$ can be taken as a quadratic form in $(x_1, \dots, x_k)$ only.

Now assume that we are given a complex Lagrangian $\lambda \subset T^*\mathbf{C}^n$ such that $V = \{\xi; (0, \xi) \in \lambda\}$ is generated by $V \cap \mathbf{R}^n$. By a change of coordinates we may assume that this is the $\xi_{k+1}, \dots, \xi_n$ plane. Then $x_{k+1} = \dots = x_n = 0$ in λ, and $\xi_1, \dots, \xi_k$ are there linear functions of $x' = (x_1, \dots, x_k)$, so λ is defined by

$$\xi_j = a_j(x'),\ j = 1, \dots, k, \quad x_j = 0,\ j = k + 1, \dots, n.$$

which is precisely the situation just discussed since $\sum_1^k a_j(x')y_j = \sum_1^k a_j(y')x_j$.

The condition on λ in Proposition 5.1 is not invariant under the linear symplectic group in $T^*\mathbf{R}^n$. The following result shows how it must be strengthened to become invariant. Note that

$$i\sigma(\overline{Y}, X), \quad X, Y \in T^*\mathbf{C}^n,$$

is a Hermitian symmetric form.

Proposition 5.2 *If $\lambda \subset T^*\mathbf{C}^n$ is a complex Lagrangian plane, then the following properties are equivalent:*

(i) For every real Lagrangian plane $\mu \subset T^*\mathbf{R}^n$, with complexification $\mu_{\mathbf{C}}$, the intersection $\lambda \cap \mu_{\mathbf{C}}$ is invariant under complex conjugation.
(ii) $i\sigma(\overline{X}, X)$ is semidefinite when $X \in \lambda$.

Proof. Assume that (ii) is valid and let $X \in \lambda \cap \mu_{\mathbf{C}}$. Then $\overline{X} \in \mu_{\mathbf{C}}$, so $\sigma(\overline{X}, X) = 0$, and it follows from (ii) that $i\sigma(\overline{X}, Y) = 0$ for every $Y \in \lambda$. Hence $\overline{X} \in \lambda$ since λ is Lagrangian, so (i) is valid. Now assume that (ii) is not fulfilled. Then we can find $X \in \lambda$ with $i\sigma(\overline{X}, X) = 0$ such that $i\sigma(\overline{X}, Y) \neq 0$ for some $Y \in \lambda$. Thus $\sigma(\operatorname{Re} X, \operatorname{Im} X) = 0$, so we can choose a real Lagrangian μ containing the vectors $\operatorname{Re} X$ and $\operatorname{Im} X$. Then $X \in \lambda \cap \mu_{\mathbf{C}}$ but $\overline{X} \notin \lambda$ since λ is isotropic and $Y \in \lambda$. Hence (i) is not fulfilled.

Definition 5.3 *If S is a real symplectic vector space with complexification $S_{\mathbf{C}}$, then a Lagrangian $\lambda \subset S_{\mathbf{C}}$ is said to be positive if $i\sigma(\overline{X}, X) \geq 0$, $X \in \lambda$, and strictly positive if $i\sigma(\overline{X}, X) > 0$ when $0 \neq X \in \lambda$. The set of strictly positive Lagrangian planes will be denoted by Λ^+.*

It is obvious that the closure of Λ^+ consists of positive Lagrangian planes. To prove that every positive Lagrangian plane λ is in the closure we represent λ in the form $\{(x', 0, \partial Q/\partial x', \xi'')\}$ as in the proof of Proposition 5.1, where Q is a quadratic form in $x' = (x_1, \dots, x_k)$ and $x'' = (x_{k+1}, \dots, x_n)$. The positivity means that

$$i(\langle \overline{\partial Q(x')/\partial x'}, x' \rangle - \langle \overline{x'}, \partial Q(x')/\partial x' \rangle) = 4 \operatorname{Im} Q(\operatorname{Re} x') + 4 \operatorname{Im} Q(\operatorname{Im} x'),$$

that is, that $\operatorname{Im} Q$ is positive semidefinite in $\mathbf{R}^k$. It is now clear that λ is the limit as $\varepsilon \to 0$ of the strictly positive Lagrangians

$$\{(x', \varepsilon x'', \partial Q/\partial x' + i\varepsilon x', ix'')\}.$$

Let $\mathcal{G}$ be the set of all Gaussians in $\mathbf{R}^n$ such that the associated complex Lagrangian is positive. By Proposition 5.1 and the preceding discussion these are precisely the Gaussians which are temperate distributions, and the map $\mathcal{G} \ni u \to \overline{\Lambda}_+$ to the corresponding positive Lagrangian has complex lines with the origin removed as fibers.

Proposition 5.4 *With the topology induced by $\mathcal{S}'$ and the projection just defined, the temperate Gaussians $\mathcal{G}$ form a complex line bundle over $\overline{\Lambda}_+$ with the zero section removed.*

Proof. The Gaussian in the proof of Proposition 5.1 can be written as a Fourier transform

$$u = c \int \exp(iq(x') + i\langle x'', \xi'' \rangle)\, d\xi''$$

in the sense of distribution theory, where $c \in \mathbf{C} \setminus \{0\}$ and $\operatorname{Im} q \geq 0$ in $\mathbf{R}^k$ if and only if u is temperate. If $Q(x', \xi'')$ is an arbitrary quadratic form with $\operatorname{Im} Q \geq 0$ in $\mathbf{R}^n$, then

$$u_{c,Q} = c \int \exp(iQ(x', \xi'') + i\langle x'', \xi'' \rangle)\, d\xi'' \tag{5.3}$$

is an injective continuous function of $c \neq 0$ and Q with values in $\mathcal{S}'$ and is a Gaussian associated with the positive Lagrangian

$$\{(x', -\partial Q(x',\xi'')/\partial\xi'', \partial Q(x',\xi'')/\partial x', \xi'')\}. \tag{5.4}$$

The integral in (5.3) is defined in the sense of distribution theory. That (5.4) defines a Lagrangian is clear since

$$\langle \xi, dx\rangle = \sum_1^k \partial Q/\partial x_j dx_j - \sum_{k+1}^n \xi_i d\partial Q/\partial \xi_i$$

has the differential

$$\sum \partial^2 Q/\partial\xi_i\partial x_j d\xi_i \wedge dx_j - \sum d\xi_i \wedge \partial^2 Q/\partial\xi_i\partial x_j dx_j = 0.$$

Every Lagrangian λ close to the given Lagrangian $\{(x', 0, \partial q(x')\ /\partial x', \xi'')\}$ is of the form (5.4), for the projection $\lambda \ni (x,\xi) \mapsto (x',\xi'')$ is then surjective so we can write $x'' = \varphi(x',\xi'')$ and $\xi' = \psi(x',\xi'')$ on λ. That λ is Lagrangian means then that $0 = d(\langle\xi', dx'\rangle - \langle x'', d\xi''\rangle) = d(\langle\psi, dx'\rangle - \langle\varphi, d\xi''\rangle)$, so there is a quadratic form $Q(x',\xi'')$ with $\psi = \partial Q/\partial x'$ and $\varphi = -\partial Q/\partial\xi''$. The statement is now a consequence of the following more general result:

Proposition 5.5 *Let $Q(x,\theta)$ be a complex valued quadratic form in $\mathbf{R}^n \oplus \mathbf{R}^N$ such that* Im $Q \geq 0$ *and the linear forms $\partial Q/\partial\theta_j$, $j = 1,\dots,N$, are linearly independent over* **C**. *Then*

$$u = \int e^{iQ(x,\theta)}\, d\theta \tag{5.5}$$

can be interpreted as a Gaussian belonging to the positive Lagrangian

$$\lambda = \{(x, \partial Q(x,\theta)/\partial x); \partial Q(x,\theta)/\partial\theta = 0\}. \tag{5.6}$$

Proof. First note that the equations

$$\xi = \partial Q(x,\theta)/\partial x, \quad 0 = \partial Q(x,\theta)/\partial\theta$$

for $(x,\xi,\theta) \in \mathbf{C}^n \oplus \mathbf{C}^n \oplus \mathbf{C}^N$ define a linear space of dimension n since the equations are linearly independent. The projection $(x,\xi,\theta) \mapsto (x,\xi)$ is injective there, for if $\partial Q(x,\theta)/\partial x = 0$, $\partial Q(x,\theta)/\partial\theta = 0$ and $x = 0$ then

$$\sum_{j=1}^N \theta_j \partial^2 Q/\partial x_k\partial\theta_j = 0,\ k = 1,\dots,n;$$

$$\sum_{j=1}^N \theta_j \partial^2 Q/\partial\theta_k\partial\theta_j = 0,\ k = 1,\dots,N\ ;$$

which implies $\theta = 0$ by the assumed linear independence of $\partial Q/\partial\theta_j$, $j = 1,\dots,N$. Thus (5.6) defines a linear space of complex dimension n, and since

$$\sum_{j=1}^{n} d\xi_j \wedge dx_j = \sum_{j=1}^{n} d(\partial Q/\partial x_j) \wedge dx_j$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{N} \partial^2 Q/\partial x_j \partial\theta_k d\theta_k \wedge dx_j = \sum_{k=1}^{N} d\theta_k \wedge d\partial Q/\partial\theta_k$$

vanishes on the manifold where $\partial Q/\partial\theta = 0$, it follows that λ is Lagrangian. If $X = (x, \partial Q(x,\theta)/\partial x)$ and $\partial Q(x,\theta)/\partial\theta = 0$, then

$$i\sigma(\overline{X}, X) = i(\langle\overline{\partial Q/\partial x}, x\rangle - \langle\overline{x}, \partial Q/\partial x\rangle)$$
$$= 2\mathrm{Im}\,(\langle\partial Q/\partial x, \overline{x}\rangle + \langle\partial Q/\partial\theta, \overline{\theta}\rangle = 4\mathrm{Im}\, Q(x,\theta;\overline{x},\overline{\theta}))$$

where $Q(\cdot;\cdot)$ is the polarized symmetric bilinear form defined by Q. Hence λ is positive.

The integral (5.5) can be defined in the sense of distribution theory: If $\varphi \in \mathscr{S}$ we shall prove that

$$\langle u, \varphi\rangle = \int \varphi(x) e^{iQ(x,\theta)}\, dx\, d\theta \tag{5.5'}$$

exists as an oscillatory integral. If Im Q is positive definite the integral is absolutely convergent. Since we have just seen that $x = \partial Q/\partial x = 0$, $\partial Q/\partial\theta = 0$ imply $\theta = 0$ we can write

$$\theta_j = \sum a_{jk} x_k + \sum b_{jk} \partial Q/\partial x_k + \sum c_{jk} \partial Q/\partial\theta_k$$

and obtain in the strictly positive case for any integer $\nu > 0$ by partial integration after multiplication by $(1 - i\theta_j)(1 - i\theta_j)^{-1}$

$$\langle u, \varphi\rangle = \int e^{iQ(x,\theta)} \prod_{j=1}^{N} \Big(\big(1 - i \sum a_{jk} x_k + \sum b_{jk} \partial/\partial x_j$$
$$+ \sum c_{jk} \partial/\partial\theta_k\big)(1 - i\theta_j)^{-1}\Big)^{\nu} \varphi(x)\, dx\, d\theta.$$

When $\nu > N$ this defines a distribution in $\mathscr{S}'$ also when Im Q is just positive semidefinite, so (5.5) defines a temperate distribution depending continuously on Q. It suffices to verify that it is Gaussian in the strictly positive case. If $L(x,\xi) = 0$ on the Lagrangian (5.6), then $L(x, \partial Q/\partial x) = \sum_1^N t_j \partial Q/\partial\theta_j$ for some t_j, and we obtain

$$L(x, D)u = \int L(x, \partial Q/\partial x) e^{iQ(x,\theta)}\, d\theta$$
$$= \sum_1^N t_j \int \partial Q/\partial\theta_j e^{iQ(x,\theta)}\, d\theta = 0,$$

which completes the proof.

Proposition 5.6 *If Q is a quadratic form satisfying the hypotheses of Proposition 1.5 and $c = \partial^2 Q/\partial\theta_1^2 \neq 0$, then*

$$Q(x,\theta) = \tfrac{1}{2}c(\theta_1 + L(x,\theta'))^2 + Q_1(x,\theta'), \quad \theta' = (\theta_2,\dots,\theta_N), \tag{5.7}$$

where L is a linear form, Q_1 is a quadratic form satisfying the hypotheses of Proposition 5.5 with N replaced by $N-1$, and

$$\int e^{iQ(x,\theta)}\,d\theta = \sqrt{2\pi i/c}\int e^{iQ_1(x,\theta')}\,d\theta' \tag{5.8}$$

with the square root in the right half plane.

Proof. By completion of squares we can write Q in the form (5.7). If $\beta \in \mathbf{R}$ then $\{(\alpha+i\beta)^2; \alpha \in \mathbf{R}\}$ is a parabola surrounding the circle $\{z; |z| = \beta^2\}$, so we have

$$-\tfrac{1}{2}|c||\mathrm{Im}\, L(x,\theta')|^2 + \mathrm{Im}\, Q_1(x,\theta') \geq \inf_{\theta_1} \mathrm{Im}\, Q(x,\theta_1,\theta') \geq 0, \quad x \in \mathbf{R}^n,\ \theta' \in \mathbf{R}^{N-1}$$

which proves that Q_1 satisfies the hypotheses of Proposition 5.5. When $\mathrm{Im}\, c > 0$ we obtain (5.8) immediately by integration and the general case follows by continuity.

By repeated use of Proposition 5.6 we can reduce the number of θ variables in (5.5) until we are left with a quadratic form $Q(x,\theta)$ which is linear in θ. It is then essentially uniquely determined by λ:

Proposition 5.7 *If $Q(x,\theta)$ is a quadratic form satisfying the hypotheses in Proposition 5.5 and Q is linear in θ, then*

$$Q(x,\theta) = \langle L\theta, x\rangle + q(x), \tag{5.9}$$

where L is an injective linear transformation $\mathbf{R}^N \to \mathbf{R}^n$ and $\mathrm{Im}\, q \geq 0$. The corresponding Lagrangian is

$$\lambda = \{(x, q'(x) + L\theta); x \in \mathbf{C}^n, \theta \in \mathbf{C}^N, {}^tLx = 0\}.$$

The Gaussian distribution (5.5) is $(2\pi)^N\delta({}^tLx)e^{iq(x)}$. If $\tilde{Q}(x,\theta) = \langle\tilde{L}\theta, x\rangle + \tilde{q}(x)$ defines the same Lagrangian then $\tilde{L} = LT$ where T is a linear bijection in $\mathbf{R}^N$, and $\tilde{q} = q$ when ${}^tLx = 0$, which is equivalent to ${}^t\tilde{L}x = 0$.

Proof. Since $\mathrm{Im}\, Q(x,\theta) \geq 0$ in $\mathbf{R}^n \oplus \mathbf{R}^N$ we must have $\mathrm{Im}\,\langle L\theta, x\rangle = 0$ there, so L maps $\mathbf{R}^N$ to $\mathbf{R}^n$. The linear independence of $\partial Q/\partial\theta_j, j = 1,\dots,N$, means that L is injective. The range of L is $\{\xi; (0,\xi) \in \lambda\}$, which proves that L is uniquely determined by λ apart from an invertible factor to the right. This completes the proof.

For the sake of brevity we shall not discuss here the definition of symbols of general Gaussian distributions but pass to a study of those which are related to symplectic linear maps. If $\lambda \subset T^*\mathbf{C}^n \oplus T^*\mathbf{C}^n \cong T^*\mathbf{C}^{2n}$ is a positive Lagrangian with injective projection in each of the two factors, then the projections are bijective. In fact, if $\lambda \ni (X, Y) \mapsto X \in T^*\mathbf{C}^n$ is not surjecive, then one can find $X_0 \in T^*\mathbf{C}^n \setminus \{0\}$ such that $\sigma(X, X_0) = 0$ for all $(X, Y) \in \lambda$. Since λ is Lagrangian this means that $(X_0, 0) \in \lambda$ so the projection to the second factor is not injective. — When the projections are bijective we introduce the twisted Lagrangian

$$C = \lambda' = \{(X, Y'); (X, Y) \in \lambda\},$$

where $Y' = (y, -\eta)$ if $Y = (y, \eta)$. We regard C as the graph $\{(TY, Y); Y \in T^*\mathbf{C}^n\}$ of a symplectic linear bijection $T^*\mathbf{C}^n \to T^*\mathbf{C}^n$, that is, $\sigma(TX, TY) = \sigma(X, Y)$ for all $X, Y \in T^*\mathbf{C}^n$. We shall say that T *is positive if the corresponding Lagrangian* λ *is positive*, that is,

$$i(\sigma(\overline{TY}, TY) - \sigma(\overline{Y}, Y)) \geq 0, \quad Y \in T^*\mathbf{C}^n. \tag{5.10}$$

With T we associate the Gaussian distribution

$$\begin{aligned} K_T &= (2\pi)^{-(n+N)/2}\sqrt{\det\begin{pmatrix} Q''_{\theta\theta}/i & Q''_{\theta y} \\ Q''_{x\theta} & iQ''_{xy}\end{pmatrix}} \int e^{iQ(x,y,\theta)}\, d\theta \\ &\in \mathscr{S}'(\mathbf{R}^n \times \mathbf{R}^n), \end{aligned} \tag{5.11}$$

where Q is a quadratic form defining $\lambda = C'$ according to Proposition 5.5. We do not prescribe the sign of the square root so K_T is only determined up to the sign. Apart from that K_T is uniquely determined, independently of the choice of Q. To prove this we first observe that if the hypotheses of Proposition 5.6 are fulfilled, then the definitions using the two quadratic forms Q and Q_1 (with x replaced by (x, y)) agree. We just have to prove that

$$\det\begin{pmatrix} Q''_{\theta\theta} & Q''_{\theta y} \\ Q''_{x\theta} & Q''_{xy}\end{pmatrix} = c \det\begin{pmatrix} Q''_{1\theta'\theta'} & Q''_{1\theta' y} \\ Q''_{1x\theta'} & Q''_{1xy}\end{pmatrix}. \tag{5.12}$$

Let Q_2 be the form obtained when L is replaced by 0 in (5.7). Then the matrix in the left-hand side of (5.12) is equal to the corresponding matrix with Q replaced by Q_2 multiplied right and left by $\partial((\theta_1 + L), \theta', y)/\partial(\theta, y)$ and the transpose of $\partial((\theta_1 + L), \theta', x)/\partial(\theta, x)$, which have determinant 1. This proves (5.12), and it just remains to discuss the case where Q is linear in θ. By Proposition 5.6 two such forms Q can only differ by a substitution of $T\theta$ for θ where T is an invertible linear transformation in $\mathbf{R}^N$, and this does not affect (5.11).

Before discussing the composition of two operators with kernels of the form (5.11) we have to prove a continuity property making it well defined.

Proposition 5.8 *If* T *is a positive symplectic linear bijection in* $T^*\mathbf{C}^n$, *then the map* $\mathscr{K}_T : \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n)$ *with kernel* (5.11) *is a continuous map in* $\mathscr{S}(\mathbf{R}^n)$ *and extends to a continuous map in* $\mathscr{S}'(\mathbf{R}^n)$.

Proof. By duality it suffices to prove the statement on $\mathcal{S}$. If $(x_0, \xi_0) = T(y_0, \eta_0)$ then $(x_0, \xi_0, y_0, -\eta_0) \in \lambda = C'$, which means that

$$\langle \xi, x_0 \rangle - \langle x, \xi_0 \rangle + \langle \eta, y_0 \rangle + \langle y, \eta_0 \rangle = 0, \quad (x, \xi, y, \eta) \in \lambda.$$

Thus the corresponding differential operator annihilates K_T which means that $L_T \mathcal{K}_T = \mathcal{K}_T L$ if $L = \langle D_x, y_0 \rangle - \langle y, \eta_0 \rangle$ and $L_T = \langle D_x, x_0 \rangle - \langle x, \xi_0 \rangle$. Any product of $\mathcal{K}_T$ to the left by such operators is therefore equal to a product to the right. Since $(1 + |D|^2 + |x|^2)^{-N} \mathcal{K}_T f$ is a bounded continuous function for all $f \in \mathcal{S}$ if N is large enough, it follows that

$$x^\alpha D^\beta \mathcal{K}_T f = (1 + |D|^2 + |x|^2)^{-N} (1 + |D|^2 + |x|^2)^N x^\alpha D^\beta \mathcal{K}_T f$$

is a bounded continuous function for arbitrary α, β. This proves the statement, and the proof shows that $\mathcal{K}_T f$ depends continuously on T also.

Proposition 5.9 *If T_1 and T_2 are two positive symplectic bijections in $T^*\mathbf{C}^n$, then $T_1 T_2$ is also a positive symplectic bijection and $\mathcal{K}_{T_1 T_2} = \pm \mathcal{K}_{T_1} \mathcal{K}_{T_2}$.*

Proof. We write

$$K_{T_1} = (2\pi)^{-(n+N_1)/2} \sqrt{\det \begin{pmatrix} Q''_{1\theta\theta}/i & Q''_{1\theta y} \\ Q''_{1x\theta} & iQ''_{1xy} \end{pmatrix}} \int e^{iQ_1(x,y,\theta)} \, d\theta,$$

$$K_{T_2} = (2\pi)^{-(n+N_2)/2} \sqrt{\det \begin{pmatrix} Q''_{2\tau\tau}/i & Q''_{2\tau z} \\ Q''_{2y\tau} & iQ''_{2yz} \end{pmatrix}} \int e^{iQ_2(y,z,\tau)} \, d\tau,$$

and let $Q(x, z, y, \theta, \tau) = Q_1(x, y, \theta) + Q_2(y, z, \tau)$ with $\chi = (y, \theta, \tau)$ considered as the fiber variables, with dimension $\nu = n + N_1 + N_2$. Since $n + N_1 + n + N_2 = n + \nu$, this gives the right power of (2π) in the composition. The corresponding twisted Lagrangian is defined by

$$\{(x, Q'_x, z, -Q'_z); Q'_y = 0, Q'_\theta = 0, Q'_\tau = 0\}.$$

Then we have

$$T_2(z, -\partial Q_2(y, z, \tau)/\partial z) = (y, \partial Q_2(y, z, \tau)/\partial y) = (y, -\partial Q_1(x, y, \theta)/\partial y),$$

thus $T_1 T_2(z, -\partial Q_2(y, z, \tau)/\partial z) = (x, \partial Q_1(x, y, \theta)/\partial x)$, so it will follow that $\mathcal{K}_{T_1 T_2}$ is equal to $\mathcal{K}_{T_1} \mathcal{K}_{T_2}$ if we prove that

$$\det \begin{pmatrix} Q''_{\chi\chi}/i & Q''_{\chi z} \\ Q''_{x\chi} & iQ''_{xz} \end{pmatrix} = \det \begin{pmatrix} Q''_{1\theta\theta}/i & Q''_{1\theta y} \\ Q''_{1x\theta} & iQ''_{1xy} \end{pmatrix} \det \begin{pmatrix} Q''_{2\tau\tau}/i & Q''_{2\tau z} \\ Q''_{2y\tau} & iQ''_{2yz} \end{pmatrix},$$

for this will prove at the same time that the components of $\partial Q/\partial \chi$ are linearly independent. For reasons of continuity it suffices to prove this algebraic identity when the number of θ and τ variables can be reduced to 0, and when these variables are absent it reduces to

$$\det \begin{pmatrix} (Q''_{1yy} + Q''_{2yy})/i & Q''_{2yz} \\ Q''_{1xy} & 0 \end{pmatrix} = \det(iQ''_{1xy}) \det(iQ''_{2yz})$$

which is obvious.

The kernel of the adjoint of $\mathcal{K}_T$ is defined by $-\overline{Q(y,x,\theta)}$, and the corresponding symplectic transformation is $Y \mapsto \overline{T^{-1}\overline{Y}}$, which we shall denote by $\overline{T}^{-1}$. In particular, this is the inverse of T when T is real. The structure of a general positive symplectic linear map is clarified by the following analogue of the polar decomposition:

Proposition 5.10 *Every positive symplectic linear map in $T^*\mathbf{C}^n$ can be factored as $T_1T_2T_3$ where T_1 and T_3 are real linear symplectic maps and $T_2(x,\xi) = (x',\xi')$ where for $j = 1,\dots,n$ either*

$$(x_j',\xi_j') = (x_j\cosh\tau_j - i\xi_j\sinh\tau_j, ix_j\sinh\tau_j + \xi_j\cosh\tau_j)$$
$$\text{or}\quad (x_j',\xi_j') = (x_j, ix_j + \xi_j),$$

with $\tau_j \geq 0$. The map T_2 is defined by the sum of the quadratic forms

$$(x_j\theta_j + \tfrac{1}{2}i(x_j^2+\theta_j^2)\sinh\tau_j)/\cosh\tau_j - y_j\theta_j \quad \text{or} \quad x_j\theta_j + \tfrac{1}{2}ix_j^2 - y_j\theta_j.$$

Proof. This is contained in Theorem 4.1 of Hörmander [5] where it was proved by studying $\overline{T}^{-1}T$.

We shall now connect the preceding facts with the Weyl calculus. First we consider the Weyl operator in $\mathbf{R}^n$ with symbol $e^{iQ(x,\xi)}$ where Q is a quadratic form with $\operatorname{Im} Q \geq 0$ in $T^*\mathbf{R}^n$. The kernel of the operator is

$$K(x,y) = (2\pi)^{-n}\int e^{iQ(\frac{1}{2}(x+y),\theta)+i\langle x-y,\theta\rangle}\,d\theta \tag{5.13}$$

which is well defined since the derivatives of the exponent with respect to $\theta_1,\dots,\theta_n$ are obviously linearly independent. The corresponding twisted Lagrangian contains (x,ξ,y,η) if with $z = \frac{1}{2}(x+y)$ and some θ

$$Q'_\theta(z,\theta) + x - y = 0, \quad \tfrac{1}{2}Q'_z(z,\theta) + \theta = \xi, \quad -\tfrac{1}{2}Q'_z(z,\theta) + \theta = \eta,$$

which is equivalent to

$$x = z - \tfrac{1}{2}Q'_\theta(z,\theta),\ \xi = \theta + \tfrac{1}{2}Q'_z(z,\theta),\ y = z + \tfrac{1}{2}Q'_\theta(z,\theta),$$
$$\eta = \theta - \tfrac{1}{2}Q'_z(z,\theta).$$

If F is the Hamilton map corresponding to Q this means that

$$(x,\xi) = (I-F)(z,\theta), \quad (y,\eta) = (I+F)(z,\theta). \tag{5.14}$$

If F does not have the eigenvalue 1, hence not the eigenvalue -1 either, we obtain a symplectic linear map $T = (I-F)(I+F)^{-1}$. (The passage from F to T is an analogue of the Cayley transformation.) The kernel (5.13) is then a constant times K_T, defined by (5.11). To determine the constant we must evaluate

$$\det\begin{pmatrix} Q''_{\theta\theta}/i & \frac{1}{2}Q''_{\theta z} - I \\ \frac{1}{2}Q''_{z\theta} + I & \frac{1}{4}iQ''_{zz}\end{pmatrix} = \det\begin{pmatrix} \frac{1}{2}Q''_{\theta\theta} & \frac{1}{2}Q''_{\theta z} - I \\ \frac{1}{2}Q''_{z\theta} + I & \frac{1}{2}Q''_{zz}\end{pmatrix}.$$

Multiplication to the right by $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, which has determinant 1, shows that this determinant is equal to

$$\det\begin{pmatrix} I - \frac{1}{2}Q''_{\theta z} & \frac{1}{2}Q''_{\theta\theta} \\ -\frac{1}{2}Q''_{zz} & I + \frac{1}{2}Q''_{z\theta} \end{pmatrix} = \det(I - F).$$

Hence we have proved:

Proposition 5.11 *If Q is a quadratic form in $T^*\mathbf{R}^n$ with $\operatorname{Im} Q \geq 0$ and the Hamilton map F of Q does not have the eigenvalues ± 1, then*

$$\sqrt{\det(I - F)}(e^{iQ})^w = \mathscr{K}_{(I-F)/(I+F)}, \tag{5.15}$$

where $\mathscr{K}_T$ is the operator with kernel K_T defined in (5.11).

We shall apply this result to e^{iQ^w} where Q is a quadratic form with Hamilton map F and $\operatorname{Im} Q \geq 0$. When $\det\cosh F \neq 0$ we know from Theorem 4.3 that the Weyl symbol is equal to $\exp(i\tilde{Q})/\sqrt{\det\cosh F}$ where $\tilde{Q}(X) = \sigma(X, (\tanh F)X)$. Thus the Hamilton map of $\tilde{Q}$ is $\tanh F$, which according to Proposition 5.11 is associated with the linear symplectic map

$$(I - \tanh F)(I + \tanh F)^{-1} = e^{-2F}.$$

We have $\det(I - \tanh F) = \det(\cosh F - \sinh F)/\det\cosh F = 1/\det\cosh F$, for the determinant of e^{-F} is equal to 1 since it is a symplectic map. Now it follows from Proposition 5.11 that the Weyl operator defined by $\exp(i\tilde{Q})/\sqrt{\det\cosh F}$ is equal to $\mathscr{K}_{\exp(-2F)}$, which means that e^{iQ^w} is equal to $\mathscr{K}_{\exp(-2F)}$. We can therefore give another interpretation of the Mehler formulas of Sect. 4:

Theorem 5.12 *If Q is a quadratic form in $T^*\mathbf{R}^n$ with $\operatorname{Im} Q \geq 0$, then $\exp(iQ^w) = \mathscr{K}_{\exp(-2F)}$ where $\mathscr{K}_T$ is the operator with kernel K_T defined by* (5.11) *when T belongs to the semigroup $\mathscr{C}_+$ of positive symplectic linear maps in $T^*\mathbf{C}^n$. The semigroup generated by the contraction operators $\exp(iQ^w)$ consists of all operators $\mathscr{K}_T$ with $T \in \mathscr{C}_+$; it is a double cover of $\mathscr{C}_+$. The invertible elements in the semigroup are those with $T \in \mathscr{C}_0 \subset \mathscr{C}_+$, where $\mathscr{C}_0$ is the real symplectic group. They form a double cover of $\mathscr{C}_0$, generated by $\exp(iQ^w)$ when Q is a real quadratic form.*

Proof. If $\cosh\lambda \neq 0$ when λ is an eigenvalue of F, we have already seen that $\exp(iQ^w) = \mathscr{K}_{\exp(-2F)}$. Otherwise we write $1 = s + t$ where $0 < s < t$ and $\cosh\lambda s \neq 0$, $\cosh\lambda t \neq 0$ when λ is an eigenvalue of F. Then $\exp(itQ^w) = \mathscr{K}_{\exp(-2tF)}$ and $\exp(isQ^w) = \mathscr{K}_{\exp(-2sF)}$, and using Proposition 5.9 we conclude that $\exp(iQ^w) = \mathscr{K}_{\exp(-2F)}$.

If T is a real symplectic map close to the identity, then the equation $e^{-2F} = T$ has a skew symmetric solution F close to 0 given by the Taylor expansion of $F = -\frac{1}{2}\log(I + (T - I))$; the skew symmetry follows since T^{-1} is the adjoint of T with respect to the symplectic form. Thus $\mathscr{K}_T = \exp(iQ^w)$ if Q is the quadratic form with Hamilton map F.

Let G be the subgroup of the unitary group generated by $\exp(iQ^w)$ with real quadratic forms Q, and let $\widetilde{G}$ be the group consisting of the operators with kernel (5.11) where $T \in \mathscr{C}_0$. We have $G \subset \widetilde{G}$, and $\widetilde{G}$ is a double cover of $\mathscr{C}_0$. Since the range of the composition $G \to \widetilde{G} \to \mathscr{C}_0$ contains a neighborhood of the identity we have either $G \cong \widetilde{G}$ or $G \cong \mathscr{C}_0$, for $\mathscr{C}_0$ is connected. Now we know from the Mehler formula (4.7) that if $Q(x,\xi) = \sum_1^n s_j(x_j^2 + \xi_j^2)$ where $\sin s_j = 0$, $j = 1, \dots, n$, then $\exp(iQ^w) = I / \prod_1^n \cos(s_j)$, which is $-I$ if we take $s_1 = \pi$ and $s_j = 0$ for $j \neq 1$. Hence $G = \widetilde{G}$.

In the complex case the surjectivity is now a consequence of Proposition 5.10, for the map T_2 there is equal to e^{-2F} where $F(x,\xi) = (x',\xi')$ means that

$$x_j' = \tfrac{1}{2} i\tau_j \xi_j,\ \xi_j' = -\tfrac{1}{2} i\tau_j x_j \quad \text{or} \quad x_j' = 0,\ \xi_j' = -\tfrac{1}{2} i x_j.$$

The corresponding quadratic form is $Q(x,\xi) = \sum_1^n Q_j(x_j,\xi_j)$ where

$$Q_j(x_j,\xi_j) = \tfrac{1}{2} i\tau_j(x_j^2 + \xi_j^2) \quad \text{or} \quad Q_j(x_j,\xi_j) = \tfrac{1}{2} i x_j^2.$$

The corresponding operator is therefore a classical Mehler operator. It is a contraction operator, and it is not unitary unless $Q = 0$. If $T = T_1 T_2 T_3$ as in Proposition 5.10 it follows that T has an inverse which is a contraction operator if and only if T_2 is the identity which means that T is a real symplectic map.

The group of operators $\{\mathscr{K}_T\}$ with $T \in \mathscr{C}_0$ is isomorphic to the metaplectic group, and it seems natural to call the semigroup of operators $\{\mathscr{K}_T\}$ with $T \in \mathscr{C}_+$ the *metaplectic semigroup*. If $U = \mathscr{K}_T$ with $T \in \mathscr{C}_0$ and if $\operatorname{Im} Q \geq 0$, then the metaplectic invariance of the Weyl calculus (see [4, Theorem 18.5.9]) implies that

$$U^{-1}(e^{iQ})^w U = (e^{i\widetilde{Q}})^w,$$

where $\widetilde{Q}(X) = Q(TX)$ has the Hamilton map $\widetilde{F} = T^{-1}FT$, if F is the Hamilton map of Q. This is also a consequence of Proposition 5.11, but the full metaplectic invariance does not follow from Proposition 5.11 since all Gaussian symbols are even.

Remark. The exponential map $Q \mapsto \exp(iQ^w)$, defined when Q is a quadratic form in $\mathbf{R}^n$ with $\operatorname{Im} Q \geq 0$, is not a surjection of a neighborhood of 0 on a neighborhood of the identity in $\{\mathscr{K}_T; T \in \mathscr{C}_+\}$. In fact, assume that $n = 1$ and that Q is small; let F be the Hamilton map of Q and set $\widetilde{F} = \tanh F$, thus $F = \operatorname{arctanh} \widetilde{F}$. Suppose that $\sigma(X, \widetilde{F}X) = ax^2 + ib\xi^2$ for some small $a \in \mathbf{R}\setminus 0$ and $b > 0$, which is a form with non-negative imaginary part. Then the eigenvalues λ of $\widetilde{F} = \begin{pmatrix} 0 & ib \\ -a & 0 \end{pmatrix}$ are given by $\lambda^2 + iab = 0$, so

$$(\operatorname{arctanh} \widetilde{F})/\widetilde{F} = (\operatorname{arctanh} \lambda)/\lambda = \int_0^1 \frac{dt}{1 - \lambda^2 t^2} = \int_0^1 \frac{1 - iabt^2}{1 + a^2b^2t^4}\, dt,$$

which proves that

$$\begin{aligned}\operatorname{Im} Q(X) &= \operatorname{Im}\ \big((ax^2 + ib\xi^2)(\operatorname{arctanh}\lambda)/\lambda\big)\\ &= b\xi^2 \int_1^1 \frac{dt}{1 + a^2b^2t^4} - a^2bx^2 \int_0^1 \frac{t^2\,dt}{1 + a^2b^2t^4},\end{aligned}$$

which is not non-negative. Hence the element $\mathscr{K}_T$ obtained by normalizing the Weyl operator with symbol $\exp(i(ax^2+ib\xi^2))$ as in (5.15) is not in the local range of the exponential map. However, the factorisation in Theorem 5.12 is given explicitly by $\mathscr{K}_T = \exp(\frac{1}{2}iax^2)\exp(icD^2)\exp(\frac{1}{2}iax^2)$ where $c = ib/(1 + iab)^{-1}$, thus $\operatorname{Im} c = b/(1 + a^2b^2) > 0$.

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