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A CHARACTERIZATION OF A FREE ELEMENTARY INVERSE SEMIGROUP

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The purpose of this note is to give a characterization of a free inverse semigroup on a singleton set. It will be shown that if B is the bicyclic semigroup, then a certain inverse subsemigroup F of B x B is a free elementary inverse semigroup.

1. PRELIMINARIES. Let S be any inverse semigroup and let $x \in S$. Then x^0 will mean $1 \in S^1$. If $n \in N = \{0, 1, 2, \ldots\}$, then x^{-n} will mean $(x^{-1})^n$. This should cause no confusion since $(x^{-1})^n = (x^n)^{-1}$ [3]. The following two lemmas are proved easily by induction, or may be deduced from [3]. The corollaries follow from the lemmas by replacing x with x^{-1} and then taking inverses.

LEMMA 1.1. *If* $0 \leq b \leq s$, *then* $x^s x^{-b} x^b = x^s$.

COROLLARY 1.2. *If* $0 \leq r \leq a$, *then* $x^r x^{-r} x^a = x^a$

LEMMA 1.3. *If* $0 \leq a \leq r$, *then* $x^r x^{-r} x^a = x^r x^{-(r-a)}$.

COROLLARY 1.4. *If* $0 \leq s \leq b$, *then* $x^s x^{-b} x^b = x^{-(b-s)} x^b$.

The bicyclic semigroup B may be characterized as N x N with multiplication given by

$(p,r)(a,c) = (p + a - \min(r,a), r + c - \min(r,a))$ [1].

Then B is a bisimple inverse semigroup and so B x B, with componentwise multiplication, is also a bisimple inverse semigroup. Let

$F = \{((p,r),(q,s)) \in B \times B : p + q = r + s > 0\}$. Then F is an inverse subsemigroup of B x B.

2. THE FREE INVERSE SEMIGROUP ON A SINGLETON SET. If X is a non-empty set, a free inverse semigroup on X is a pair (I,f) such that (i) I is an inverse semigroup, (ii) $f: X \to I$, and (iii) if S is any inverse semigroup and $g: X \to S$, then there exists a unique homomorphism $h: I \to S$ such that $fh = g$. It has been shown in [4] and again in [2] that a free inverse semigroup on any non-empty set always exists.

Let F be the inverse semigroup described above and let $f:\{u\} \to F$ by $uf = ((1,0),(0,1))$.

THEOREM 2.1. (F,f) *is a free inverse semigroup on* $\{u\}$.

PROOF. Let S be any inverse semigroup and let $g:\{u\} \to S$, say $ug = y$. Let $h:F \to S$ by $((p,r),(q,s))h = y^{-q}y^{p+q}y^{-r}$.

To see that h is a homomorphism, let $\alpha = ((p,r),(q,s))$, and $\beta = ((a,c),(b,d)) \in F$. Then

(i) $(\alpha\beta)h = y^{-(q+b-\min(s,b))}y^{p+a-\min(r,a)+q+b-\min(s,b)}y^{-(r+c-\min(r,a))}$, and

(ii) $(\alpha h)(\beta h) = y^{-q}y^{p+q}y^{-r}y^{-b}y^{a+b}y^{-c} = y^{-q}y^{s}\left(y^{r}y^{-r}\right)\left(y^{-b}y^{b}\right)y^{a}y^{-c} = y^{-q}\left(y^{s}y^{-b}y^{b}\right)\left(y^{r}y^{-r}y^{a}\right)y^{-c}$.

1. Assume $b \le s$ and $a \le r$. Then, by Lemmas 1.1 and 1.3, $(\alpha h)(\beta h) = y^{-q}y^{s}y^{r}y^{-(r-a)}y^{-c} = y^{-q}y^{p+q}y^{-(r+c-a)} = (\alpha\beta)h$.

2. Assume $b \le s$ and $r \le a$. Then, by Lemma 1.1 and Corollary 1.2, $(\alpha h)(\beta h) = y^{-q}y^{s}y^{a}y^{-c} = y^{-q}y^{p+q-r+a}y^{-c} = (\alpha\beta)h$.

3. Assume $s \le b$ and $a \le r$. Then, by Corollary 1.4 and Lemma 1.3, $(\alpha h)(\beta h) = y^{-q}y^{-(b-s)}y^{b}y^{r}y^{-(r-a)}y^{-c} = y^{-(q+b-s)}y^{b+p+q-s}y^{-(r+c-a)} = (\alpha\beta)h$.

4. Assume $s \le b$ and $r \le a$. Then, by Corollaries 1.4 and 1.2, $(\alpha h)(\beta h) = y^{-q}y^{-(b-s)}y^{b}y^{a}y^{-c} = y^{-(q+b-s)}y^{p+a-r+q+b-s}y^{-c} = (\alpha\beta)h$.

Further, $u(fh) = ((1,0),(0,1))h = y^{-0}y^{1+0}y^{-0} = y = ug$.

Finally, to show that h is unique, it is sufficient to show that $((1,0),(0,1))$ generates F. But $((p,r),(q,s)) = ((1,0),(0,1))^{-q}((1,0)(0,1))^{p+q}((1,0),(0.1))^{-r}$ and so the proof is complete.

REMARK. Let $N = \{0, 1, 2, . . .\}$ and consider max as a multiplication on N. Then N is a semilattice and N is uniform ($Na \cong Nb$ for all $a,b \in N$).

Let $E = N \times N$. If $(p,q),(r,s) \in E$, an isomorphism $\alpha = [(p,q);(r,s)]$ of $E(p,q) = Np \times Nq$ onto $E(r,s)$ is induced by isomorphisms of Np onto Nr and of Nq onto Ns. This isomorphism is given by
$(x,y)[(p,q);(r,s)] = (x + r - p, y + s - q)$. Thus E is uniform and so the semigroup $T_E = T$ of all principal ideal isomorphisms is inverse and bisimple [5]. If
$\alpha = [(p,q);(r,s)]$ and $\beta = [(a,b);(c,d)] \in T$, then $\Delta(\alpha\beta)$, the domain of $\alpha\beta$, is $(E(r,s) \cap E(a,b))[(r,s);(p,q)]) =$
$E(\max(r,a) + p - r, \max(s,b) + q - s)$. Thus
$[(p,q);(r,s)][(a,b);(c,d)] =$
$[(\max(r,a) + p - r, \max(s,b) + q - s);$
$(\max(r,a) + c - a, \max(s,b) + d - b)]$.

The author's original characterization of a free elementary inverse semigroup was given in terms of
$F' = \{[(p,q);(r,s)] \in T:p + q = r + s > 0\}$, an inverse subsemigroup of T. Indeed;$\theta:((p,r),(q,s)) \to [(p,q);(r,s)]$ is an isomorphism of $B \times B$ onto T. This isomorphism was pointed out to the author by Professor Mario Petrich.

It is not difficult to characterize Green's relations on F and $\Lambda(F)$, the lattice of congruences on F. These results are not included here, however, since they are special cases of results to appear in [2].

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