Math. Z. 216, 471–482 (1994)

Mathematische Zeitschrift

Mean curvature vector and symplectic topology of Lagrangian submanifolds in Einstein-Kähler manifolds

Yong-Geun Oh

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

Received 15 December 1992; in final form 14 June 1993

1 Introduction

A Lagrangian submanifold L is an n-dimensional submanifold of a symplectic manifold (M^{2n}, ω) on which the symplectic form ω vanishes. When (M, ω) carries a Kähler structure, i.e., possesses an integrable almost complex structure such that the bilinear form

$$g(X, Y)=\langle X, Y\rangle:=\omega(X, JY)$$

defines a Riemannian metric, the associated Riemannian properties of Lagrangian submanifolds have been studied by various authors in relation to the study of minimal Lagrangian submanifolds. (See [B, C, HL, O1, T]). Concerning the interplay between the symplectic and Riemannian geometries of Lagrangian submanifolds, Morvan [M] (see also [HL]) proved that the mean curvature vector $\vec{H}_i$ represents the Maslov class of the Lagrangian immersion $i\colon L\to\mathbb{C}^n$: the one form $\frac{1}{\pi}\alpha_{\vec{H}_i}$ on L defined by

$$(1.1)\qquad \alpha_{\vec{H}_i}:=\vec{H}_i \lrcorner\, \omega|_{TL}=\langle J\vec{H}_i, \cdot\rangle|_{TL}$$

represents the Maslov class $\mu\in H^1(L, \mathbb{Z})$. Morvan's result was generalized in a way by Dazord (see also [B]) who showed that on Kähler manifolds, the one form $\alpha_{\vec{H}_i}$ on L satisfies

$$(1.2)\qquad d\alpha_{\vec{H}_i}=i^*\rho$$

where $i\colon L\to(M, \omega)$ is a Lagrangian immersion and ρ is the Ricci form of the Kähler metric g. In particular, if (M, ω, J) is Einstein-Kähler, i.e., if

$$\rho=c\,\omega, \qquad \text{for } c\in\mathbb{R},$$

the one form $\alpha_{\vec{H}_i}$ on L is closed and so defines a real cohomology class on L. One corollary of the closedness of the one form $\alpha_{\vec{H}_i}$ is that the Lagrangian submanifolds stay Lagrangian under the mean curvature flow on Einstein-Kähler manifolds.

In the present paper, we further analyze the one form $\alpha_{\vec{H}}$ and prove that this one form is *characteristic* in the symplectic topological sense that the cohomology class $[\alpha_{\vec{H}_t}]$ is preserved under *Hamiltonian isotopies* (or *exact isotopies*) of the Lagrangian immersion $i: L \to (M, \omega)$.

Theorem I *Let (M, ω, J) be Einstein-Kähler and let $i_0: L \to (M, \omega)$ be a Lagrangian immersion. Then under the global Hamiltonian isotopy $\phi = \{\phi_t\}_{0 \leq t \leq 1}$ on (M, ω) the one-forms $\alpha_{\vec{H}_t}$ on L represent the same cohomology class, where $\vec{H}_t$ is the mean curvature vector of the immersion*

$$i_t := \phi_t \circ i_0 .$$

This result is an immediate consequence of the computations done in [O2] in the course of computing the time variation of $\alpha_{\vec{H}}$ under Hamiltonian deformations (see Remark 2.5 in Sect. 2 in the present paper). In Sect. 2, we will give a more illuminating geometric proof which also provides the frame work for the proof of Theorem II below. The most primitive case of this theorem is the case of closed curves on S^2, where Gauss-Bonnet formula implies the theorem: For any simple closed curve C on S^2 with the standard metric, let D be one of the components of $S^2 - C$. Then the Gauss-Bonnet formula tells us

$$\int_D K\, dA + \int_C \kappa\, ds = 2\pi$$

where $K \equiv 1$ and κ is the (signed) geodesic curvature of C. Therefore if we vary C in such a way that the area of D is not changed, $\int_C \kappa\, ds$ stay constant. In other words, the one forms $\kappa\, ds$ on S^1 define the same cohomology class in $H^1(S^1, \mathbb{R})$ if we consider the curves C as a map from S^1.

This theorem has an interesting consequence on symplectic geometry of Lagrangian submanifolds in Einstein-Kähler manifolds. It concerns *symplectic topology* of embedded *minimal* Lagrangian submanifolds as a special case. To describe this, we define two subgroups Γ_ω, $\Gamma_{\omega,L}$ of $\mathbb{R}$ by

$$\Gamma_\omega = \{[\omega](A) \mid A \in H_2(M, \mathbb{Z})\}$$

$$\Gamma_{\omega,L} = \{[\omega](B) \mid B \in H_2(M, L, \mathbb{Z})\}.$$

We call (M, ω) **prequantizable** if Γ_ω is either trivial or discrete. We call a Lagrangian submanifold L **cyclic** if $\Gamma_{\omega,L}$ is a discrete subgroup of $\mathbb{R}$. Note that Γ_ω and $\Gamma_{\omega,L}$ are countable subgroups of $\mathbb{R}$ and that Γ_ω is a subgroup of $\Gamma_{\omega,L}$ for any L (and so cyclic Lagrangian submanifolds exist only in prequantizable symplectic manifolds). We abuse the notations Γ_ω, $\Gamma_{\omega,L}$ also to denote their positive generators respectively when they are discrete groups.

Theorem II *Let (M, ω, J) be Einstein-Kähler with non-zero (constant) scalar curvature and suppose the one form $\alpha_{\vec{H}_i}$ associated to a given Lagrangian embedding $i: L \to (M, \omega)$ defined as in* (1.1) *is exact. Then L is cyclic and the following holds:*

(i) *When L is orientable, $n_L \mid \Gamma_{c_1}$.*

(ii) *When L is not orientable, $n_L | 2\Gamma_{c_1}$ where*

$$n_L = \#(\Gamma_{\omega,L}/\Gamma_\omega) = \Gamma_\omega/\Gamma_{\omega,L} \text{ and}$$

$$\Gamma_{c_1} = \text{the positive generator of the group } \{c_1(A) \mid A \in H_2(M, \mathbb{Z})\}$$

$$c_1 = \text{the first Chern class of } (M, J).$$

In particular, the above apply to compact minimal Lagrangian submanifolds.

The integer Γ_{c_1} is called the **index** of the Kähler manifold (M, ω, J) in algebraic geometry. For the case $M = \mathbb{C}P^n$ with the standard Kähler structure, we have $\Gamma_{c_1} = n+1$ and $n_{\mathbb{R}P^n} = 2$, $n_{T^n} = n+1$ where $\mathbb{R}P^n \subset \mathbb{C}P^n$ is the standard embedding of $\mathbb{R}P^n$, and T^n is the Clifford torus (see Proposition 3.7). Since $\mathbb{R}P^n$ is orientable if and only if n is odd, the above theorem is nicely illustrated by these minimal Lagrangian submanifolds in $\mathbb{C}P^n$.

We expect that these facts will have some important consequences both on the Hamiltonian volume minimization problem introduced in [O1, O2] by the author and on the study of mean curvature evolutions of Lagrangian submanifolds as well as on the classification of minimal Lagrangian submanifolds in $\mathbb{C}P^n$.

2 Canonical bundle and mean curvature vector

In this section, we will assume that (M, ω, J) is an Einstein-Kähler manifold unless otherwise stated. We first recall a theorem by Dazord [D].

Theorem 2.1 (Dazord) *Let i: $L \to (M, \omega)$ be a Lagrangian immersion on a Kähler manifold (M, ω, J) and ρ be the Ricci-form of (M, ω, J). Then the one form defined by*

$$\alpha_{\vec{H}_i} := (\vec{H}_i \lrcorner\, \omega)|_{TL}$$

satisfies the identity

$$d\alpha_{\vec{H}_i} = i^* \rho$$

where $\vec{H}_i$ is the mean curvature vector of the immersion i. In particular, $\alpha_{\vec{H}_i}$ is closed if (M, ω, J) is Einstein-Kähler i.e., if $\rho = c\,\omega$, c constant.

If we denote by K the canonical line bundle of a Kähler manifold (M, ω, J), i.e., the bundle of $(n, 0)$-forms $\Lambda^{n,0}(T^*_{\mathbb{C}} M)$, then the Ricci form ρ represents its curvature. Therefore on an Einstein-Kähler manifold, $i^* K$ is a flat bundle with respect to the induced connection from (M, ω, J) for any Lagrangian submanifold (M, ω). Indeed, this bundle is a topologically trivial bundle on any *orientable* Lagrangian submanifold L whose trivialization is given by the unique complex extension, denoted by Ω, as $(n, 0)$-form over L from the volume form on L with respect to the induced metric on L.

We now study the question when $i^* K$ is trivial as a flat bundle, i.e., when $i^* K$ carries a nowhere vanishing parallel section or equivalently when the holonomy of $i^* K$ vanishes. The following proposition relates the mean curvature vector and the holonomy of $i^* K$. A similar computation was implicitly carried out in [HL].

Proposition 2.2 *Let L be an oriented Lagrangian submanifold and Ω be the unique complex extension of the volume form on L. Then we have, for any $X \in T_x L$,*

$$\nabla_X \Omega = i\,\alpha_{\vec{H}_i}(X)\,\Omega, \qquad i = \sqrt{-1}$$

i.e.,

$$\nabla \Omega = i\,\alpha_{\vec{H}_i} \otimes \Omega$$

where ∇_X is the covariant derivative with respect to the induced connection on $i^* K$.

Proof. Since L is oriented, we choose a positively oriented orthonormal local frame $\{E_1, \ldots, E_n\}$ and the corresponding Darboux frame $\{E_1, \ldots, E_n, F_1, \ldots, F_n\}$ over L, i.e., $J E_i = F_i$. We may choose E_i's so that

$$\nabla_{E_i} E_j = 0 \quad \text{at} \quad x. \tag{2.1}$$

Let $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$ be its dual frame. Then Ω can be written with respect to this frame as

$$\Omega = (\alpha_1 + i\beta_1) \wedge \ldots \wedge (\alpha_n + i\beta_n).$$

Therefore, for any $X \in T_x L$,

$$\nabla_X \Omega = \sum_{j=1}^{n} (\alpha_1 + i\beta_1) \wedge \ldots \wedge \nabla_X(\alpha_j + i\beta_j) \wedge \ldots \wedge (\alpha_n + i\beta_n).$$

Here by the assumption (2.1), we have

$$\begin{aligned}
&\nabla_X \alpha_j(E_k) = 0 \\
&\nabla_X \alpha_j(F_k) = -\alpha_j(\bar{\nabla}_X F_k) = -\alpha_j(\bar{\nabla}_X (J E_k)) = -\alpha_j(J \bar{\nabla}_X E_k) \\
&\qquad = -\alpha_j(J B(X, E_k)) = \beta_j(B(X, E_k))
\end{aligned}$$

and therefore

$$\nabla_X \alpha_j = \sum_k \beta_j(B(X, E_k))\,\beta_k. \tag{2.2}$$

Similarly, we compute

$$\nabla_X \beta_j = -\sum_k \beta_j(B(X, E_k))\,\alpha_k. \tag{2.3}$$

Therefore, we have

$$\begin{aligned}
\nabla_X(\alpha_j + i\beta_j) &= \sum_k \beta_j(B(X, E_k))(\beta_k - i\alpha_k) \\
&= -i \sum_k \beta_j(B(X, E_k))(\alpha_k + i\beta_k)
\end{aligned}$$

and so

$$\begin{aligned}
\nabla_X \Omega &= -i \sum_{j=1}^{n} (\alpha_1 + i\beta_1) \wedge \ldots \wedge \Big(\sum_k \beta_j(B(X, E_k))(\alpha_k + i\beta_k)\Big) \wedge \ldots \wedge (\alpha_n + i\beta_n) \\
&= -i \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \delta_{jk}\, \beta_j(B(X, E_k)) \right) \Omega = -i \sum_{j=1}^{n} \beta_j(B(X, E_j))\,\Omega.
\end{aligned}$$

However,

$$\sum_{j=1}^{n} \beta_j(B(X, E_j)) = \sum_{j=1}^{n} \langle B(X, E_j), F_j \rangle$$

$$= \sum_{j=1}^{n} \langle B(X, E_j), J E_j \rangle = \sum_{j=1}^{n} \langle B(E_j, E_j), J X \rangle$$

$$= \langle \tilde{H}, J X \rangle = -\langle J \vec{H}, X \rangle = -\alpha_{\vec{H}}(X).$$

In the third identity above, we used the symmetry property of the second fundamental form B, i.e.,

$$\langle B(X, Y), J Z \rangle = \langle B(X, Z), J Y \rangle$$

of Lagrangian submanifolds in Kähler manifolds (see e.g. [O1]). This finishes the proof. Q.E.D.

This gives the following immediate corollary.

Corollary 2.3 *The holonomy of the bundle $i^* K$ over a loop $\gamma \subset L$ with respect to the induced connection is given by*

$$\exp(i \int_{\gamma} \alpha_{\vec{H}_i}).$$

We now assume $\rho = c \cdot \omega$, $c \neq 0$ and denote by E the holomorphic line bundle over M whose curvature form is the indivisible positive integral class proportional to $[\omega]$. Then we can express K as

$$K = (E)^{\otimes k} \quad \text{or} \quad (E^*)^{\otimes k} \tag{2.4}$$

when M is simply connected and we can at least choose such line bundle E so that (2.4) holds in general (The k in (2.4) is called the **index** of M in algebraic geometry). We will always assume that we have made this choice in the rest of this paper. From the definition Γ_{c_1} in the introduction, it follows

$$k = \Gamma_{c_1}. \tag{2.5}$$

Since K is the prequantization line bundle with respect to the Ricci form ρ when $c \neq 0$, any holonomy preserving deformation of $i^* E$ preserves the holonomy of $i^* K$ from (2.4). In other words, any Hamiltonian isotopy of the immersion i: $L \to (M, \omega)$ preserves the holonomy of $i^* K$ (see for the discussion for Hamiltonian isotopy in Sect. 3).

Theorem 2.4 *Let i: $L \to (M, \omega)$ be a Lagrangian immersion and let $\phi = \{\phi_t\}_{0 \leq t \leq 1}$ be a Hamiltonian isotopy of (M, ω). Define $i_t = \phi_t \circ i$ and let $\alpha_{\vec{H}_t}$ be the one-form associated to the mean curvature vector $\vec{H}_t$ of the immersion*

$$i_t \colon L \to (M, \omega, J).$$

Then the one forms $\alpha_{\vec{H}_t}$ on L define the same cohomology class on L.

Proof. First note that for any given loop $\gamma \subset L$, the integrals $\int_\gamma \alpha_{\vec{H}_t}$ vary continuously with respect to t. On the other hand, the values

$$\exp(i \int_\gamma \alpha_{\vec{H}_t})$$

stay constant by Corollary 2.3. These imply that the integrals $\int_\gamma \alpha_{\vec{H}_t}$ stay constant for a given loop $\gamma \subset L$ with respect to t. Hence $\alpha_{\vec{H}_t}$ define the same cohomology class in $H^1(L, \mathbb{R})$. Q.E.D.

Remark 2.5 Theorem 2.4 is an immediate consequence of the time variation formula of $\alpha_{\vec{H}_t}$,

$$\frac{\partial \alpha_{\vec{H}_t}}{\partial t} = -\Delta_t \alpha_{V_t} + c\,\alpha_{V_t},$$

where Δ_t is the Hodge-Laplacian of the metric induced from the immersion $i_t: L \to (M, \omega, J)$ and

$$V_t = \frac{d}{dt}(\phi_t \circ i).$$

This formula is an immediate consequence of the Eqs. (16) and the one right after (25) in the author's paper [O2]. However the present proof is more geometric and shows the interplay between Riemannian and symplectic geometries of Lagrangian submanifolds in Einstein-Kähler manifolds in a clearer way.

3 Action integral and prequantization

We first begin with a general discussion on the action integral and prequantization on general symplectic manifolds (M, ω). When the symplectic form ω is exact, i.e. has the form

$$\omega = -d\beta$$

for some one form β, the action integral A along the loop γ is defined by the integral of β along γ. If γ bounds a surface S mapped into M, then by Stokes' theorem we have

$$A(\gamma) = -\int_S \omega. \tag{3.1}$$

The equation can be used to define the action integral A on loops on M up to an element of the *period group* (See [W] for more details).

Definition 3.1 The **period group**, denoted by $\Gamma_\omega = \Gamma(M, \omega) \subset \mathbb{R}$ is defined to be the image of $[\omega] \times H_2(M, \mathbb{Z})$ under the integration pairing $H^2(M, \mathbb{R}) \times H_2(M, \mathbb{Z}) \to \mathbb{R}$. In other words,

$$\Gamma_\omega = \{[\omega](A) \in \mathbb{R} \mid A \in H_2(M, \mathbb{Z})\}.$$

Remark 3.2 When M is simply connected, Γ_ω coincides with the spherical period group

$$\{[\omega](u) \in \mathbb{R} \mid u: S^2 \to M\}.$$

The period group is either trivial, a discrete subgroup or a countable dense subgroup of $\mathbb{R}$. When Γ_ω is discrete, we abuse the notation Γ_ω to be the positive generator of the group itself. Following Weinstein [W], when Γ_ω is trivial or discrete (such a symplectic manifold is called **prequantizable**). We can interpret the action integral as a holonomy of some principal G_ω-bundle where

$$G_\omega := \mathbb{R}/\Gamma_\omega$$

as follows: it is well-known (see e.g. [K]) that when a symplectic manifold (M, ω) is prequantizable, there is a principal G_ω-bundle $\pi: Q \to M$ with a connection form θ such that its curvature form $d\theta$ satisfies

$$d\theta = \pi^* \omega.$$

Then the action integral of a loop γ is equal to the holonomy of the connection θ around γ.

When (M, ω, J) is an Einstein-Kähler manifold, i.e.,

$$\rho = c \cdot \omega \tag{3.2}$$

where ρ is the Ricci-form of (M, ω, J), we may assume by multiplying a constant appropriately that $[\omega] \in H^2(M, \mathbb{R})$ defines an indivisable integral class in $H^2(M, \mathbb{Z})$ proportional to $[\omega]$. Then the above prequantization is topologically isomorphic to a principal S^1-bundle associated to a holomorphic line bundle E over M whose curvature becomes ω. If we denote K the canonical line bundle of (M, ω, J), i.e., the bundle $\Lambda^{n,0}(T^*_{\mathbb{C}} M)$, then we may assume as we remarked in Sect. 2 that

$$\begin{aligned} K &= E^{\otimes k} \quad \text{when} \quad c > 0 \\ K &= (E^*)^{\otimes k} \quad \text{when} \quad c < 0, \end{aligned} \tag{3.3}$$

where $k = \Gamma_{c_1}$.

Now we restrict the prequantization G_ω-bundle to a given Lagrangian submanifold. In this section, we assume that Lagrangian submanifolds are *embedded*. Given any Lagrangian submanifold $L \subset (M, \omega)$ the connection θ on the bundle $Q|_L$ is flat over L because

$$d\theta = \pi^* \omega$$

and L is Lagrangian. We now define

$$G_L(x) := \text{Image of } \{\pi_1(L, x) \to G_\omega\},$$

i.e., the holonomy group of $Q|_L$ and x. Since L is assumed to be connected, $G_L(x)$ are all isomorphic and so we denote the common group by G_L. (Compare this with [W] where G_L is meant to be the quotient of G_ω by the image of holonomy.) This group is either cyclic or a countable dense subgroup of G_ω.

Definition 3.3 Let (M, ω) be a prequantizable symplectic manifold. We call a Lagrangian submanifold $L \subset (M, \omega)$ **cyclic** if G_L is a discrete subgroup of G_ω or equivalently if the group

$$\Gamma_{\omega, L} := \{[\omega](B) \mid B \in H_2(M, L, \mathbb{Z})\}$$

is discrete.

Lemma 3.4 *For any two loops γ_0 and γ_1 with $\gamma_1 = \phi_1(\gamma_0)$ where $\phi = \{\phi_t\}_{0 \leqq t \leqq 1}$ is a Hamiltonian isotopy of (M, ω), then*

$$\int_c \omega = 0$$

where c is the 2-chain given by $c = \{\phi_t \circ \gamma_0\}_{0 \leqq t \leqq 1}$.

Proof. Parametrize γ_0 by the parameter $s \in [0, 1]$. Then

$$\begin{aligned}\int_c \omega &= \int_0^1 \int_0^1 \omega\left(\frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}\right) ds\, dt \\ &= \int_0^1 \int_0^1 \omega\left(X_{H_t}(\phi_t \circ \gamma_0), \frac{\partial}{\partial s}(\phi_t \circ \gamma_0)\right) ds\, dt \\ &= \int_0^1 \int_0^1 dH_t\left(\frac{\partial}{\partial s}(\phi_t \circ \gamma_0)\right) ds\, dt = \int_0^1 \int_0^1 \frac{\partial}{\partial s} H_t(\phi_t \circ \gamma_0)\, ds\, dt = 0.\end{aligned}$$

Here H_t denote the Hamiltonian functions generating the isotopy $\phi = \{\phi_t\}_{0 \leqq t \leqq 1}$. Q.E.D.

Corollary 3.5 *Let γ_0, γ_1 be as in Lemma 3.4. Then the action integrals satisfy*

$$A(\gamma_1) = A(\gamma_0)$$

as elements in G_ω i.e., the holonomies around γ_0 and γ_1 of the prequantization bundle are the same.

In the remaining section, we describe two basic examples of cyclic Lagrangian submanifolds that are indeed *minimal submanifolds* with respect to the associated metric on (M, ω, J).

Examples (1) Let (M, ω) be a Hermitian symmetric space of compact type with an invariant Kähler form and the invariant almost complex structure. Then it becomes an Einstein-Kähler manifold and so (M, ω) is a prequantizable symplectic manifold with respect to the Kähler form. Let $L = \mathrm{Fix}\, \tau$ for an anti-holomorphic involutive isometry. Then it is easy to check that L is a cyclic Lagrangian submanifold. In particular, the totally geodesic $\mathbb{R}P^n \subset \mathbb{C}P^n$ is cyclic and in this case $G_L \cong \mathbb{Z}_2$. Indeed, we have the following general proposition.

Proposition 3.6 *Let (M, ω) be a prequantizable symplectic manifold and $\tau: M \to M$ be an anti-symplectic involution, i.e., it satisfies*

$$\tau^* \omega = -\omega \quad \textit{and} \quad \tau^2 = \mathrm{id}.$$

Suppose that the fixed point set $L := \mathrm{Fix}\,\tau$ *is nonempty. Then* L *is a cyclic Lagrangian submanifold such that* G_L *is either trivial or* $\mathbb{Z}_2$.

Proof. It is easy to show that L is Lagrangian (see e.g., [O1] and we shall just show that it is cyclic. We ambiguously denote by Γ_ω the positive generator for the period group Γ_ω of P. Then we shall prove that for any ω: $(D^2, \partial D^2) \to (P, L)$, D^2 the unit disk, Γ_ω divides $2[\omega](w)$, which shows that G_L is either trivial or $\mathbb{Z}_2$. Since $w|_{\partial D^2} = \tau \circ w|_{\partial D^2}$, we can define the so-called **double** of w, $u: S^2 \to P$ by

$$u(z) = \begin{cases} w(z), & z \in D^2 \\ \tau \circ w(\bar{z}), & z \in \bar{D}^2 \end{cases}$$

where $S^2 = D^2 \cup \bar{D}^2$ and $\bar{D}^2$ is the unit disk with opposite orientation to D^2. Then we have

$$\int_{S^2} u^* \omega = \int_{D^2} w^* \omega + \int_{\bar{D}^2} (\tau \circ w)^* \omega$$

$$= \int_{D^2} w^* \omega - \int_{\bar{D}^2} w^* \omega = \int_{D^2} w^* \omega + \int_{D^2} w^* \omega = 2 \int_{D^2} w^* \omega$$

i.e.,

$$[\omega](u) = 2[\omega](w)$$

and so Γ_ω divides $2[\omega](w)$ as it does $[\omega](u)$ for any sphere u. Q.E.D.

(2) One more interesting example is the *Clifford torus* $T^n \subset \mathbb{C}P^n$: consider the isometric embedding

$$T^{n+1} := \underbrace{S^1(1/\sqrt{n+1}) \times \ldots \times S^1(1/\sqrt{n+1})}_{n+1 \text{ times}} \subset \mathbb{C}^{n+1}.$$

This embedding is Lagrangian in $\mathbb{C}^{n+1}$. Since the standard Hopf-action restricts to both the above torus and S^{2n+1}, we take the quotients of these. The torus $T^n := T^{n+1}/S^1 \subset \mathbb{C}P^n := S^{2n+1}/S^1$ is Lagrangian and we call this torus the **Clifford torus**. In this case, we have

Proposition 3.7 *The Clifford torus above is cyclic in* $\mathbb{C}P^n$ *with*

$$G_{T^n} \cong \mathbb{Z}_{(n+1)}.$$

Proof. Note that $\pi_1(T^n)$ is isomorphic to $\mathbb{Z}^n$ and its generators are given by the boundaries of the following maps

$$w_k: (D^2, \partial D^2) \to (\mathbb{C}P^n, T^n)$$

$$w_k(z) = [1, \underbrace{0, \ldots, z,}_{k\text{-th}} \ldots, 0], \qquad k = 1, \ldots, n,$$

in the homogeneous coordinates of $\mathbb{C}P^n$. It is then enough to prove

$$[\omega](\mathbb{C}P^1) = (n+1)[\omega](w_k), \qquad k = 1, \ldots, n$$

where $\mathbb{C}P^1$ is any complex line in $\mathbb{C}P^n$. It is obvious that $[\omega](w_k)$ are all the same for k. Therefore we have only to compare $[\omega](\mathbb{C}P^1)$ and $[\omega](w_1)$, which

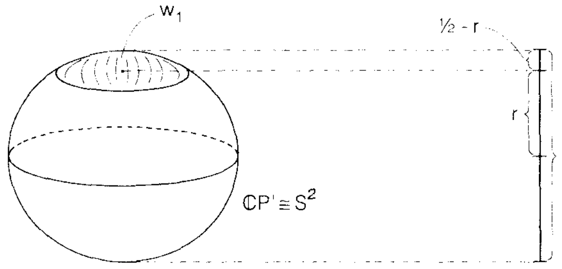

Fig. 1

are nothing but the areas of $\mathbb{C}P^1$ and w_1, respectively, since they are holomorphic. Here we choose for $\mathbb{C}P^1$ the complex line given by the equations

$$z_2 = z_3 = \ldots = z_n = 0$$

in the homogeneous coordinates $[z_0, z_1, \ldots, z_n]$ of $\mathbb{C}P^n$. Then w_1 is just a part of the complex line $\mathbb{C}P^1$, whose boundary ∂w_1 is a round circle in $\mathbb{C}P^1 \cap T^n$ (see Fig. 1).

In our description of the Clifford torus above, the $\mathbb{C}P^1$ with the induced metric is isometric to $S^2(\frac{1}{2})$ and so has area π. Now it is not too difficult to prove that the length of ∂w_1 is $\frac{2\pi\sqrt{n}}{n+1}$ (see [O1]) which implies that the boundary circle ∂w_1 has the radius $\frac{\sqrt{n}}{n+1}$, that in turn implies $r = \frac{n-1}{2(n+1)}$ and so $\frac{1}{2} - r = \frac{1}{n+1}$ in Fig. 1. Now by Archemedian law, it implies

$$\frac{[\omega](w_1)}{[\omega](\mathbb{C}P^1)} = \frac{\text{Area}(w_1)}{\text{Area}(\mathbb{C}P^1)} = \frac{1}{n+1}.$$

Hence the proof. Q.E.D.

4 The case when $\alpha_{\vec{H}}$ is exact

In this section, we study some symplectic topological properties of the embedded Lagrangian submanifolds whose associated form $\alpha_{\vec{H}}$ is exact. This case includes the case $\vec{H} \equiv 0$, i.e., the case of minimal Lagrangian submanifolds. In this section, we again assume $i: L \to (M, \omega)$ is a Lagrangian embedding and identify L with its image through the embedding. The main theorem we prove is the following.

Theorem 4.1 *Let (M, ω, J) be an Einstein-Kähler manifold with the constant $c \neq 0$, and let $L \subset (M, \omega, J)$ be a Lagrangian submanifold such that the one form $\alpha_{\vec{H}}$ is exact. Then L is cyclic and*

(i) *when L is orientable, $n_L \mid \Gamma_{c_1}$*
(ii) *when L is non-orientable, $n_L \mid 2\Gamma_{c_1}$*
where $n_L = \Gamma_\omega / \Gamma_{\omega, L}$.

Proof. Since the proof for the case $c<0$ is similar, we prove the theorem only for the case $c>0$. Denote $\Gamma_{c_1}=k$. Then

$$K=E^{\otimes k} \tag{4.1}$$

from (3.3). When L is orientable, we first prove that the bundle $K|_L$ is trivial as a flat bundle, i.e., there exists a nowhere vanishing *parallel* section of $K|_L$, provided $\alpha_{\vec{H}}$ is exact. Since $\alpha_{\vec{H}}$ is exact, there exists a function f such that

$$\alpha_{\vec{H}}=d f.$$

Using Proposition 2.2, it is immediate to prove that $e^{-if}\Omega$ is parallel. Indeed,

$$\begin{aligned}\nabla (e^{-if}\Omega) &= i\, d f\, e^{-if}\Omega + e^{-if}\nabla \Omega\\ &= -i e^{-if}(d f-\alpha_{\vec{H}})\cdot \Omega=0.\end{aligned}$$

Hence $K|_L$ is trivial and so its holonomy is trivial. Therefore from (4.1), we have

$$a^k=\mathrm{id},$$

for any element $a\in G_L$. Since G_L is an abelian subgroup of $G_\omega=\mathbb{R}/\Gamma_\omega$, this immediately implies that G_L is finite, i.e. cyclic and its order $n_L=\#(G_L)=\Gamma_\omega/\Gamma_{\omega,L}$ divides $k=\Gamma_{c_1}$.

When L is not orientable, K does not have to be trivial in general but $K^{\otimes 2}$ is so. In fact, for any given orthonormal local frame $\{E_1, \ldots, E_n\}$, its associated Darboux frame $\{E_1, \ldots, E_n, F_1, \ldots, F_n\}$ and its dual frame $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$, the expression

$$\{(\alpha_1+\gamma\beta_1)\wedge \ldots \wedge(\alpha_n+i\beta_n)\}\otimes\{(\alpha_1+\gamma\beta_1)\wedge \ldots \wedge(\alpha_n+i\beta_n)\}$$

does not depend on the choice of $\{E_1, \ldots, E_n\}$ and defines a global section of $K^{\otimes 2}$. The same computations we carried out in the proof of Proposition 2.2 proves that this is indeed parallel. Hence we have proved that $K^{\otimes 2}$ is a trivial flat bundle, which implies (ii) in the same way as in (i). Q.E.D.

Since any minimal Lagrangian submanifold has vanishing one form $\alpha_{\vec{H}}$, in particular, exact $\alpha_{\vec{H}}$, this theorem has the following corollary.

Corollary 4.2 *Let (M, ω, J) be on Einstein-Kähler manifold with $c\neq 0$, i.e. not Ricci-flat and L be a minimal Lagrangian submanifold. Then the same conclusion as in Theorem* 4.1 *holds.*

This leaves an interesting question when (M, ω, J) is Ricci-flat i.e. $c=0$ (e.g., when (M, ω, J) is a K_3-surface).

Question. What can we say about compact minimal Lagrangian submanifolds in Ricci-flat Kähler manifolds (if they exist at all)? For example, are they cyclic? Note that in this case, we lose the relationship between the prequantization of (M, ω, J) and the canonical bundle of (M, J), which has been the crucial ingredient to prove Theorem 4.2.

We have already shown two examples of minimal Lagrangian submanifolds in $\mathbb{C}P^n$, one $\mathbb{R}P^n$ and the other T^n. When $n=2$, we have

$$\Gamma_{c_1}=3, \qquad n_{\mathbb{R}P^2}=2, \qquad n_{T^2}=3$$

and $\mathbb{R}P^2$ is not orientable. Therefore these two examples seem to exhaust possible n_L dividing Γ_{c_1} or $2\Gamma_{c_1}$. This leads us to the following conjecture.

Conjecture. *$\mathbb{R}P^2$ and T^2 in $\mathbb{C}P^2$ are the only compact minimal Lagrangian submanifolds.*

We may even ask the same question for $\mathbb{C}P^n$ for general n. More generally, the following classification seems to be an interesting question to study since the system of minimal and Lagrangian equations is overdetermined and since our theorem provides some a priori informations on their symplectic topological behavior.

Problem. Try to classify all possible compact minimal Lagrangian submanifolds in compact Hermitian symmetric spaces. Is there any group theoretical mechanism to construct minimal Lagrangian submanifolds?

Acknowledgement. The idea of this work was evolved while the author was a member of the Institute for Advanced Study and was visiting IHES. We would like to thank both institutions for their financial support and hospitalities. This research is supported in part by NSF Grant DMS-9296078.

References

[B] Bryant, R.L.: Minimal Lagrangian submanifolds of Einstein-Kähler manifolds. In: Chaohao, G. et al. (eds.) Differential Geometry and Differential Equations. (Lect. Notes Math., vol. 1255, pp. 1–12) Berlin Heidelberg New York: Springer 1987

[C] Chen, B.-Y.: Geometry of Submanifolds and its Applications. Science University of Tokyo (1981)

[D] Dazord, P.: Sur la géometrie des sous-fibrés et des feuilletages lagrangiense. Ann. Sci. Éc. Norm. Super., IV. Sér. **13**, 465–480 (1981)

[HL] Harvey, R., Lawson, B.: Calibrated geometries. Acta Math. **148**, 47–157 (1982)

[K] Kostant, B.: Quantization and unitary representations. In: Taam, C.T. (ed.) Lectures in modern analysis and applications III. (Lect. Notes Math., vol. 170, pp. 187–208) Berlin Heidelberg New York: Springer 1970

[M] Morvan, J.-M.: Classe de Maslov d'une immersion lagrangienne et minimalité. C.R. Acad. Sci. **292**, 633–636 (1981)

[O1] Oh, Y.-G.: Second variation and stabilities of minimal Lagrangian submanifolds. Invent. Math. **101**, 501–519 (1990)

[O2] Oh, Y.-G.: Volume minimization of Lagrangian submanifolds under Hamiltonian deformations. Math. Z. **212**, 175–192 (1993)

[T] Takeuchi, M.: Stability of certain minimal submanifolds of compact Hermitian symmetric spaces. Tohoku Math. J. **36**, 293–314 (1984)

[W] Weinstein, A.: Connections of Berry and Hannay type for moving Lagrangian submanifolds. Adv. Math. **82**, 133–159 (1990)