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Isometric embedding of the 2-sphere with non negative curvature in $\mathbb{R}^3$

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1 Introduction

We are interested in this work in the following question raised by H. Weyl [15]. Let g be a smooth Riemanian metric on the 2-sphere S^2. Does there exist a smooth global isometric embedding of the Riemanian manifold (S^2, g) in $\mathbb{R}^3$ endowed with the flat metric ?

When the Gauss curvature of the metric is strictly positive everywhere, a positive answer has been given through the works of H. Weyl [15] and L. Nirenberg [11], who proved the existence of a $C^{k-1,\alpha}, 0 < \alpha < 1$, isometric embedding provided that g is $C^k, (k \geqq 3)$. (See also E Heinz [8] and the works of A.D. Alexandrov [1], A.V. Pogorelov [12] for a different approach.)

In a recent work, J.A. Iaia [10] considered the case when the positive Gauss curvature vanishes at one point $P \in S^2$ and ΔK is non negative in a small neighborhood of P. (Here Δ is the Laplacian on S^2 associated to the metric g.) He proved under these conditions the existence of a $C^{1,1}$ isometric embedding provided $g \in C^4$.

The purpose of our work is to extend this result to the general case, namely:

Theorem 1.1 *Let g be a C^4 Riemanian metric on S^2. Suppose that the Gauss curvature satisfies*

$$K \geqq 0 \text{ on } S^2 . \tag{1.1}$$

Then there exists a $C^{1,1}$, isometric embedding $X : (S^2, g) \to \mathbb{R}^3$. Moreover X is $C^{3,\alpha}, 0 < \alpha < 1, (C^\infty)$ in $S^2 \backslash K^{-1}(0)$ (if g is C^∞).

Let us note that the Gauss–Bonnet theorem tells us that for any C^2 metric on S^2 its Gauss curvature is strictly positive somewhere on the sphere.

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Concerning higher regularity, Iaia [10] observed that if, at a point P, K has a non degenerate minimum, and if a C^2 isometric embedding X has a mean curvature which vanishes at P, then X cannot be C^3. It is worthwhile to point out the Pogorelov gave an example of a geodesic disk, with non negative curvature, not admitting a local C^2 isometric embedding in $\mathbb{R}^3$ at its center.

Let us now outline the paper. Approximating the metric g by a sequence of smooth metrics g_ε with strictly positive Gauss curvature and using Nirenberg's theorem, we get a sequence (X_ε) of isometric embeddings. The problem is then to give absolute upper bounds for the derivatives up to the second order of X_ε. Classical results (see [5]) show that $\rho_\varepsilon = -\frac{1}{2}\|X_\varepsilon\|^2$ satisfies, in every local chart, a non linear second order p.d.e. of Monge–Ampère type :

$$\det\ (\nabla_{ij}\rho_\varepsilon + g_{ij}^\varepsilon) = K_\varepsilon(\det g_{ij}^\varepsilon)\ (-2\rho_\varepsilon - |\nabla\rho_\varepsilon|^2)\ .$$

Of course the problems come from the fact that the right hand side may vanish. An important observation is that actually the quantity $(-2\rho_\varepsilon^2 - |\nabla\rho_\varepsilon|^2)$ is bounded below by a strictly positive constant independant of ε. This is a consequence of a geometric lemma of S.Y. Cheng–S.T. Yau [3] about lower bounds for the radius of balls which can be inscribed in a convex compact body. On the other hand, the classical upper bounds for the second derivatives, as in L. Caffarelli-L. Nirenberg-J. Spruck [2] or C.M. Corona [4] for example, are given in terms of upper bounds for $\frac{1}{K_\varepsilon}$ which is here irrelevant. We use instead a technique which is inspired by the work of J. Hong [9] and an interior estimate proved by E. Heinz [7].

2 Differential geometry formulas

For sake of completeness we recall in this section some classical facts which will be used in the proof. (See [5], [11].)

Let Σ be a smooth convex surface with positive Gauss curvature locally given by a smooth $X : U \subset \mathbb{R}^2 \to \Sigma$. Local coordinates will be denoted by $u = (u_1, u_2)$. Let $(g_{ij}(u))$ and $(\ell_{ij}(u))$ be the components of the metric and of the second fundamental form of Σ. We set $G = \det g_{ij}$.

The orientation of Σ is so chosen that the inner unit normal to the surface at any point is given by $X_3 = \frac{1}{\sqrt{G}}(X_{u_1} \wedge X_{u_2})$.

The mean and Gauss curvatures of the surface H and K, which are then positive, are given by

$$K = \frac{\det\ \ell_{ij}}{G}, \quad H = \frac{1}{2}\sum_{i,j=1}^{2}\ell_{ij}g^{ij}\ , \tag{2.1}$$

where $(g^{ij}) = (g_{ij})^{-1}$. The structure equations are then

$$\frac{\partial^2 X}{\partial u_i \partial u_j} = \sum_{k=1}^{2}\Gamma_{ij}^k\frac{\partial X}{\partial u_k} + \ell_{ij}X_3\ , \tag{2.2}$$

where Γ_{ij}^k are the Christoffel symbols of the metric.

Choosing the origin as the center of the largest ball which may be included in the convex body bounded by Σ we define

$$\rho(u) = -\frac{1}{2}\|X(u)\|^2 , \tag{2.3}$$

where $\|\cdot\|$ denotes the Euclidian norm on $\mathbb{R}^3$.

Following [11] we shall show that ρ satisfies an equation of Monge–Ampère type. By (2.2) and (2.3) we get

$$\begin{cases} \frac{\partial\rho}{\partial u_i} = -\langle X, \frac{\partial X}{\partial u_i}\rangle \\ \frac{\partial^2\rho}{\partial u_i\partial u_j} = \sum_{k=1}^{2}\Gamma_{ij}^k\frac{\partial\rho}{\partial u_k} - \ell_{ij}\langle X,X_3\rangle - g_{ij}. \end{cases} \tag{2.4}$$

Denoting by $\nabla_{ij}\rho$ the second covariant derivative of ρ we get

$$\ell_{ij}\langle X,X_3\rangle = \nabla_{ij}\rho - g_{ij} , \tag{2.4$'$}$$

and therefore by (2.1) we get

$$K\langle X,X_3\rangle^2 = \frac{1}{G}\det(\nabla_{ij}\rho + g_{ij}) . \tag{2.5}$$

Now the expression $\langle X,X_3\rangle^2$ represents the square of the distance from the origin to the tangent plane to Σ at the point $X(u)$. we have

$$\begin{aligned} \langle X,X_3\rangle^2 &= \|X\|^2 - \|X\wedge X_3\|^2 = \|X\|^2 - \left\|X\wedge\frac{X_{u_1}\wedge X_{u_2}}{\sqrt{G}}\right\| \\ &= \|X\|^2 - \frac{1}{G}\|\langle X,X_{u_1}\rangle X_{u_2} - \langle X,X_{u_2}\rangle X_{u_1}\|^2 \\ &= -2\rho - \frac{1}{G}\left(g_{22}\left(\frac{\partial\rho}{\partial u_1}\right)^2 - 2g_{12}\frac{\partial\rho}{\partial u_1}\cdot\frac{\partial\rho}{\partial u_2} + g_{11}\left(\frac{\partial\rho}{\partial u_2}\right)^2\right) \end{aligned}$$

and therefore

$$\langle X,X_3\rangle^2 = -2\rho - \sum_{i,j=1}^{2} g^{ij}\nabla_i\rho\cdot\nabla_j\rho . \tag{2.6}$$

It follows from (2.5) and (2.6) that ρ satisfies the equation

$$\det(\nabla_{ij}\rho + g_{ij}) = KG\left(-2\rho - \sum_{i,j=1}^{2} g^{ij}\nabla_i\rho\cdot\nabla_j\rho\right) . \tag{2.7}$$

Moreover, using the mean curvature H we get another equation for ρ. Indeed we have $H = \frac{1}{2}\sum_{i,j} g^{ij}\ell_{ij}$, and by (2.4)$'$

$$\sum_{i,j} g^{ij}\ell_{ij}\langle X,X_3\rangle^2 = -\sum_{i,j} g^{ij}\nabla_{ij}\rho - \sum_{i,j} g^{ij}g_{ij} = -\Delta_g\rho - 2$$

$$\Delta_g\rho + 2 = 2H\sqrt{-2\rho - \sum_{i,j=1}^{2} g^{ij}\nabla_i\rho\cdot\nabla_j\rho} . \tag{2.8}$$

Here $\Delta_g = \sum_{i,j} g^{ij}\nabla_{ij}$ is the Laplace operator with respect the metric (g_{ij}).

We shall use in the proof a result of S.Y. Cheng–S.T. Yau that we recall now in the particular case of the dimension two.

Lemma 2.1 (Cheng-Yau [3]). *Let Σ be a compact convex C^4 hypersurface in $\mathbb{R}^3$. Let K be its Gauss curvature function defined on S^2. Then we can find a positive constant r which depends only on an upper bound of $\int_{S^2} \frac{d\omega}{K(\omega)}$ and a lower bound of $\inf_{u \in S^2} \int_{S^2} \max(0, \langle u, \omega \rangle) \frac{d\omega}{K(\omega)}$ such that we can put a ball of radius r inside the convex body bounded by Σ.*

We shall also use an interior estimated proved by Heinz [7] which we recall now. Let us set $B_r = \{u \in \mathbb{R}^2 : |u| \leqq r\}$.

Lemma 2.2 *Let (Σ, g) be a closed convex surface in $\mathbb{R}^3$ given around $p \in \Sigma$ by $X : B_r \to \Sigma$ with $X(0) = p$. Assume that its Gauss curvature satisfies*

$$K(u) \geqq a > 0 \quad \forall u \in B_r \,. \tag{2.9}$$

Then there exists a positive constant C depending only on $diam(\Sigma, g)$, the maximum over B_r of $(\det g_{ij})^{-1}, r, \|g\|_{C^3(B_r)}, \|K\|_{C^2(B_r)}$ and $\|X\|_{C^0(B_r)}$ such that

$$| D^2X(u) | \leqq C \quad \forall u \in B_{r/2} \,. \tag{2.10}$$

This lemma follows from Satz 3 in Heinz [7] since we have $\int\int_{(\Sigma,g)} H d\theta = \int\int_{(\Sigma,g)} \langle X, \vec{n} \rangle K d\theta$ where $\vec{n}$ is the outward unit normal and $d\theta$ is the area element of Σ.

3 Regularisation of the metric

The purpose of this section is to approximate our metric by smooth metrics with strictly positive curvature.

Lemma 3.1 *Let g_0 be a C^4 metric on S^2 whose Gauss curvature satisfies*

$$K_0 \geqq 0 \text{ on } S^2 \,. \tag{3.1}$$

Then there exists a sequence (g_ε) of C^∞ metrics on S^2 with curvature K_ε such that

$$\begin{cases} \text{i)} & (g_\varepsilon) \text{ tends to. } g_0 \text{ in } C^4 \\ \text{ii)} & K_\varepsilon \text{ tends to } K_0 \text{ in } C^2 \\ \text{iii)} & K_\varepsilon > 0 \text{ on } S^2 \text{ for small } \varepsilon. \end{cases} \tag{3.2}$$

Proof. We first approximate g_0 by a sequence of C^4 metrics.

Let us set $F = \{P \in S^2 : K_0(P) = 0\}$. We may assume $F \neq \phi$ otherwise our result reduces to that of Nirenberg. Moreover the Gauss–Bonnet theorem implies that $F \neq S^2$. Let $\mathcal{O}_1 \subset \mathcal{O}_2$ be open sets in S^2 contained in $S^2 \backslash F$ and let $\theta \in C^\infty(S^2)$ be such that: $0 \leqq \theta \leqq 1, \theta = 1$ on $\mathcal{O}_1, \theta = 0$ on $S^2 \backslash O_2$.

Since the right hand side has mean zero on the sphere, the equation

$$\Delta_{g_0} v = \theta - \frac{1}{|S^2|_{g_0(S^2, g_0)}} \int \theta(\omega) d\omega \tag{3.3}$$

has a C^∞ solution v.

Let us set

$$g_\varepsilon = e^{2\varepsilon v} g_0 \tag{3.4}$$

It is well known that the Gauss curvature of g_ε is given by

$$e^{2\varepsilon v} K_\varepsilon = K_0 - \varepsilon \Delta_{g_0} v \ . \tag{3.5}$$

Let $\delta = \inf_{\overline{\mathcal{C}}_2} K_0$. Then $\delta > 0$, and if ε is so small that $\varepsilon \sup_{S^2} |\Delta_{g_0} v| \leqq \frac{1}{2}\delta$ we have $K_\varepsilon > 0$ on $\mathcal{C}_2$. Moreover, on $S^2 \backslash \mathcal{C}_2$ we have $\Delta_{g_0} v = -\frac{1}{|S^2|_{g_0}} \int_{(S^2, g_0)} \theta(\omega) d\omega < 0$ and $K_0 \geqq 0$. Therefore $K_\varepsilon > 0$ on S^2. This proves iii). The statements i) and ii) are easy.

Now for a fixed ε, since K_ε is strictly positive, we can use the mollifier technique discussed in Green–Wu [6] to approximate in the C^4-topology the C^4 metric g_ε by a sequence of C^∞ metrices $(g_{\varepsilon,n})$ with strictly positive Gauss curvature, which completes the proof of the Lemma 3.1

4 Proof of the Theorem

Consider the Riemanian manifolds (S^2, g_ε) where g_ε is given by the Lemma 3.1. Since its Gauss curvature is strictly positive, the theorem L. Nirenberg [11] implies that there exists a global C^∞ isometric embedding $X_\varepsilon : (S^2, g_\varepsilon) \to \mathbb{R}^3$. We shall set $\Sigma_\varepsilon = X_\varepsilon(S^2)$. Then Σ_ε is a strictly convex hypersurface which bounds a strictly convex body.

Bounds for X_ε and ∇X_ε

Without loss of generality we may assume that the origin is at the center of the largest ball inscribed in the convex body bounded by Σ_ε, translating if necessary this isometric embedding X_ε. Of course this will not change the bounds of $|\nabla X_\varepsilon|$ and $|\nabla^2 X_\varepsilon|$. It follows from the convexity of Σ_ε that for $\varepsilon \in (0, \varepsilon_0)$,

$$|X_\varepsilon| \leqq \operatorname{diam}(S^2, g_\varepsilon) \leqq \max_{S^2} e^{2\varepsilon v} \operatorname{diam}(S^2, g_0) \leqq C_0 \tag{4.1}$$

for some constant C_0 independent of ε.

Now since $\langle X_{\varepsilon i}, X_{\varepsilon j} \rangle = \mathrm{e}^{2\varepsilon v} g_{oij}$ we get

$$|\nabla_{g_0} X_\varepsilon|^2_{g_0} \leqq \sup_{S^2} \mathrm{e}^{2\varepsilon v} \leqq C_2 \ . \tag{4.2}$$

From (4.1), (4.2) we get, with a constant C independent of ε,

$$\|X_\varepsilon\|_{C^1{}_{g_0}} \leqq C \ . \tag{4.3}$$

Bounds of the second derivatives

As soon as we have a bound for the C^1 norm of X_ε we can show that we can put inside the body bounded by Σ_ε a ball whose radius is bounded below by an absolute positive constant. Indeed, by Lemma 2.1 it is sufficient to get absolute bounds for $\inf_{u\in S^2}\int_{S^2}\max(0,\langle u,\omega\rangle)\frac{d\omega}{K_\varepsilon(\omega)}$ and $\int_{S^2}\frac{d\omega}{K_\varepsilon(\omega)}$.

First of all since $K_\varepsilon = e^{-2\varepsilon v}(K_0 - \varepsilon\Delta v)$ one has $0 < K_\varepsilon \leqq A$ with A independent of ε. Therefore for $u \in S^2$

$$\int_{S^2}\max(0,\langle u,\omega\rangle)\frac{d\omega}{K_\varepsilon(\omega)} \geqq \int_{S^2}\frac{1}{A}\max(0,\langle u,\omega\rangle)\,d\omega = \frac{2\pi}{A}\,. \tag{4.4}$$

On the other hand, since the Gauss map $\Sigma_\varepsilon \xrightarrow{n} S^2$ (where $n(x)$ is the unit normal at x to Σ_ε) is a global diffeomorphism one has

$$\int_{S^2}\frac{d\omega}{K_\varepsilon(\omega)} = \int_{\Sigma_\varepsilon} d\sigma_\varepsilon = \int_{(S^2,g_0)} e^{2\varepsilon v}d\sigma_0 \leqq C\text{area}(S^2,g_0)\,. \tag{4.5}$$

It follows then, from (2.6), (2.7) and Lemma 2.1, that our function $\rho_\varepsilon = -\frac{1}{2}\|X_\varepsilon\|^2$ satisfies

$$\det(\nabla_{ij}\rho_\varepsilon + g_{\varepsilon ij}) = K_\varepsilon G_\varepsilon(-2\rho_\varepsilon - |\nabla\rho_\varepsilon|^2_{g_\varepsilon}) \tag{4.6}$$

$$-2\rho_\varepsilon - |\nabla\rho_\varepsilon|^2_{g_\varepsilon} \geqq r_0^2 > 0\,, \tag{4.7}$$

where $|\nabla\rho_\varepsilon|^2_{g_\varepsilon} = \sum g_\varepsilon^{ij}\nabla_i\rho_\varepsilon\nabla_j\rho_\varepsilon$ and r_0 is independent of ε.

Indeed the left hand side of (4.7) represents the square of the distance from the origin to the tangent plane to Σ_ε at $X_\varepsilon(P)$. Here we took the origin to be the center of the largest ball which can be inscribed in the convex body bounded by Σ_ε.

Consider now that C^∞ function on S^2 defined by

$$w_\varepsilon = \Delta_{g_\varepsilon}\rho_\varepsilon \exp\left(\frac{\lambda}{2}|\nabla\rho_\varepsilon|^2_{g_\varepsilon}\right)\,, \tag{4.8}$$

where λ is a real constant staisfying

$$\lambda C_0^2 \leqq \frac{1}{4} \tag{4.8$'$}$$

and where C_0 is defined in (4.1).

From now we skip the subscript ε keeping in mind that all our upper bounds must be independent of ε and we write K instead of K_0. Let us introduce some conventions and notations. First of all from now on all our derivatives will be covariant derivatives and we shall sometimes write ρ_i, ρ_{ij} instead of $\nabla_i\rho, \nabla_j\nabla_i\rho$ etc $\cdots$ Moreover we shall write

$$(4.9)\qquad \begin{cases} F(z_{ij}) = \det z_{ij}, \ \ z_{ij} = \rho_{ij} + g_{ij} \\ f = K\,G(-2\rho - |\nabla\rho|_g^2) \\ F^{ij} = \dfrac{\partial F}{\partial z_{ij}}, F^{ijpq} = \dfrac{\partial^2 F}{\partial z_{ij}\partial z_{pq}} \\ \varDelta_g = \varDelta \ , \end{cases}$$

and we shall use the Einstein summation convention.

Differentiating two times the equation (4.8) we get (since $\nabla_k g_{ij} = 0$)

$$(4.10)\qquad F^{ij}\rho_{ijk\ell} + F^{ijpq}\rho_{ijk}\rho_{pq\ell} = f_{k\ell} \ .$$

Now from the Ricci formulas we get

$$(4.11)\qquad \begin{aligned} \rho_{ijk\ell} = \rho_{k\ell ij} &+ R^m_{ijk}\rho_{m\ell} + R^m_{ij\ell}\rho_{mk} + R^m_{\ell ik}\rho_{mj} + R^m_{ik\ell}\rho_{mj} + R^m_{kj\ell}\rho_{mi} \\ &+ \nabla_\ell R^m_{ijk}\rho_m + \nabla_j R^m_{\ell ik}\rho_m + \nabla_j R^m_{ik\ell}\rho_m \ , \end{aligned}$$

where R^m_{ijk} are the components of the Riemann tensor.

Now multiplying (4.9) by $g^{k\ell}$, using (4.10), setting $L = F^{ij}\nabla_j\nabla_i$ and recalling that $\varDelta = g^{k\ell}\nabla_l\nabla_k$ we get

$$(4.12)\qquad L\varDelta\rho = \varDelta f - g^{k\ell}F^{ijpq}\rho_{ijk}\rho_{pq\ell} - g^{k\ell}\psi_{k\ell}$$

with

$$(4.13)\qquad \psi_{k\ell} = F^{ij}\left(R^m_{ijk}\rho_{m\ell} + \cdots\cdots + \nabla_j R^m_{ik\ell}\rho_m\right) \ .$$

With w defined in (4.8) we get

$$(4.14)\qquad \begin{aligned} Lw = e^{\frac{\lambda}{2}|\nabla\rho|^2}\Big[& L\varDelta\rho + 2\lambda F^{ij}(\varDelta\rho)_i g^{k\ell}\rho_{kj}\rho_\ell + \varDelta\rho[\lambda g^{k\ell}f_k\rho\ell \\ & + \lambda g^{k\ell}F^{ij}R^m_{kij}\rho_m\rho_\ell + \lambda F^{ij}g^{k\ell}\rho_{ki}\rho_{\ell j} + \lambda^2 F^{ij}g^{k\ell}g^{pq}\rho_{ki}\rho_\ell\rho_{pj}\rho_q]\Big] \ . \end{aligned}$$

Here we have used $\rho_{kij} = \rho_{ijk} + R^m_{ikj}\rho_m$ and $F^{ij}\rho_{ijk} = f_k$.

On the other hand from (4.9) we get

$$(4.15)\qquad \begin{aligned} \varDelta f = \underset{1)}{\varDelta(KG)(-2\rho - |\varDelta\rho|^2)} &+ 2g^{k\ell}(KG)_k(-\underset{2)}{2\rho_\ell} - \underset{3)}{2g^{ij}\rho_{i\ell}\rho_j}) \\ &+ KGg^{k\ell}(-\underset{4)}{2\rho_{k\ell}} - \underset{5)}{2g^{ij}\rho_{ik\ell}\rho_j} - \underset{6)}{2g^{ij}\rho_{ik}\rho_{j\ell}}) \ . \end{aligned}$$

We shall denote in that follows by $0(1)$ a term which is uniformly bounded with respect to ε.

Since g_ε tends to g in C^4, K_ε tends to K in C^2, and since we have already uniform bounds for ρ_ε and its first order derivatives, it follows that

$$(4.16)\qquad \begin{cases} 1) + 2) = 0(1) \\ 3) + 4) = 0(1)\varDelta\rho \\ \qquad 6) = -2KG(\varDelta\rho)^2 + 0(1) + 0(1)\varDelta\rho \ . \end{cases}$$

Moreover, using $\rho_{ik\ell} = \rho_{k\ell i} + R^m_{ki\ell}\rho_m$ we see that

$$(4.17)\qquad 5) = -2KGg^{ij}(\Delta\rho)_i\rho_j + 0(1)\ .$$

It follows that

$$(4.18)\qquad \Delta f = -2KGg^{ij}(\Delta\rho)_i\rho_j - 2KG(\Delta\rho)^2 + 0(1) + 0(1)\Delta\rho\ .$$

Now let p be a point where w attains its maximum on S^2. Assume that, with the constant C_0 mentioned in (4.1), we have

$$(4.19)\qquad K(p)C_0^2 \geqq \frac{1}{16}\ .$$

It follows from the interior estimates given in Lemma 2.2 that we have absolute bounds for the second derivatives of X in a neighborhood of p. It follows from (2.3), (4.3) and (4.8) that we have an absolute bound for $w_\varepsilon(p)$ and therefore for $\sup_{S^2} w_\varepsilon$ from which we deduce an absolute bound for $\Delta\rho_\varepsilon$ on S^2.

Thus we may assume in the sequel that

$$(4.20)\qquad K(p)C_0^2 < \frac{1}{16}\ .$$

We shall work in a normal coordinate system centered at p. This means that

$$(4.21)\qquad g_{ij}(p) = \delta_{ij},\quad G(p) = 1,\quad \Gamma_{ij}^k(p) = 0$$

and the covariant derivatives become, at p, the usual ones.

Moreover, since the equation (4.6) is invariant by rotation, we may assume that

$$(4.22)\qquad \rho_{12}(p) = 0\ .$$

It follows that

$$(4.23)\qquad F^{12} = 0 \quad \text{at } p\ .$$

On the other hand, since w has at p a maximum, we get $\nabla_i w(p) = 0$ which implies

$$(4.24)\qquad (\Delta\rho)_i = -\lambda\rho_i\rho_{ii}\Delta\rho \text{ at } p,\quad \text{for } i = 1,2\ .$$

Now it is easy to see that the second term in the right hand side of (4.12) can be written at p as

$$(4.25)\qquad -g^{k\ell}F^{ijpq}\rho_{ijk}\rho_{pq\ell} = 2\sum_{k=1}^{2}(\rho_{12k}^2 - \rho_{11k}\rho_{22k})\ .$$

Using (4.12) to (4.25) we get at p

$$\begin{aligned}(4.26)\quad \exp\left(\frac{1}{2}\lambda|\nabla\rho|^2\right)Lw =\ & 2\sum_{k=1}^{2}(\rho_{12k}^2 - \underbrace{\rho_{11k}\rho_{22k}}_{1)}) - \underbrace{2Kg^{ij}(\Delta\rho)_i\rho_i}_{2)} \\ & - \underbrace{2K(\Delta\rho)^2}_{3)} - \underbrace{g^{kk}\psi_{kk}}_{4)} + \underbrace{2\lambda F^{ii}(\Delta\rho)_i\rho_{ii}\rho_i}_{5)} \\ & + \underbrace{\lambda g^{ii}f_i\rho_i\Delta\rho}_{6)} + \underbrace{\lambda F^{ii}\rho_{ii}^2\Delta\rho}_{7)} \\ & + \underbrace{\lambda^2F^{ii}\rho_{ii}^2\rho_i^2\Delta\rho}_{8)} + 0(1) + 0(1)\Delta\rho\ .\end{aligned}$$

First of all we have

$$-g^{kk}\psi_{kk} = 2F^{ii}R^{i}_{kik}\rho_{kk} + 0(1)\Delta\rho + 0(1)$$

and therefore

$$(4.27)\qquad 4) = 2K(\Delta\rho)^2 + 0(1) + 0(1)\Delta\rho\ .$$

Since $f = -2KG_\rho - KG|\nabla\rho|^2$ we get at $p, f_i = -2K\rho_i\rho_{ii} + 0(1)$, so

$$6) = -2KGg^{ii}\lambda\rho_i^2\rho_{ii}\Delta\rho + 0(1)\lambda\Delta\rho = 2Kg^{ii}(\Delta\rho)_i\rho_i + 0(1)\lambda\Delta\rho$$

$$(4.28)\qquad 2) + 6) = 0(1)\lambda\Delta\rho\ .$$

Now from (4.24) we get

$$\begin{aligned}5) + 8) = -\lambda^2F^{ii}\rho_{ii}^2\rho_i^2\Delta\rho = &- \lambda^2F^{ii}(\rho_{ii}+1)\rho_{ii}\rho_i^2\Delta\rho + \\ &+ \lambda^2F^{ii}(\rho_{ii}+1)\rho_i^2\Delta\rho - \lambda^2F^{ii}\rho_i^2\Delta\rho\ .\end{aligned}$$

But for each $i = 1, 2$, we have at p

$$(4.29)\qquad F^{ii}(\rho_{ii} + 1) = f,\quad F^{11} + F^{22} = \Delta\rho + 2\ .$$

It follows that

$$(4.30)\qquad 5) + 8) \geqq -\lambda^2(1 + f)|\nabla\rho|^2(\Delta\rho)^2 + 0(1)\lambda^2\Delta\rho\ .$$

Let us look at 7),

$$7) = \lambda F^{ii}(\rho_{ii} + 1)\rho_{ii}\Delta\rho - \lambda F^{ii}(\rho_{ii} + 1)\Delta\rho + \lambda\sum_{i=1}^{2}F^{ii}\Delta\rho\ .$$

From (4.29) we get

$$(4.31)\qquad 7) = \lambda(1 + f)(\Delta\rho)^2 + \lambda 0(1)\Delta\rho\ .$$

Let us estimate the term 1). We first observe that if at p we have

$$(4.32)\qquad \Delta\rho + 2 \leqq \max(1, \sqrt{8f})\ .$$

then we are done since the right hand side of (4.32) is uniformly bounded. So we can assume that

$$(4.33)\qquad \Delta\rho + 2 \geqq \max(1, \sqrt{8f})\quad \text{at } p\ .$$

Since $(\rho_{11} - \rho_{22})^2 = (\Delta\rho)^2 - 4\rho_{11}\rho_{22} = (\Delta\rho + 2)^2 - 4f$ we get

$$(4.34)\qquad (\rho_{11} - \rho_{22})^2 \geqq \frac{1}{2}(\Delta\rho + 2)^2 \geqq \frac{1}{2}\ .$$

Now differentiating the equation (4.6) with respect to ∇_k we get at p

$$\begin{cases} F^{11}\rho_{11k} + F^{22}\rho_{22k} = fk \\ \rho_{11k} + \rho_{22k} = (\Delta\rho)_k. \end{cases}$$

It follows that

(4.35) $$\begin{cases} \rho_{11k} = \dfrac{f_k - F^{22}(\Delta\rho)_k}{\rho_{22} - \rho_{11}} \\ \rho_{22k} = \dfrac{f_k - F^{11}(\Delta\rho)_k}{\rho_{11} - \rho_{22}} \end{cases}$$

Now using Ricci formulas we get

$$1) = 2\left(\rho_{112}^2 - \rho_{111}\rho_{221} + \rho_{221}^2 - \rho_{112}\rho_{222} + 2R_{121}^m\rho_m\rho_{112} + 2R_{212}^m\,\rho_m\rho_{221} + 0(1)\right) .$$

Using (4.35) and $F^{11}\cdot F^{22} = f, F^{11} + F^{22} = \Delta\rho + 2$ We see that

$$\frac{1}{2}1) = (\rho_{112} + R_{121}^m\rho_m)^2 + (\rho_{221} + R_{212}^m\rho_m)^2 + \sum_{i=1}^{2}\frac{f_i^2 + f(\Delta\rho)_i^2 - f_i(\Delta\rho)_i(\Delta\rho + 2)}{(\rho_{22} - \rho_{11})^2} + 0(1)$$

Since

$$\frac{f_i(\Delta\rho)_i(\Delta\rho + 2)}{(\rho_{22} - \rho_{11})^2} = \frac{2\lambda K\rho_i^2\rho_{ii}^2\Delta\rho(\Delta\rho + 2)}{(\rho_{22} - \rho_{11})^2} + 0(1)\lambda$$

we deduce that

(4.36) $$\frac{-f_i(\Delta\rho)_i(\Delta\rho + 2)}{(\rho_{22} - \rho_{11})^2} \geqq -4\lambda K|\nabla\rho|^2(\Delta\rho)^2 + 0(1)\lambda\Delta\rho + 0(1)\lambda .$$

It follows that

(4.37) $$1) \geqq -8\lambda K|\nabla\rho|^2(\Delta\rho)^2 + 0(1)\lambda\Delta\rho + 0(1)\lambda .$$

From (4.26) to (4.36) we get

(4.38) $$\exp(\cdots)Lw \geqq \lambda(\Delta\rho)^2((1 - \lambda|\nabla\rho|^2)(1 + f) - 8K|\nabla\rho|^2) + 0(1)\lambda\Delta\rho .$$

Now from (4.7), (2.3) and (4.1) we get

$$|\nabla\rho|^2 \leqq -2\rho = \|X\|^2 \leqq C_0^2 .$$

Therefore we deduce from (4.8)′, (4.21) that

(4.39) $$(1 - \lambda|\nabla\rho|^2)(1 + f) - 8K|\nabla\rho|^2 \geqq (1 - \lambda C_0^2)(1 + f) - 8KC_0^2 \geqq \frac{1}{4} .$$

From (2.4)′, since $\langle X, X_3\rangle$ is negative and the second fundamental form (ℓij) is positive, the matrix $(\nabla_{ij}\rho + g_{ij})$ is positive. This implies that the operator L is elliptic with positive symbol. It follows that at p, since w is maximum, we have $Lw \leqq 0$. Using (4.38) and (4.39) we deduce that $\Delta\rho = 0(1)$ at p, which implies a uniform bound for $\Delta_{g_\varepsilon}\rho_\varepsilon$ on S^2.

It follows from (2.8) and (4.7) that

(4.40) The mean curvature H_ε is uniformly bounded on S^2 .

Now from the Gauss equations we see easily that $\Delta_{g_\varepsilon} X_\varepsilon \cdot \Delta_{g_\varepsilon} X_\varepsilon = 4H_\varepsilon^2$. So (4.40) implies

(4.41) $$\|\Delta_{g_\varepsilon} X_\varepsilon\|^2 \text{is uniformly bounded .}$$

On the other hand we have

$$|\nabla_{g_\varepsilon}^2 X_\varepsilon|_{g_\varepsilon}^2 = g_\varepsilon^{ij} g_\varepsilon^{k\ell} X_{\varepsilon ik} X_{\varepsilon j\ell} \; .$$

Choose normal coordinates at a point Q. It follows that

$$|\nabla_{g_\varepsilon}^2 X_\varepsilon|_{g_\varepsilon}^2 = X_{\varepsilon 11} \cdot X_{\varepsilon 11} + 2X_{\varepsilon 12} \cdot X_{\varepsilon 12} + X_{\varepsilon 22} \cdot X_{\varepsilon 22} \; ,$$

Since $X_{\varepsilon 12} \cdot X_{\varepsilon 12} = \ell_{\varepsilon 12}^2 < \ell_{\varepsilon 11} \ell_{\varepsilon 22} = X_{\varepsilon 11} \cdot X_{\varepsilon 22}$, we have at Q

$$|\nabla_{g\varepsilon}^2 X_\varepsilon|_{g\varepsilon}^2 \leqq (X_{\varepsilon 11} + X_{\varepsilon 22})^2 = (\Delta x_\varepsilon)^2 + (\Delta y_\varepsilon)^2 + (\Delta z_\varepsilon)^2 = \Delta_{g\varepsilon} X_\varepsilon \cdot \Delta_{g\varepsilon} X_\varepsilon \; ,$$

where $X_\varepsilon = (x_\varepsilon, y_\varepsilon, z_\varepsilon)$. It follows from (4.36) that, with an absolute constant M,

$$|\nabla_{g_0}^2 X_\varepsilon|_{g_0}^2 = e^{4\varepsilon v} |\nabla_{g_\varepsilon}^2 X_\varepsilon|_{g_\varepsilon}^2 \leqq M \; ,$$

which completes the proof of the bounds of the second derivatives of X_ε.

Moreover near a point where K_0 is strictly positive it follows from (4.6) and (4.7) that ρ satisfies a uniformly elliptic Monge–Ampère equation. It follows from [8] and [14] that $\rho \in C^{3,\alpha}$. It follows from (2.8) and (4.7) that $H \in C^{1,\alpha}$. Since every component of X satisfies the equation $\Delta_z = 2H\sqrt{1 - |\nabla z|^2}$ (see [10]) it follows that $X \in C^{3,\alpha}$. This completes the proof of Theorem 1.1

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Note added in Proof. After completion of our manuscript, we are informed that Pengfei Guan and Yan Yan Li recently got the same result but the method they used to estimate the mean curvature is different from ours.