Math. Z. 217, 577–606 (1994)

Mathematische Zeitschrift

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Applications of symplectic homology I

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Received 5 March 1993; in final form 11 August 1993

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1 Introduction

1.1 Some general remarks

The purpose of this paper is to illustrate the symplectic homology construction in [7] via some applications concerned with ellipsoids and symplectic polydisks.Moreover we show how a symplectic capacity, see[19, 3, 4, 16, 15, 14, 22],can be constructed from the symplectic homology theory.A few results have been announced in [16]. We assume the reader to be familiar with the results and notation in [7]. For basic symplectic geometry we refer to [25].

* Andreas Floer died on May 15th, 1991

** Supported in parts by DFG–SFB 237 DAAD–Procope

1.2 Symplectic homology of ellipsoids

Although ellipsoids and polydisks are the most simple shapes in symplectic geometry we shall see that there are still some interesting open questions. Let us define a set $\Gamma \subset (0, +\infty)^n$ by

$$\Gamma = \{r \in (0, +\infty)^n \mid r = (r_1, \dots, r_n),\ r_1 \leq r_2 \leq \dots \leq r_n\} \ .$$

We denote by $\Gamma_{reg} \subset \Gamma$ the subset of those r such that the numbers $r_1^2, \dots, r_n^2$ are linearely independent over the integers. An element of $r \in \Gamma_{\rm reg}$ will be called a regular element of Γ. Given $r \in \Gamma$ we denote by $E(r)$ the open ellipsoid defined by

$$E(r) = \{z \in \mathbb{C}^n \mid \sum \left|\frac{z_k}{r_k}\right|^2 < 1\} \ .$$

Generally an ellipsoid in $\mathbb{C}^n$ is a set of the form

$$E = q^{-1}((-\infty, 1)) \ ,$$

where q is a positive definite (reel) quadratic form. The linear symplectic group Sp acts on the collection of all ellipsoids, say $\mathscr{E}$ via

$$Sp \times \mathscr{E} \longrightarrow \mathscr{E} : (\Psi, E) \longrightarrow \Psi(E) \ .$$

The quotient space $\hat{\mathscr{E}}$ of $\mathscr{E}$ by the Sp–action can be identified with Γ via the map

$$\Gamma \longrightarrow \hat{\mathscr{E}} : r \longrightarrow [E(r)] \ .$$

This is an exercise in linear symplectic algebra, see [17]. Hence an ellipsoid $E \in \mathscr{E}$ has a natural symplectic numerical invariant $r(E) \in \Gamma$.

The map $\mathscr{E} \longrightarrow \Gamma : E \longrightarrow r(E)$ is a "fibration" with the fibre over a given $r \in \Gamma$ consisting of all linearlely equivalent ellipsoids. Ellipsoids are very good objects to demonstrate the difference between linear and nonlinear symplectic geometry.

The following result is well–known, see [17] for example.

Proposition 1 *For two ellipsoids $E, F \in \mathscr{E}$ the following statements are equivalent*

(i) There exists $\Psi \in Sp$ with $\Psi(E) \subset F$
(ii) $r(E) \leq r(F)$.

Here $a \leq b$ precisely means $a_i \leq b_i$ for $i = 1, \dots, n$. As a corollary we have

Corollary 1 *$r \leq r'$ iff $\Psi(E(r)) \subset E(r')$ for some $\Psi \in Sp$.*

This motivates immediately a nonlinear generalization. Let us denote by Diff(ω) the group of smooth symplectic diffeomorphisms on $\mathbb{C}^n$. We put $\leq_1 := \leq$ and define a second order relation by

Definition 1 *We define an order relation $\leq_2$ on Γ by*

$$r \leq_2 r' :\Longleftrightarrow \quad \begin{array}{l} \textit{There exists } \Psi \in \mathrm{Diff}(\omega) \textit{ with} \\ \Psi(E(r)) \subset E(r') \ . \end{array}$$

It is not difficult to show that

Lemma 1 *If $r \leq_2 r'$ and $r' \leq_2 r$ then $r = r'$.*

Obviously the map $(\Gamma, \leq_1) \longrightarrow (\Gamma, \leq_2) : r \longrightarrow r$ is order preserving. However the inverse does not preserve the order. To see this we take a symplectic embedding of $B^2(2)$ into $E(1, \ldots, 1)$, (for the flexibility of symplectic embeddings with positive codimension see [11, 12]). Then the image has a symplectic tubular neighbourhood, i. e. there exists a symplectic embedding $B^{2n-2}(\epsilon) \times B^2(2) \longrightarrow E(1, \ldots, 1)$ for small $\epsilon > 0$. In particular we have a symplectic embedding $\tilde{\Psi}$ of $E(\epsilon, \ldots, \epsilon, 2)$ into $E(1, \ldots, 1)$. Given any $\epsilon' \in (0, \epsilon)$ and $\tau \in (0, 2)$ there exists $\Psi_{\epsilon', \tau} \in \mathrm{Diff}(\omega)$ such that $\Psi_{\epsilon', \tau} \mid E(\epsilon', \ldots, \epsilon', \tau) = \Psi \mid E(\epsilon', \ldots, \epsilon', \tau)$. (That is the well–known extension after restriction principle for symplectic maps on simple shapes, see [3]). Hence we have proved $(\epsilon', \ldots, \epsilon', \tau) \leq_2 (1, \ldots, 1)$ for every $\epsilon' \in (0, \epsilon)$, $\tau \in (0, 2)$. In particular

Proposition 2 *For all sufficiently small $\epsilon > 0$ we have*

$$\left(\epsilon, \ldots, \epsilon, \frac{3}{2}\right) \leq_2 (1, \ldots, 1) \ .$$

Proposition 2 shows that a considerable amount of symplectic rigidity is lost if we pass from the linear to to the nonlinear theory. But still we know that there is a considerable amount of rigidity left (Gromov width, capacities). To compare $\leq_1$ and $\leq_2$ should give a better idea about rigidity and flexibility in symplectic geometry.

The symplectic homology theory turns out to be a useful tool in comparing $\leq_1$ and $\leq_2$. In contrast to proposition 2 we have in $\mathbb{C}^2$:

Theorem 1 *For $n = 2$, i. e. ellipsoids in $\mathbb{C}^2$, let $\sigma \in \left(\frac{1}{\sqrt{2}}, 1\right)$. The following statements are equivalent.*

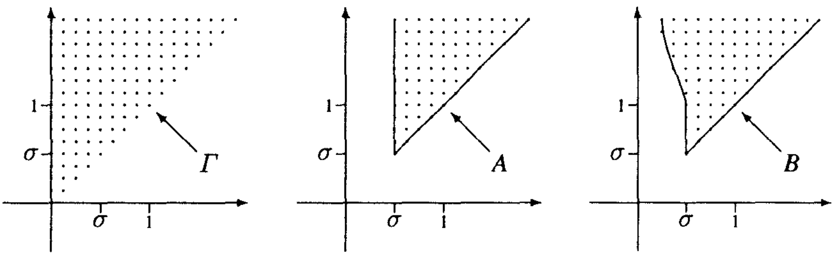
(i) $(\sigma, \sigma) \leq_1 r \leq_1 r' \leq_1 (1, 1)$
(ii) $(\sigma, \sigma) \leq_2 r \leq_2 r' \leq_2 (1, 1)$

In other words; the order intervals for $i = 1, 2$

$$[(\sigma, \sigma), (1, 1)]_i = \{r \mid (\sigma, \sigma) \leq_i r \leq_i (1, 1)\}$$

are identical and the orderings $\leq_1$ and $\leq_2$ on them coincide.

The following figure shows the sets $A = \{r \mid (\sigma, \sigma) \leq_1 r\}$ and $B = \{r \mid (\sigma, \sigma) \leq_2 r\}$.



The proof of Theorem 1 is quite difficult and involves results by McDuff, [20], and the symplectic homology theory from [7].

The obvious question is why do we need the condition $\sigma \in \left(\frac{1}{\sqrt{2}}, 1\right)$? To a certain extend it is possible to understand this constraint geometrically. In passing from the linear symplectic theory to the nonlinear theory the new flexibility is the possibility of "folding" a set. Symplectic capacities measure to some extend the area of two dimensional symplectic projections of a set. If one folds a set some projection might double in size. If for example $r = (\sigma, \sigma)$ then the smallest projection is of area $\pi\sigma^2 > \frac{\pi}{2}$. If one "folds" some projection could have area around $2\pi\sigma^2 > \pi$. The folded set would not fit into the ball $B^4(1)$. So to some extend the pinching condition in Theorem 1 excludes folding. If folding is excluded the linear and nonlinear theory coincide. This is of course extremely heuristic, but we believe it points in the right direction. Here is a "test problem".

Problem 1 Given any $\sigma \in \left(0, \frac{1}{\sqrt{2}}\right)$ does there exist a r_σ such that

$$\begin{aligned} (\sigma, \sigma) &\leq_1 \; r_\sigma \, , \; r_\sigma \not\leq_1 (1,1) \\ r_\sigma &\leq_2 \; (1,1) \; ? \end{aligned}$$

In other words: is it true that Theorem 1 is sharp?

Next we turn to the main topic of this section, namely the computation of the symplectic homology of an ellipsoid. Since symplectic homology is an invariant under symplectic diffeomorphism in $\mathscr{D}$, see [7], it suffices to compute it for the normal form $E(r)$. We shall work for simplicity only with $\mathbb{Z}_2$–coefficients. Given $r \in \Gamma$ we denote by $\sigma(r)$ the subset of $(0, +\infty)$ defined by

$$\sigma(r) = \left\{ k\pi r_j^2 \mid k \in \mathbb{N}^* \, , \, j = 1, \ldots, n \right\} \; .$$

Here $\mathbb{N}^* = \{1, 2, 3, \ldots\}$. We have a map

$$\phi : \mathbb{N}^* \times \{1, \ldots, n\} \longrightarrow \sigma(r) : (k, j) \longrightarrow \pi k r_j^2 \; .$$

Given $d \in \sigma(r)$ we define $m(d) = \#\phi^{-1}(r)$ and call it the multiplicity of d. Consider the sequence (d_l) , $d_l = d_l(r)$, $l \in \mathbb{N}^*$

$$(d_0 =) 0 < d_1 \leq d_2 \leq d_3 \ldots$$

consisting of all elements in $\sigma(r)$, written in increasing order and each element repeated according to its multiplicity.

We need the notation of a $\mathbb{R}$–filtered chain complex. For every number $\mu \in (-\infty, +\infty]$ suppose a subcomplex (C^μ, ∂^μ) of (C, ∂) is given with $C^{+\infty} = C$. We define

$$C^{[a,b)} = C^b / C^a$$

with the induced boundary operator $\partial^{[a,b)}$. For given $r \in \Gamma$ let $(d_j)_{j \in \mathbb{N}}, \mathbb{N} = \{0, 1, \ldots\}$ be the associated sequence. We define

$$\begin{aligned} C^\mu(r) &= 0 \quad \text{for} \quad \mu \le 0 \\ C^\mu(r) &= (\mathbb{Z}_2, n) \quad \text{for} \quad 0 < \mu \le d_1 \\ C^\mu(r) &= (\mathbb{Z}_2, n) \oplus \ldots \oplus (\mathbb{Z}_2, n + 2m(\mu; r)) \quad \text{for} \quad d_1 \le \mu < +\infty \\ C^{+\infty}(r) &= \oplus_{l=0}^\infty (\mathbb{Z}_2, n + l) \ , \end{aligned}$$

where $m(\mu, r) = \sup\{l \mid d_l < \mu\}$. Next we define the boundary map as follows. If $\mu \le d_1$ it is the zero map, if $d_1 < \mu < +\infty$ it is given by the following diagram

$$\ldots \leftarrow 0 \leftarrow (\mathbb{Z}_2, n) \xleftarrow{\text{Id}} (\mathbb{Z}_2, n+1) \quad \xleftarrow{0} (\mathbb{Z}_2, n+2) \xleftarrow{\text{Id}} \ldots \xleftarrow{\text{Id}} (\mathbb{Z}_2, n+2m-1) \xleftarrow{0} (\mathbb{Z}_2, n+2m) \\ \leftarrow 0 \leftarrow \ldots \tag{1}$$

with $m = m(\mu, r)$ and $C^{+\infty}$ is the direct limit complex. We have the following result

Theorem 2 *Let $r \in \Gamma$ and $-\infty < a \le b \le +\infty$. Then*

$$S^{[a,b)}(E(r)) = H_* \left(C^{[a,b)}(r) \ , \ \partial^{[a,b)}(r) \right) \ .$$

As an example let $r = (1, \ldots, 1)$ so that $E(r) = B^{2n}(1)$ is the open unit ball around zero in $\mathbb{C}^n$. The sequence (d_l) is given by

$$0, \ \underbrace{\pi, \ldots, \pi}_{n\text{–times}}, \ \underbrace{2\pi, \ldots, 2\pi}_{n\text{–times}}, \ \underbrace{3\pi, \ldots, 3\pi}_{n\text{–times}}, \ 4\pi, \ldots$$

To compute the symplectic homology of $B^{2n}(1)$ consider the following diagram

$$\ldots \leftarrow 0 \leftarrow (\mathbb{Z}_2, n; 0) \xleftarrow{\text{Id}} (\mathbb{Z}_2, n+1; \pi) \leftarrow \ldots \xleftarrow{0} (\mathbb{Z}_2, 3n, \pi) \\ \xleftarrow{\text{Id}} (\mathbb{Z}_2, 3n+1, 2\pi) \leftarrow \ldots$$

Here $(\mathbb{Z}_2, k; d)$ has the following meaning: $k \in \mathbb{Z}$ is the grading dimension, and with $C = (\mathbb{Z}_2, k; d)$ we have $C^\mu = 0$ for $\mu \le d$, $C^\mu = (\mathbb{Z}_2, k)$ for $\mu > d$. For example let $\mu = 2\pi$. Then the $C^{2\pi}$–part of the above filtered complex is

$$\begin{aligned} &\ldots \leftarrow 0 \leftarrow (\mathbb{Z}_2, n) \xleftarrow{\text{Id}} (\mathbb{Z}_2, n+1) \xleftarrow{0} (\mathbb{Z}_2, n+2) \\ &\ldots \xleftarrow{0} (\mathbb{Z}_2, 3n) \leftarrow 0 \leftarrow \ldots \end{aligned} \tag{2}$$

since only the groups $(\mathbb{Z}_2, n, 0), \ldots, (\mathbb{Z}_2, 3n, \pi)$ live strictly below level 2π. The groups $(\mathbb{Z}_2, 3n+1, 2\pi)$ and up are replaced by zero in order to obtain $C^{2\pi}$. In order to continue our example we compute $C^{[0,2\pi)}(B^{2n}(1))$. The relevant complex is (2) since $C^0 = 0$. Hence

$$C^{[0,2\pi)}(B^{2n}(1)) = (\mathbb{Z}_2, 3n) \ .$$

More generally we compute from Theorem 2

Corollary 2 *Let $-\infty < a \leq b \leq +\infty$. For $a \leq 0$ and $k\pi < b \leq (k+1)\pi$, $k \in \mathbb{N}$, we have*

$$S^{[a,b)}(B^{2n}(1)) = (\mathbb{Z}_2, n + 2kn) \ .$$

For $0 \leq j\pi < a \leq (j+1)\pi \leq k\pi < b \leq (k+1)\pi$ with $j,k \in \mathbb{N}$ we have

$$S^{[a,b)}(B^{2n}(1)) = (\mathbb{Z}_2, n + 2jn + 1) \oplus (\mathbb{Z}_2, n + 2kn) \ .$$

Further for $0 \leq j\pi < a \leq (j+1)\pi$ and $b = +\infty$

$$S^{[a,+\infty)}(B^{2n}(1)) = (\mathbb{Z}_2, n + 2jn + 1) \ .$$

In all other cases it is zero.

1.3 Symplectic Homology of Polydisks

For $r \in \Gamma$ denote by $D^{2n}(r)$ the open polydisk

$$D^{2n}(r) = B^2(r_1) \times \ldots \times B^2(r_n) \ .$$

Our aim is to compute its symplectic homology. From the construction in [7] it is clear that the symplectic homology has certain product properties. This combined with the knowledge of the symplectic homology of a disk suffices to compute it for polydisks. Before we give a formula we give two applications. The first is Gromov's polydisk conjecture, [11].

Theorem 3 *Let $r, r' \in \Gamma$ and assume the open polydisks are symplectomorphic. Then $r = r'$.*

Another application is concerned with the topology of the space of symplectic embeddings of $D^4(r)$ into $D^4(1,1)$.

The study of such symplectic embedding spaces is in general very difficult. In particular McDuff studied in [20] symplectic embeddings of $B^4(\delta)$ into $B^4(1)$ for $\delta \in (0,1)$ and showed that they are all symplectically isotopic. This result is not known if $n \geq 3$.

However we will be able to show that polydisks behave quite differently.

Theorem 4 *Let $n = 2$ and $r \in \Gamma$ and assume $r_1^2 + r_2^2 > 1$ and $r_1, r_2 < 1$. Then there are at least two different symplectic isotopy classes of symplectic embeddings into $D^4(1,1)$.*

In order to derive the statement for the symplectic homology of the polydisk recall that the diagram for a 2–disk of radius r is

$$\ldots \leftarrow 0 \leftarrow 0 \leftarrow (\mathbb{Z}_2, 1; 0) \stackrel{\mathrm{Id}}{\leftarrow} (\mathbb{Z}_2, 2; \pi r^2) \stackrel{0}{\leftarrow} (\mathbb{Z}_2, 3; \pi r^2) \leftarrow \ . \tag{3}$$

Again the numbers $0, \pi r^2$ etc. mean that the groups $(\mathbb{Z}_2, 1), (\mathbb{Z}_2, 2)$ etc. live above a certain level.

The relevant diagram for a polydisk is then the "tensor product" of the diagram as in (3) with different radii. The rule for the tensor product is

$$(G, k, b) \otimes (H, k', b') = (G \otimes H, k + k', b + b') \ .$$

Let us denote for $r \in (0, +\infty)$ by $(C^{\mu}(r), \partial^{\mu})$ the complex for the r–disk in $\mathbb{C}$. Define

$$\tilde{C}^{\infty}(r_1, \ldots, r_n) = C^{\infty}(r_1) \otimes \ldots \otimes C^{\infty}(r_n)$$

with obvious boundary operator, which is the usual tensor product of chain complexes. For $l \in (-\infty, +\infty)$ we define

$$\tilde{C}^{l}(r_1, \ldots, r_n) = \bigcup_{l = \sum l_j} C^{l_1}(r_1) \otimes \ldots \otimes C^{l_n}(r_n) \ ,$$

which is a subcomplex of $\tilde{C}^{\infty}(r_1, \ldots, r_n)$.

Finally we put $\tilde{C}^{[\mu,\nu)}(r) = \tilde{C}^{\nu}(r) / \tilde{C}^{\mu}(r)$ for $r \in \Gamma$. Then

Theorem 5 *For $-\infty < a \leq b \leq +\infty$ we have*

$$S^{[a,b)}(D^{2n}(r)) = H_* \left(\tilde{C}^{[a,b)}(r) \, , \, \partial^{[a,b)} \right) \ .$$

This is already quite complicated to calculate and we restrict ourselves to the computations of particular cases in order to prove Theorems 3 and 4.

1.4 Symplectic capacities

One of the very fruitful notions in symplectic geometry is that of a symplectic capacity. This is a map which associates to a symplectic $2n$–dimensional manifold a number $c(M, \omega) \in (0, +\infty]$ (assuming $M \neq \emptyset$) satisfying the following axioms

- $c(M, \alpha\omega) = |\alpha| \ c(M, \omega)$ for $\alpha \neq 0$
- If there exists a symplectic embedding of (M, ω) into (N, τ) then $c(M) \leq c(N)$ (4)
- $c(B^2(1) \times \mathbb{C}^{n-1}$, standard$) < +\infty$

The success of the variational theory of Hamiltonian systems was very important for the development of the symplectic capacity theory, see [21, 26, 23, 18, 2]. Many constructions of symplectic capacities are known [19, 3, 4, 22, 24, 11, 17]. It is also well–known that the existence of a single capacity implies for example the C^0–rigidity phenomenon detected by Eliashberg, see [5, 11, 12].

Here we show that one can construct a symplectic capacity for open sets in $\mathbb{C}^n$ from the symplectic homology theory. For $0 < \epsilon_1 \leq \epsilon_2 < +\infty$ and $b \geq \epsilon_2$ we have the natural maps

$$S_{n+1}^{[\epsilon_1,b)}(U) \longrightarrow S_{n+1}^{[\epsilon_2,b)}(U) \ .$$

We obtain an inverse system and define

$$S_{n+1}^{(0,b)}(U) := \lim_{\leftarrow} \left(S_{n+1}^{[\epsilon,b)}(U)\right)_{\epsilon\in(o,b]} \ .$$

We shall prove

Proposition 3 *If* $\pi r^2 \in (0,b)$ *then*

$$S_{n+1}^{(0,b)}\left(B^{2n}(r)\right) \cong \mathbb{Z}_2 \ .$$

Next we consider for $\epsilon \leq \epsilon'$ the natural map

$$S_{n+1}^{(0,b)}\left(B^{2n}(\epsilon')\right) \longrightarrow S_{n+1}^{(0,b)}\left(B^{2n}(\epsilon)\right)$$

since $B^{2n}(\epsilon) \subset B^{2n}(\epsilon')$. We pass to the direct limit $\epsilon \longrightarrow 0$ obtaining the group Θ.

Proposition 4 *For* $b > 0$ *we have* $\Theta \cong \mathbb{Z}_2$.

Given any $z \in U$ and $\Psi \in \mathscr{D}$ with $\Psi(0) = z$ we have for $\epsilon > 0$ sufficiently small $\Psi(B^{2n}(\epsilon)) \subset U$. Hence we obtain an induced map

$$\Psi^* : S_{n+1}^{(0,b)}(U) \longrightarrow S_{n+1}^{(0,b)}(B^{2n}(\epsilon))$$

and passing to the direct limit we obtain a map

$$\Psi^\natural : S_{n+1}^{(0,b)}(U) \longrightarrow \Theta \ .$$

Using the isotopy invariance the map $\Psi^\natural$ only depends on U and the connected component of $z \in U$.

Hence for every connected bounded open set U there is a characteristic map denoted by σ_U

$$\sigma_U : S_{n+1}^{(0,b)}(U) \longrightarrow \Theta \ .$$

The capacity of a connected set U is defined by

$$c(U) = \inf\{b \mid \sigma_U : S_{n+1}^{(0,b)}(U) \longrightarrow \Theta \quad \text{is onto}\} \ .$$

If U has components $(U_\lambda)_{\lambda\in\Lambda}$ we define $c(U) := \sup_{\lambda\in\Lambda} c(U_\lambda)$. For unbounded U we define $c(U)$ by exhaustion through bounded open sets. We have

Theorem 6 *i) If* $\Psi \in \mathscr{D}$ *and* $\Psi(U) \subset V$ *the* $c(U) \leq c(V)$.
ii) $c(\alpha U) = |\alpha|^2 c(U)$ *for* $\alpha \in \mathbb{R} \setminus \{0\}$.
iii) $c(B^{2n}(1)) = c(B^2(1) \times \mathbb{C}^{n-1}) = \pi$.

1.5 Some Remarks

The previous results, though we only study very simple objects in symplectic geometry, show already the richness of this structure. What is missing in symplectic geometry is an idea about *what one can do with a symplectic map.* Symplectic homology is useful in showing that certain things can not be done. It would be interesting to know how sharp this theory is, because then borderline examples of things which cannot be done could be pin pointed and used in order to find the zone of transition between flexibility and rigidity in the theory.

In symplectic homology II, [8], the theory is extended to more general manifolds and in [10] we give some further applications. We study for example the action spectrum of the boundary of a symplectic manifold and show its rigidity.

2 Computation of symplectic homology

2.1 Generic approximations

In order to carry out our computation it is important to have a set of cofinal Hamiltonians which can be controlled.

Let $q : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a real positive definite quadratic form. We have already said something in section 1.2 about the symplectic properties of ellipsoids in the linear theory.

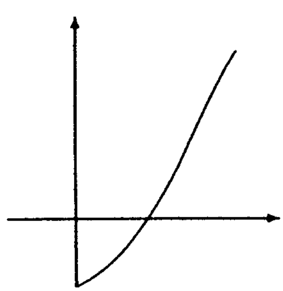
Let $r \in \Gamma_{\text{reg}}$ and $E = E(r)$. We study Hamiltonians of the form

$$H(z) = \rho(q(z)) \ ,$$

where $q(z) = \sum_{j=1}^{n} \left| \frac{z_l}{r_j} \right|^2$ and $\rho : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is smooth. Here $\mathbb{R}^+ = [0, +\infty)$. We assume that ρ has the following properties

$$\begin{array}{lcl} \rho'(s) & =: & \rho'(\infty)(=\text{const}) \quad \text{for} \quad s \geq s_0 \\ \rho''(s) & > & 0 \quad \text{for} \quad 0 \leq s < s_0 \\ \rho'(0) & > & 0 \ . \end{array} \tag{5}$$

Hence ρ looks like



Moreover we assume that

$$\begin{aligned} -i\dot{z} &= \rho'(\infty)q'(z) \\ z(0) &= z(1) \end{aligned} \tag{6}$$

does not have any solution other than the zero solution. This is a nonresonance condition on H at infinity. The above system splits into

$$-i\dot{z}_j = 2\rho'(\infty)\frac{1}{r_j^2}z_j \ , \ j = 1, \ldots, n \ .$$

Hence we deduce that

$$2\rho'(\infty)\frac{1}{r_j^2} \notin 2\pi\mathbb{Z} \quad \text{for} \quad j = 1, \ldots, n \ .$$

That means

$$\rho'(\infty) \notin \bigcup_{j=1}^{n} \left(\pi r_j^2 \mathbb{Z}\right) \ . \tag{7}$$

Lemma 2 *Let ρ be as described above and assume z is non constant and solves*

$$\begin{aligned} -i\dot{z} &= \rho'(q(z))q'(z) \\ z(0) &= z(1) \ . \end{aligned}$$

Then the geometric multiplicity of the kernel of the linear unbounded selfadjoint operator in $L^2(0,1;\mathbb{C}^n)$ with domain $H^{1,2}(S^1,\mathbb{C}^n)$

$$h \to -i\dot{h} - \rho''(q(z))\left\langle q'(z), h\right\rangle q'(z) - \rho'(q(z))q''(z)h$$

is one. ($S^1 = \mathbb{R}/\mathbb{Z}$).

Proof Since $r \in \Gamma_{\text{reg}}$ we immediately see that z has the form

$$z(t) = (0, \ldots, 0, z_j(t), 0 \ldots 0)$$

for some $j \in \{1, \ldots, n\}$ with $z_j \not\equiv 0$. Moreover z_j solves

$$\begin{aligned} -i\dot{z}_j &= 2\rho'(q(z))\frac{z_j}{r_j^2} \\ z_j(0) &= z_j(1) \ . \end{aligned}$$

Obviously $q(z) = \text{const} =: \tau$ and therefore

$$\rho'(\tau) \in \pi r_j^2 \mathbb{N}^* \ , \ \mathbb{N}^* = \{1, 2, 3, \ldots\} \ . \tag{8}$$

Hence

$$z_j(t) = e^{2\pi ikt} z_j(0)$$

with

$$k = \frac{\rho'(\tau)}{\pi r_j^2} \ . \tag{9}$$

Now consider the operator L defined as before by

$$\begin{aligned} h \longrightarrow -i\dot{h} \quad &- \quad \rho''(\tau)\left\langle z(0)e^{2\pi ikt}\tfrac{2}{r_j^2}, h\right\rangle \left(0,\ldots,\tfrac{2}{r_j^2}z_j(0)e^{2\pi ikt},0,\ldots,0\right) \\ &- \quad \rho'(\tau)q''(z)h \ . \end{aligned}$$

L splits into the direct sum $L = L_1 \oplus \ldots \oplus L_n$ with

$$\begin{aligned} L_m h = -i\dot{h} \quad &- \quad \rho'(\tau)\frac{2}{r_m^2}h \ , \ m \neq j \ , \\ L_j h = -i\dot{h} \quad &- \quad \rho''(\tau)\langle z_j(0)e^{2\pi ikt}\left(\frac{2}{r_j^2}\right) \ , \ h\rangle \frac{2}{r_j^2}z_j(0)e^{2\pi ikt} \\ &- \quad \rho'(\tau)\frac{2}{r_j^2}h \ . \end{aligned}$$

For $m \neq j \quad L_m h = 0$ implies

$$h(t) = h(0)e^{2\pi ik\left(\frac{r_j}{r_m}\right)^2 t} \ .$$

Since $r \in \Gamma_{\text{reg}}$ and $h(0) = h(1)$ we must have $h(0) = 0$. This shows that $L_1, \ldots L_{m-1}, L_{m+1}, \ldots, L_n$ are isomorphisms as operators from

$$H^{1,2}(S^1, \mathbb{C}) \longrightarrow L^2(S^1, \mathbb{C}) \ .$$

Next assume $L_j h = 0$. That is

$$\begin{aligned} 0 = -i\dot{h} \quad &- \quad \rho''(\tau)\left\langle z_j(0)e^{2\pi ikt}\left(\tfrac{2}{r_j^2}\right), h\right\rangle \left(\tfrac{2}{r_j^2}\right) z_j(0)e^{2\pi ikt} \\ &- \quad 2\pi kh \ , \end{aligned} \tag{10}$$

where $k \in \mathbb{N}^*$ is given by (9). Multiplying (10) by $e^{-2\pi ikt}$ we obtain

$$(-i\dot{h} - 2\pi kh)e^{-2\pi ikt} = \rho''(\tau)\langle z_j(0)e^{2\pi ikt}\left(\frac{2}{r_j^2}\right) \ , \ h\rangle \left(\frac{2}{r_j^2}\right) z_j(0) \ .$$

Hence

$$-i\frac{d}{dt}\left[he^{-2\pi ikt}\right] = \rho''(\tau)\frac{4}{r_j^4}\operatorname{Re}\left(\overline{z_j(0)} \cdot \left(he^{-2\pi ikt}\right)\right) z_j(0) \ .$$

Therefore with $\alpha = \overline{z_j(0)}he^{-2\pi ikt}$ and $\lambda = 4\rho''(\tau)\frac{|z_j(0)|^2}{r_j^4}$ we have

$$-i\dot{\alpha} = \lambda \operatorname{Re}(\alpha) \ .$$

Since $\alpha(0) = \alpha(1)$ and $\alpha \neq 0$ this implies

$$\begin{aligned} \alpha(0) \quad &= \quad \alpha(0) + i\lambda \operatorname{Re}(\alpha(0)) \\ \alpha(0) \quad &\neq \quad 0 \ . \end{aligned}$$

Consequently $\operatorname{Re}(\alpha(0)) = 0$ and $\operatorname{Im}(\alpha(0)) \neq 0$. Therefore $\alpha(t) = \alpha(0)$ for $t \in [0, 1]$ and

$$h(t)\overline{z_j(0)}e^{-2\pi ikt} = i\sigma \ , \ \sigma \in \mathbb{R}$$

for all $t \in [0, 1]$. This gives

$$\begin{aligned} h(t) &= i\sigma z_j(0)e^{2\pi ikt} \\ &= \frac{\sigma}{2\pi k}\frac{d}{dt}z_j(t) \ . \end{aligned}$$

Summing up we have shown that if $L_m h = 0$ then $h \in \mathbb{R}\left(\frac{d}{dt}z\right)$. □

We remark further that if $\rho'(0) > 0$ is small enough the only constant solution of the Hamiltonian system

$$\begin{aligned} \dot{z} &= \rho'(q(z))q'(z) \\ z(0) &= z(1) \end{aligned}$$

— the zero solution — has a linearization which is an isomorphism.

Given any $\tilde{H} \in \mathcal{N}_{\text{reg}}(E(r))$ there exists a ρ as described before and a $\tau > 0$ such that

$$\begin{aligned} &\rho \circ q \geq \tilde{H} + \tau \\ &\rho \circ q \mid \overline{E(r)} < 0 \end{aligned}$$

and 0 is a nondegenerate solution for the Hamiltonian system $H = \rho \circ q$.

Our next step consists in taking a precise perturbation of $\rho \circ q$, say Δ, such that $\rho \circ q + \Delta \in \mathcal{N}_{\text{reg}}(E(r))$ and satisfies $\rho \circ q + \Delta \geq \tilde{H} + \frac{\tau}{2}$. In addition this perturbation should have a very controlled behaviour as far as the 1-periodic solutions are concerned. We define the action for a solution x by

$$\Phi(x) = \frac{1}{2}\int_0^1 \langle -i\dot{x}, x\rangle \, dt - \int_0^1 \rho(q(x))dt \ .$$

Consider the set

$$\Sigma = \{k\pi r_j^2 \mid j = 1, \ldots, n \ ; \ k \in \mathbb{N}^*\} \ .$$

We observe that if $r \in \Gamma_{\text{reg}}$ the map

$$\phi : \{1, \ldots, n\} \times \mathbb{N}^* \to \Sigma : (j, k) \to k\pi r_j^2$$

is a bijection. We order the elements of Σ in increasing order

$$\Sigma = \{d_1 < d_2 < d_3 < d_4 \ldots\} \ .$$

We define $d_0 = 0$. Now assume $\phi(j, k) = d_m$. Then for $m = 1, 2, \ldots$

$$\begin{aligned} z^m(t) &= (0, \ldots, 0, z_j e^{2\pi ikt}, 0 \ldots 0) \\ q(z^m(t)) &= 1 \ , \ t \in S^1 \end{aligned}$$

solves

$$\begin{aligned} -i\tfrac{d}{dt}z^m &= d_m q'' z^m \\ z^m(0) &= z^m(1) \ . \end{aligned} \tag{11}$$

Here q'' is the linearization of the gradient q' of q. Define $m_0 = \sup\{m \mid d_m < \rho'(\infty)\}$. For every $m = 1, \ldots, m_0$ we find a unique positive number λ_m such that $\hat{z}^m = \lambda_m z^m$ solves

$$\dot{z} = X_{\rho\circ q}(z)\ ,\ z(0) = z(1)\ .$$

Then up to a phase shift and putting $\hat{z}^0 = 0$ we know that $\hat{z}^0, \ldots, \hat{z}^{m_0}$ are all 1–periodic solutions. We compute

$$\Phi(\hat{z}^m) = \int_0^1 \frac{1}{2}\langle -i\dot{\hat{z}}^m, \hat{z}^m\rangle\, dt - \int_0^1 \rho\circ q(\hat{z}^m) dt\ .$$

We have

$$\Phi(\hat{z}^0) = -\rho(0)\ ,$$

and for $m = 1, \ldots, m_0$ with $d_m = \phi(j,k)$

$$\begin{aligned}
\Phi(\hat{z}^m) &= k\pi\, |\hat{z}^m(0)|^2 - \rho(q(\hat{z}^m(0)))\\
&= d_m \left|\frac{\hat{z}^m(0)}{r_j}\right|^2 - \rho\left(\left|\frac{\hat{z}^m}{r_j}\right|^2\right)\\
&= \rho'(\tau_m)\tau_m - \rho(\tau_m)
\end{aligned} \tag{12}$$

with $\tau_m = \left|\frac{\hat{z}^m(0)}{r_j}\right|^2$. For $\tau \in [0, s_0)$ we compute

$$\begin{aligned}
\frac{d}{d\tau}\left(\rho'(\tau)\tau - \rho(\tau)\right) &= \rho''(\tau)\tau + \rho'(\tau) - \rho'(\tau)\\
&= \rho''(\tau)\tau\\
&> 0\ .
\end{aligned}$$

Since $d_m = \rho'(\tau_m)$ for $m = 1, \ldots, m_0$ and $0 < d_1 < d_2 \ldots < d_{m_0}$ we infer

$$0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 \ldots < \tau_{m_0}\ . \tag{13}$$

Therefore

$$\Phi(\hat{z}^0) < \Phi(\hat{z}^1) < \ldots < \Phi(\hat{z}^{m_0})\ . \tag{14}$$

Assume $\hat{z}^m(t) = (0, \ldots, 0, \hat{z}^m(0)e^{2\pi ikt}, 0, \ldots, 0)$ for $d_m = \phi(j,k)$. We define

$$\Psi(t) = e^{2\pi ikt}\,\mathrm{Id}$$

and put

$$\hat{z}^m = \Psi(t)\hat{a}^m\ .$$

Then

$$\hat{a}(t) = (0, \ldots, 0, \hat{z}_j^m(0), 0, \ldots, 0)$$

and we deduce via the substitution $\Psi a = z$

$$\begin{aligned}
0 &= -i(\dot{\Psi}a + \Psi\dot{a}) - \rho'(q(\Psi a))q'(\Psi a)\\
&= \Psi\left[-i\dot{a} + \Psi^{-1}\dot{\Psi}a - \rho'(q(a))q'(a)\right]\ .
\end{aligned}$$

Therefore, if z is a 1–periodic solution the map "a" satisfies

$$0 = -i\dot{a} + 2\pi ka - \rho'(q(a))q'(a) \ . \tag{15}$$

Observe that for every solution "a" of (15) the map $e^{i\theta}a$ for $\theta_1 \in [0, 2\pi]$ is also a solution of (15). Let $m \geq 1$. We have in view of the previous discussion a critical circle $S = \{e^{i\theta}\hat{a}^m \mid \theta \in \mathbb{R}\}$. We find a C^∞– small perturbation h supported as close as we wish to the circle S such that $h \mid S$ has precisely two different critical points and is a Morse function. We do this for every critical circle and define

$$H(t,z) = \left(\sum_{j=1}^{m_0} h_j(\Psi_j(t)^{-1}z)\right) + \rho(q(z)) \ .$$

By construction H is an arbitrarily small C^∞ perturbation of $\rho \circ q$. Moreover we may choose the h_j's in such a way that the perturbation only takes place near the critical circles. In addition we may choose the h_j's in such a way that for a proper choice of ρ

$$\begin{aligned} H &\geq \tilde{H} + \tau \\ H \mid \overline{E(r)} &< 0 \ , \end{aligned}$$

where $\tilde{H}$ is a given function in $\mathscr{H}_{\mathrm{reg}}(E(r))$ and $\tau > 0$ is sufficiently small. If ρ is sufficiently flat at zero we may also assume 0 is a nondegenerate critical point. For every $m \in \{1, \ldots, m_0\}$ the transformation $\hat{z}^m \longrightarrow \hat{a}^m$, $\Psi\hat{a}^m = \hat{z}^m$, transforms a critical circle of one problem into a critical circle of another problem. The critical circle $S = \{e^{i\theta}\hat{a}^m \mid \theta \in \mathbb{R}\}$ is split into two nondegenerate critical points (in view of lemma 5 and the fact that h is a Morse function on S and C^∞–small) under the perturbation h_m. A straightforward computation gives therefore for the corresponding critical points $\hat{z}^m_\pm$

$$\Phi_H(\hat{z}^m_\pm) = \Phi_{\rho\circ q}(\hat{z}^m) - h_m(\hat{a}^m_\pm) \ .$$

We take our notation in such a way that

$$\Phi_H(\hat{z}^m_+) > \Phi(\hat{z}^m_-) \ .$$

Assuming the perturbation H of $\rho \circ q$ to be small enough we have

$$\begin{aligned} 0 &< \Phi_H(0 = \hat{z}^0) \\ &< \Phi_H(\hat{z}^1_-) < \Phi_H(\hat{z}^1_+) < \ldots \\ &< \Phi_H(\hat{z}^{m_0}_-) < \Phi_H(\hat{z}^{m_0}_+) \ . \end{aligned} \tag{16}$$

The Conley–Zehnder index, see [1, 6, 7], can easily be computed to be

$$\begin{aligned} \mathrm{Ind}(\hat{z}^0, H) &= n \\ \mathrm{Ind}(\hat{z}^1_-, H) &= n + 1 \\ &\vdots \\ \mathrm{Ind}(\hat{z}^{n_0}_+, H) &= n + 2m_0 \ . \end{aligned} \tag{17}$$

We sum up the previous discussion as follows

Proposition 5 *Given a ρ as described in (5) and (6) with ρ sufficiently flat at 0 there exists an arbitrarily small S^1–dependent C^∞–perturbation of ρ supported as close as we wish to the m_0–many nontrivial critical circles such that the perturbed Hamiltonian $H : S^1 \times \mathbb{C}^n \longrightarrow \mathbb{R}$ has besides the zero solution also $2m_0$ nondegenerate critical points with energies and Conley–Zehnder index as just described, see (16),(17).*

2.2 The boundary operator

We shall work exclusively with $\mathbb{Z}_2$–coefficients to avoid the more elaborate orientation questions. We assume that the Hamiltonian $H \in \mathcal{N}_{\text{reg}}(E(r))$ is of the kind we just constructed in 2. 1. By construction we have for every regular pair (H, J) the following diagram

$$\begin{aligned} 0 \leftarrow \quad & (\mathbb{Z}_2, n, \Phi_H(0)) \leftarrow (\mathbb{Z}_2, n+1, \Phi_H(\hat{z}_-^1)) \leftarrow \\ & (\mathbb{Z}_2, n+2, \Phi_H(\hat{z}_+^1)) \leftarrow \dots \quad , \end{aligned} \tag{18}$$

where the boundary maps have to be determined.

Consider the Hamiltonian H_0 defined by

$$H_0(z) = \rho'(\infty) q(z) + c$$

for a suitable constant $c \in \mathbb{R}$ such that

$$H(z) = H_0(z) \quad \text{for} \quad |z| \quad \text{large} \ .$$

Then $H_0 \in \mathcal{N}_{\text{reg}}$ and H_0 has only zero as a critical point with $\text{Ind}(0, H_0) = (\mathbb{Z}_2, n + 2m_0)$. Hence for every regular $\hat{J}$ and numbers a and b with $a \le \Phi_{H_0}(0) = -c < b$ we have

$$S^{[a,b)}(H_0, \hat{J}) = (\mathbb{Z}_2, n + 2m_0) \ .$$

Following 4. 5 in [7] we may take a homotopy L between H_0 and H_1, which is constant for large $|z|$, and a regular $\tilde{J}$. If $a << 0 << b$, i. e. a sufficiently negative a and a sufficiently positive b we have following [7] for the gap g and the size $d(L)$ of L

$$d(L) < \frac{1}{2} g(H, H_0, [a, b]) \ .$$

Using Proposition 35 in [7] we obtain an isomorphism

$$S^{[a,b)}(H, J) \xrightarrow{\ \sim\ } S^{[a,b)}(H_0, J) = (\mathbb{Z}_2, n + 2m_0) \ .$$

This gives us precisely the knowledge about the arrows in (18) which we need. Namely we must have

$$\begin{aligned} \dots 0 \quad & \leftarrow (\mathbb{Z}_2, n, \Phi_H(0)) \xleftarrow{\text{Id}} (\mathbb{Z}_2, n+1, \Phi_H(\hat{z}_-^1)) \\ & \xleftarrow{0} (\mathbb{Z}_2, n+2, \Phi_H(\hat{z}_+^1)) \xleftarrow{\text{Id}} (\mathbb{Z}_2, n+3, \Phi_H(\hat{z}_-^2)) \dots \\ & \xleftarrow{0} (\mathbb{Z}_2, n+2m_0, \Phi_H(\hat{z}_+^{m_0})) \leftarrow 0 \leftarrow \dots \end{aligned} \tag{19}$$

or in short hand

$$\underset{0}{\bullet} \xleftarrow{\mathrm{Id}} \underset{\hat{z}_-^1}{\bullet} \xleftarrow{0} \underset{\hat{z}_+^1}{\bullet} \xleftarrow{\mathrm{Id}} \underset{\hat{z}_-^2}{\bullet} \longleftarrow \dots \tag{20}$$

In view of (19) we are able to compute $S^{[a,b)}(H,J)$ for every choice of numbers $-\infty < a \leq b \leq +\infty$. The crucial point is to understand the induced morphism between groups associated to (H,J) and $(K,\tilde{J})$ respectively, provided $(H,J) \leq (K,\tilde{J})$. This is done in the next subsection

2.3 The directed system

Let ρ_1, ρ_2 satisfy (5) and (6) in 2. 1 such that for a suitable $\hat{s} > 0$

$$\begin{aligned} \rho_1(s) = \rho_2(s) &\quad \text{for} \quad 0 \leq s \leq \hat{s} \\ \rho_1(s) \leq \rho_2(s) &\quad \text{for} \quad s \in [0,+\infty) \\ \rho_j \circ q \mid \overline{E(r)} < 0 &\quad \text{for} \quad j = 1,2 \end{aligned}$$

and $r \in \Gamma_{\mathrm{reg}}$ as in the previous subsection.

Our perturbation method allows us to obtain Hamiltonians H_1, H_2 satisfying $H_1 \leq H_2$ in such a way that H_1 has periodic solutions

$$0, \hat{z}_-^1, \hat{z}_+^1, \dots, \hat{z}_-^{m_1}, \hat{z}_+^{m_1} \tag{21}$$

and H_2 has the periodic solution given in (21) and in addition the periodic solutions

$$\hat{z}_-^{m_1+1}, \hat{z}_+^{m_1+1}, \dots, \hat{z}_-^{m_1}, \hat{z}_+^{m_2} \ . \tag{22}$$

We fix suitable regular J and $\tilde{J}$ and take a monotone homotopy between (H_1, J) and $(H_2, \tilde{J})$, which we call $(L, \hat{J})$, see [7]. The associated partial differential equation is

$$\begin{aligned} &u_s - \hat{J}(s,t,u)u_t - (\nabla_{\hat{J}} L)(s,t,u) = 0 \\ &u(s,*) \longrightarrow x \in \mathscr{P}_{H_1} \quad \text{as} \quad s \longrightarrow -\infty \\ &u(s,*) \longrightarrow y \in \mathscr{P}_{H_2} \quad \text{as} \quad s \longrightarrow +\infty \ , \end{aligned} \tag{23}$$

where $\mathrm{Ind}(x, H_1) = \mathrm{Ind}(y, H_2)$. In view of (21) and (22) it follows immediately that (23) only has solutions if $x = y$ and that the only solution in that case is $u(s,t) = x(t) = y(t)$.

Hence we have for the particular choice of H_1, H_2 just described the diagram:

$$\begin{tikzcd} (H_1,J) & \bullet \arrow[d] & \bullet \arrow[l] \arrow[d] & \cdots\cdots & \bullet \arrow[d] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & & & \\ (H_2,\tilde{J}) & \underset{0}{\bullet} & \underset{z_-^1}{\bullet} \arrow[l] & \cdots\cdots & \underset{z_+^{m_1-1}}{\bullet} & \underset{z_-^{m_1}}{\bullet} \arrow[l] & \underset{z_+^{m_1}}{\bullet} \arrow[l] & \underset{z_-^{m_1+1}}{\bullet} \arrow[l] & \cdots\cdots \quad \underset{z_-^{m_2}}{\bullet} & \underset{z_+^{m_2}}{\bullet} \arrow[l] \end{tikzcd}$$

Given $-\infty < a \leq b \leq +\infty$ with $a < +\infty$ let ρ_0 and ρ_1 be as described in 2. 1 and 2. 2 such that $\rho_0(s) = \rho_1(s)$ for $s \geq s_0$ and $\rho_0(s) < \rho_1(s)$ for $0 \leq s < s_0$ for a suitable s_0. Let $\rho_\tau = (1-\tau)\rho_0 + \tau\rho_1$. If $b < +\infty$ define j_- and j_+ by

$$d_{j_--1}(r) < a \leq d_{j_-} \leq \ldots d_{j_+} < b \leq d_{j_++1}$$

and if $b = +\infty$ put $j_+ = +\infty$ and define j_- in the obvious way. Consider for fixed τ the set consisting of all numbers $\rho'_\tau(s)s - \rho_\tau(s)$, where $\rho'_\tau(s) = d_j(r)$ for some $j \in \mathbb{N}$. We know that this is a nondecreasing sequence $d_j(r;\tau)$. Let us put $d_0(r,\tau) = -\rho_\tau(0)$.

Lemma 3 *Let the data be as described above and assume for all* $\tau \in [0,1]$ *we have*

$$d_{j_--1}(r,\tau) < a < d_{j_-}(r,\tau)$$

and if $b < +\infty$ *in addition*

$$d_{j_+}(r,\tau) < b < d_{j_++1}(r,\tau)$$

for all $\tau \in [0,1]$. *Let* H_0 *and* H_1 *be small generic perturbations of* ρ_0 *and* ρ_1 *respectively as described in 2.1 and 2.2, such that* $H_0 \leq H_1$. *Then for a generic choice of calibrated almost complex structures (perhaps t–dependent) we have that*

$$S^{[a,b)}(H_0, J) \longrightarrow S^{[a,b)}(H_1, \tilde{J}) \tag{24}$$

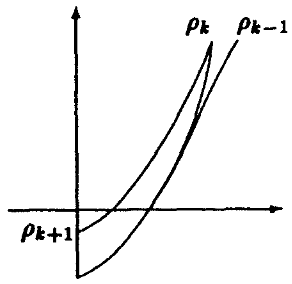
is an isomorphism.

Proof Using Proposition 35 in [7] and our assumption the monotonicity map in (23) can be written as a product of small distance isomorphism and therefore is an isomorphism. In fact if H_τ is a small generic perturbation of ρ_τ then if $|\tau_2 - \tau_1|$, $\tau_1 < \tau_2$ is small Proposition 35 in [7] says that the monotonicity map $S^{[a,b)}(H_{\tau_1}, J_{\tau_1}) \longrightarrow S^{[a,b)}(H_{\tau_2}, J_{\tau_2})$ is an isomorphism. (24) can be written as a product of such maps.

Using the discussion in this section it is not very difficult to construct for given a and b a monotonic sequence of (ρ_k) such that $\rho_k \circ q \mid \overline{E(r)} < 0$ and for every $H \in \mathscr{H}_{\text{reg}}$ there exists a $k \in \mathbb{N}^*$ and $\tau > 0$ with

$$\rho_k \circ q \geq H + \tau \ .$$

Moreover the ρ_k have the property that for two consecutive elements $\rho_k \leq \rho_{k+1}$ and suitable generic approximation H_k and H_{k+1} either the first part of our discussion in 2. 3 applies or the result in lemma 3. Hence the monotonicity map is the composition of injections and isomorphisms. Moreover the methods in section 2 show that we have a control about the critical levels for Φ_{H_k} between a and b. The following figure illustrates $\rho_{k-1} \leq \rho_k \leq \rho_{k+1}$.



2.4 Computations for ellipsoids and polydisks

The results in 2. 1, 2. 2, 2. 3 apply immediately that for a regular $r \in \Gamma_{\text{reg}}$ Theorem 2 holds. Next consider $r \in \Gamma$. We take a sequence $r^l \in \Gamma_{\text{reg}}$ with $r^l \longrightarrow r$. Without loss of generality let us assume $r^l \leq r$ (for the coordinate - wise ordering).

Let us assume $-\infty < a \leq b \leq +\infty$ with $a < +\infty$ and $a \neq d_j(r)$ for all $j = 0, 1, \ldots$ (with $d_0(r) = 0$) and if $b \neq +\infty$ we assume the same for b. For l sufficiently large we have $d_j(r^l) \neq a, b$ for all j. Consider the diagram

$$E((1-\delta)r) \hookrightarrow E(r^l) \hookrightarrow E(r) \hookrightarrow E\left((1+\delta)r^l\right)$$

for $\delta > 0$ small and l large. This induces a diagram

$$\begin{tikzcd}
S^{[a,b)}(E((1+\delta)\cdot r^l)) \arrow[rr] \arrow[dr] & & S^{[a,b)}(E(r^l)) \arrow[d] \\
& S^{[a,b)}(E(r)) \arrow[ur] \arrow[r] & S^{[a,b)}(E((1-\delta)\cdot r))
\end{tikzcd}$$

If we know that the horizontal arrows are isomorphisms for $r^l \longrightarrow r$ and $\delta \longrightarrow 0$ the proof of Theorem 2 is complete. However this result can be easily reduced to a variant of Lemma 3 and therefore ultimately to Proposition 35 in [7]. This completes the proof of Theorem 2 if a and b avoid the sequence $(d_j(r))_{j\in\mathbb{N}}$.

Next assume $d_{j_0}(r) = a$ for some $j \in \mathbb{N}$. We may assume without loss of generality that $d_{j_0-1}(r) < a$. Pick $\epsilon \in (0, a - d_{j_0-1}(r))$. We have the exact sequence

$$\begin{tikzcd} S^{[a-\epsilon,a)}(E(r)) \arrow[rr] \arrow[dr] & & S^{[a-\epsilon,b)}(E(r)) \\ & S^{[a,b)}(E(r)) \arrow[ur] & \end{tikzcd}$$

In order to compute $S^{[a-\epsilon,a)}(E(r))$ start with a ρ satisfying $\rho \circ q \mid \overline{E(r)} < 0$ and (5) and (6) in 2.1 In addition we assume $\rho'(s) = \rho'(\infty)$ for all $s \in \mathbb{R}^+$ with $\rho(s) \geq 0$, and moreover $|\rho'(s)|$ small for $s \leq 1$. Then

$$\rho'(\tau)\tau - \rho(\tau) > a$$

if $\rho'(\tau) = d_{j_0}(r)$ and $\rho'(\tau)\tau - \rho(\tau) < a - \epsilon$ if $\rho'(\tau) = d_{j_0}(r)$ for a suitable choice of ρ. We note that generic perturbations of such ρ's constitute a cofinal set.

Hence $S^{[a-\epsilon,a)}(H, J) = 0$ for a cofinal set. This shows that

$$S^{[a-\epsilon,b)}(E(r)) \simeq S^{[a,b)}(E(r)) \ .$$

The same argument works for b. This completes the proof of Theorem 2.

The recipe in the computation for ellipsoids was to take special Hamiltonians. Given a polydisk $D^{2n}(r) = B^2(r_1) \times \ldots \times B^2(r_n)$ we consider Hamiltonians of the form $H = H_1 \oplus \ldots \oplus H_n$,i. e.

$$H(z) = \sum H_j(z_j)$$

and almost complex structures of the form $J = J_1 \oplus \ldots \oplus J_n$, where $J_l(z) = i$ for $|z|$ large, $z \in \mathbb{C}$. Clearly $J(z) \not\equiv i$ in general for $|z|$ large, $z \in \mathbb{C}^n$. However one can derive apriori estimates as in [7] and show that $S^{[a,b)}(H, J)$ is isomorphic to $S^{[a,b)}(H, \hat{J})$ with $\hat{J}(z) = i$ for $z \in \mathbb{C}^n$, $|z|$ large. Note that this is an assumption in [7]. Alternatively we may consider (H, i) with $H = H_1 \oplus \ldots \oplus H_n$ which is also possible by [9]. It is now obvious that we obtain a cofinal family of complexes by considering the tensor products of those for disks. Then the discussion for the ellipsoid case immediately implies Theorem 4.

2.5 Some properties of induced homomorphisms

Next we study induced homomorphisms, in particular we given some criteria which imply their nontriviality.

Proposition 6 *Let $\Phi \in \mathscr{D}, \hat{r} \in (0, +\infty)$ and $r \in \Gamma$ such that $\Phi(B^{2n}(\hat{r})) \subset E(r)$. For $a \in (0, \pi\hat{r}^2]$ and $b \in (\pi r_1^2, +\infty]$ the linear map*

$$S_{n+1}^{[a,b)}(E(r)) \xrightarrow{\Phi^*} S_{n+1}^{[a,b)}(B^{2n}(\hat{r}))$$

is an isomorphism.

Proof We note that using Gromov width, [11, 12], or capacity we have necessarily $\hat{r} \leq r_1$ and consequently $a < b$. In view of corollary 2 we deduce that $S_{n+1}^{[a,b)}(B^{2n}(\hat{r})) \cong \mathbb{Z}_2$ and the same for $S_{n+1}^{[a,b)}(E(r))$. It is almost obvious that (using the previously constructed cofinal system)

$$S_n^{[0,a)}(E(r)) \longrightarrow S_n^{[0,a)}(B^{2n}(\hat{r}))$$

is an isomorphism, see the proof of Theorem 1 for the argument. Since $S^{[0,+\infty)}(E(r))$ and $S^{[0,+\infty)}(B^{2n}(\hat{r}))$ vanish it follows from the exact triangle that

$$S_{n+1}^{[a,+\infty)}(E(r)) \xrightarrow{\sim} S_{n+1}^{[a,+\infty)}(B^{2n}(\hat{r}))$$

is an isomorphism. Another application of the exact sequence gives the following commutative diagram

$$\begin{tikzcd}
S_{n+1}^{[a,+\infty)}(E(r)) \arrow[r, "\sim", "\Phi^*"'] & S_{n+1}^{[a,+\infty)}(B^{2n}(\hat{r})) \\
S_{n+1}^{[a,b)}(E(r)) \arrow[u, "\sim"] \arrow[r, "\Phi^*"] & S_{n+1}^{[a,b)}(B^{2n}(\hat{r})) \arrow[u, "\sim"']
\end{tikzcd}$$

Hence the bottom arrow is an isomorphism, too. □

Proposition 7 *Let* $\sigma \in \left(\frac{1}{\sqrt{2}}, 1\right]$ *and consider*

$$\mathrm{Id} : (\mathbb{C}^n, B^{2n}(r)) \longrightarrow (\mathbb{C}^n, B^{2n}(1)) \ .$$

Assume $a \in (0, \pi\sigma^2)$ *and* $b \in (\pi, \pi + \epsilon)$ *with* $\pi + \epsilon < 2\pi\sigma^2$. *Then*

$$\mathrm{Id}^* : S^{[a,b)}(B^{2n}(1)) \longrightarrow S^{[a,b)}(B^{2n}(\sigma))$$

is an isomorphism.

The arguments are similar to the ones used in subsection 2.3. Again the proof is built on two ingredients, the factorization of the monotonicity map through small distance isomorphisms and the fact that the actions of closed characteristics do not interfer with the numbers a and b for all balls $B^{2n}(r)$ with radii $r \in [\sigma, 1]$, which allows in 2. 1 to construct suitable regular Hamiltonians whose associated critical values do not interfer with the numbers a and b.

Again along the same lines one can show the following

Proposition 8 *Assume* $n = 2$ *and* $0 < a \leq \pi r_1^2 \leq \pi r_2^2 < \pi < b < \pi(r_1^2 + r_2^2)$. *Then* $\mathrm{Id} : (\mathbb{C}^2, D^4(r_1, r_2)) \longrightarrow (\mathbb{C}^2, D^4(1,1))$ *induces an isomorphism*

$$\mathrm{Id}^* : S_3^{[a,b)}(D^4(1,1)) \longrightarrow S_3^{[a,b)}(D^4(r_1, r_2)) \ .$$

For the proof one notes that the groups are isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The crucial observation is that the action of closed chararcteristics on $\partial D^4(r_1, r_4)$ are of the form $\pi(jr_1^2 + kr_2^2)$ for $k, j \in \mathbb{N}, k + j \geq 1$. (Clearly ∂D^4 is not smooth, but our approximation by suitable Hamiltonians of the form $H_1 \oplus H_2$ will result in functionals having critical levels close to that numbers). If we deform $D^4(r_1, r_2)$ through polydisks in $D^4(1, 1)$ in a monotonic way the numbers a and b satisfying the above inequalities will not be crossed by a critical level belonging to a critical point of index 2, 3 or 4 (for suitable approximations in the sense of 2. 1 or 2. 2). Precisely this allows the outlined recipe of generic approximation and small distance isomorphisms to work.

There is of course some underlying more general argument behind all this which will be established in [9].

3 Applications

3.1 Properties of the nonlinear order structure

Next we prove Theorem 1. The direction **i)** $\Longrightarrow$ **ii)** is trivial. So we assume that **ii)** holds. By assumption we have the following diagram of symplectic embeddings

$$B^4(\sigma) \overset{s}{\hookrightarrow} E(r) \overset{s}{\hookrightarrow} E(r') \overset{s}{\hookrightarrow} B^4(1) \ , \tag{25}$$

where all the maps are induced by globally defined symplectic maps. Without loss of generality we may assume that all maps are in $\mathscr{D}$. By a result of McDuff, [20], the map $B^4(\sigma) \overset{s}{\hookrightarrow} B^4(1)$ defined by (25) is symplectically isotopic to the standard inclusion. Using the well–known "extension after restriction" principle, see for example [3], the restriction of this isotopy to $B^4(\sigma')$ with $\sigma' \in \left(\frac{1}{\sqrt{2}}, \sigma\right)$ can be considered as the restriction of an isotopy in $\mathscr{D}$. Without loss of generality we may assume $\sigma' = \sigma$ and the following: There are maps $\alpha, \beta, \gamma, \Psi \in \mathscr{D}$ such that

$$\begin{aligned} \alpha(B^4(\sigma)) &\subset E(r) \\ \beta(E(r)) &\subset E(r') \\ \gamma(E(r')) &\subset B^4(1) \\ \Psi &= \gamma \circ \beta \circ \alpha \end{aligned}$$

and there exists $(\Psi_s)_{s \in [0,1]} \subset \mathscr{D}$ with $\Psi_0 = \mathrm{Id}$, $\Psi_1 = \Psi$ and

$$\Psi_s(B^4(\sigma)) \subset B^4(1) \qquad \text{for all} \quad s \in [0, 1] \ .$$

Using the isotopy invariance of symplectic homology we have

$$\begin{tikzcd}
S^{[a,b)}(B^4(1)) \arrow[r, "\mathrm{Id}^*"] \arrow[d, "\gamma^*"'] & S^{[a,b)}(B^4(\sigma)) \\
S^{[a,b)}(E(r')) \arrow[r, "\beta^*"'] & S^{[a,b)}(E(r)) \arrow[u, "\alpha^*"']
\end{tikzcd} \tag{26}$$

Note that Id^* is not necessarily identity since $\mathrm{Id} : (\mathbb{C}^2, B^4(\sigma)) \longrightarrow (\mathbb{C}^2, B^4(1))$.

First we take $a = \pi\sigma^2$ and $b = +\infty$. Then Theorem 2 gives

$$\begin{aligned}
S^{[\pi\sigma^2,+\infty)}(B^4(1)) &= (\mathbb{Z}_2, 3) \\
S^{[\pi\sigma^2,+\infty)}(B^4(\sigma)) &= (\mathbb{Z}_2, 3) \ .
\end{aligned}$$

We observe that $\mathrm{Id}^* : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ is the identity. This follows from the following observation. We have $S^{[0,+\infty)}(*) = 0$ for $* = B^4(1)$ and $B^4(\sigma)$ and $S^{[0,\pi\sigma^2)}(*) = (\mathbb{Z}_2, 2)$ for $* = B^4(1)$ and $B^4(\sigma)$. Hence it follows from the exact triangle

$$\begin{tikzcd}
S^{[0,\pi\sigma^2)}(B^4(1)) \arrow[rr, "\mathrm{Id}^*"] & & S^{[0,\pi\sigma^2)}(B^4(\sigma)) \\
S^{[\pi\sigma^2,+\infty)}(B^4(1)) \arrow[rr, "\mathrm{Id}^*"] \arrow[u, "\sim"', "\deg = -1"] & & S^{[\pi\sigma^2,+\infty)}(B^4(1)) \arrow[u, "\sim"']
\end{tikzcd}$$

This precisely gives

$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[rr] & & \mathbb{Z}_2 \\
\mathbb{Z}_2 \arrow[rr] \arrow[u, "\sim"] & & \mathbb{Z}_2 \arrow[u, "\sim"']
\end{tikzcd}$$

The top arrow however is the identity and therefore Id^* is the identity for the bottom arrow. The top arrow is given by the number of connecting orbits connecting 0 with 0. Using the cofinal system we constructed before and a suitable monotone homotopy it follows easily the the connecting orbit is precisely the map $(s, t) \longrightarrow 0$. This implies immediately that γ^* is injective and α^* surjective. Therefore

$$\begin{aligned} \dim S_3^{[\pi\sigma^2,+\infty)}(E(r')) &\geq 1 \\ \dim S_3^{[\pi\sigma^2,+\infty)}(E(r)) &\geq 1 \ . \end{aligned}$$

Since the dimension is apriori at most one we deduce

$$\dim S_3^{[\pi\sigma^2,+\infty)}(E(r)) = 1 = \dim S_3^{[\pi\sigma^2,+\infty)}(E(r')) \ .$$

From Theorem 2 it follows that

$$\begin{aligned} \pi r_1^2 &\geq \pi\sigma^2 \\ \pi(r_1')^2 &\geq \pi\sigma^2 \ . \end{aligned}$$

(We note this could also be obtained by using symplectic capacities).

For volume reasons we must have

$$r_1 \leq 1 \ , \ r_1' \leq 1$$

(since $r_2 \leq r_1$ and $r_2' \geq r_1'$).

Hence we know that

$$\sigma \leq r_1 \leq 1 \quad \text{and} \quad \sigma \leq r_1' \leq 1 \ .$$

From Proposition 7 we know that for every $\Phi \in \mathscr{D}$ with $\Phi(B^4(\epsilon)) \subset E(r)$ and for $a \in (0, \pi\epsilon^2]$, $b \in (\pi r_1^2, +\infty)$ the following map

$$S_3^{[a,b)}(E(r)) \longrightarrow S_3^{[a,b)}(B^4(\epsilon))$$

is onto. Note that $S_3^{[a,b)}(B^4(\epsilon)) = \mathbb{Z}_2$ by Theorem 2. Next let us show $r_1 \leq r_1'$. Arguing indirectly let us assume $r_1 > r_1'$. We pick $b \in (\pi(r_1')^2 \ , \ \pi r_1^2)$, and $a \in (0, \pi\sigma^2)$. Then we have the commutative diagram

$$\begin{tikzcd} S_3^{[a,b)} \arrow[r, "\beta^*"] \arrow[d, "(\beta\circ\alpha)^*"'] & S_3^{[a,b)}(E(r)) \arrow[dl, "\alpha^*"] \\ \mathbb{Z}_2 & \end{tikzcd}$$

From Theorem 2 we know that $S_3^{[a,b)}(E(r')) = \mathbb{Z}_2$ and $S_3^{[a,b)}(E(r)) = 0$. This gives a contradiction.

Hence we know so far that

$$\sigma \leq r_1 \leq r_1' \leq 1 \ .$$

Next we take in (26) $a = \pi\sigma^2$ and for $\epsilon > 0$ small $b \in (\pi, \pi+\epsilon)$. From Theorem 2 we know that

$$\begin{aligned} S^{[a,b)}(B^4(1)) &= (\mathbb{Z}_2, 3) \oplus (\mathbb{Z}_2, 6) \\ S^{[a,b)}(B^4(\sigma)) &= (\mathbb{Z}_2, 3) \oplus (\mathbb{Z}_2, 6) \ . \end{aligned}$$

Studying (26) for the above choices of a and b in dimension 6 we obtain

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r, "\mathrm{Id}"] \arrow[d, "\gamma^*"'] & \mathbb{Z}_2 \\ S_6^{[a,b)}(E(r')) \arrow[r, "\beta^*"'] & S_6^{[a,b)}(E(r)) \arrow[u, "\alpha^*"'] \end{tikzcd} \tag{27}$$

It follows from Proposition 7 that the top arrow is indeed the identity. Consequently $S_6^{[a,b)}(E(r')) \neq 0$ and $S_6^{[a,b)}(E(r)) \neq 0$. If $r_2' > 1$ we take $\epsilon > 0$ above so small that $b < \pi(r_2')^2$. Then $S_6^{[a,b)}(E(r')) = 0$ by Theorem 2 giving a contradiction. Hence $r_2' \leq 1$. The same argument shows $r_2 \leq 1$. Now assume $r_2' < r_2 \leq 1$ arguing indirectly. Taking $b \in \left(\pi(r_2')^2, \pi r_2^2\right)$ we deduce that $\alpha^*\beta^*$ in (27) is zero.

On the other hand we know that the composition

$$S_6^{[a,b)}(E(r')) \xrightarrow{\beta^*} S_6^{[a,b)}(E(r)) \xrightarrow{\alpha^*} S_6^{[a,b)}(B^4(\sigma))$$

for $a = \pi\sigma^2$ and $\pi < b < \pi + \epsilon$, $\epsilon > 0$ small enough, is an isomorphism. Since from the exact sequence for the triplet $a \leq b' \leq b$ we have

$$\begin{aligned} S_6^{[a,b')}(B^4(\sigma)) &\simeq S_6^{[a,b)}(B^4(\sigma)) \\ S_6^{[a,b')}(E(r')) &\simeq S_6^{[a,b)}(E(r')) \end{aligned}$$

provided $\pi(r_2')^2 \leq b' \leq b$, we see that

$$\mathbb{Z}_2 \cong S_6^{[a,b')}(E(r')) \xrightarrow{(\beta\circ\alpha)^*} S_6^{[a,b')}(B^4(\sigma)) \cong \mathbb{Z}_2$$

is an isomorphism. Taking b' such that $\pi(r_2')^2 \leq b < \pi r_2$ we obtain $(\beta \circ \alpha)^* \neq 0$ contrary to our assumption. So, summing up we have proved

$$(\sigma, \sigma) \leq_1 r \leq_1 r' \leq_1 (1, 1) \ .$$

3.2 Polydisk classification

We define for $a \in \mathbb{R}$ and $r \in \Gamma$

$$d_r(a) = \lim_{\epsilon \longrightarrow 0} \dim S_{n+1}^{[a-\epsilon, a+\epsilon)}(D(r)) \ .$$

In view of Theorem 4 we have

$$d_r(a) = \# \left\{ j \mid \pi r_j^2 = a \right\} \ .$$

We observe that the map

$$\Gamma \longrightarrow \quad \mathrm{Maps}\,(\mathbb{R}, \{1, \ldots, n\}) : r \longrightarrow d_r$$

is an injection. So it suffices to show that $D(r) \underset{s}{\simeq} D(r')$ implies $d_r = d_{r'}$.

We note that for every $a \in \mathbb{R}$ there exists a $\epsilon(a) > 0$ such that for $\epsilon \in (0, \epsilon(a)]$

$$\dim S_{n+1}^{[a-\epsilon,a+\epsilon)}(D(r)) = d_r(a) \ .$$

This follows from $S_{n+1}^{[a+\epsilon_1,a+\epsilon_2)}(D(r)) = 0$ for $0 < \epsilon_1 \leq \epsilon_2$, ϵ_2 small etc. , using the exact triangle. One obtains this from studying a suitable cofinal set of Hamiltonians as in the ellipsoid case.

For every $\delta > 0$ small, there exists by the extension after restriction principle a $\Psi_\delta \in \mathscr{D}$ with

$$\Psi_\delta \mid D^{2n}((1-\delta)r) = \rho \mid D^{2n}((1-\delta)r) \ ,$$

where $\rho : D^{2n}(r) \simeq D^{2n}(r')$ is the given symplectic diffeomorphism. In particular $\Psi_\delta\left(D^{2n}((1-\delta)r)\right) \subset D^{2n}(r')$. We find a $\delta' > 0$ small, such in fact $\Psi_\delta(D^{2n}((1-\delta)r)) \subset D^{2n}((1-\delta')r')$. Again we find a $\Phi_{\delta'} \in \mathscr{D}$ with

$$\Phi_{\delta'} \mid D^{2n}((1-\delta')r') = \rho^{-1} \mid D^{2n}((1-\delta')r') \ .$$

Hence we have

$$\Phi_{\delta'} \circ \Psi_\delta(x) = x \qquad \text{for} \quad x \in D((1-\delta)r) \ .$$

Take any symplectic isotopy of $\alpha = \Phi_{\delta'} \circ \Psi_\delta$ in $\mathscr{D}$, say $(\alpha_s)_{s\in[0,1]}$ with $\alpha_0 = \mathrm{Id}$ and $\alpha_1 = \alpha$. For $t \in [0,1)$ define, following [13],

$$\alpha_s^t(x) = (1-t)\alpha_s\left((1-t)^{-1}x\right) \ .$$

We note that $\alpha_s^0 = \alpha_s$ and that $\mathrm{supp}(\alpha_s^t) \subset (1-t)B_R(0)$ for R sufficiently large independent of t and s. Taking a suitable path in $[0,1] \times [0,1)$ connecting $(0,0)$ with $(1,0)$, say $\tau \longrightarrow (a(\tau), b(\tau))$ we define $\tilde{\alpha}_\tau(x) = \alpha_{a(\tau)}^{b(\tau)}$ and may assume that $\tilde{\alpha}_\tau(D(1-\delta)r)) \subset D(r)$ for all $\tau \in [0,1]$. Therefore $(\Phi_{\delta'} \circ \Psi_\delta)^* = \mathrm{Id}^* : S^{[a,b)}(D^{2n}(r)) \longrightarrow S^{[a,b)}(D^{2n}((1-\delta)r)$. Consequently we have the commutative diagram

$$\begin{tikzcd}
S_{n+1}^{[a-\epsilon,a+\epsilon)}(D^{2n}(r)) \arrow[rr, "\mathrm{Id}^*"] \arrow[dr, "\Phi_{\delta'}^*"'] & & S_{n+1}^{[a-\epsilon,a+\epsilon)}(D^{2n}((1-\delta)r) \\
& S_{n+1}^{[a-\epsilon,a+\epsilon)}(D^{2n}((1-\delta')r')) \arrow[ur, "\Psi_\delta^*"'] &
\end{tikzcd} \tag{28}$$

For $\epsilon > 0$ small we have $\dim S_{n+1}^{[a-\epsilon,a+\epsilon)}(D^{2n}(r)) = d_r(a)$. Let this $\epsilon > 0$ be fixed. For $\delta > 0$ small enough we have also $\dim S_{n+1}^{[a-\epsilon,a+\epsilon)}(D^{2n}((1-\delta)r)) = d_r(a)$ and Id^* is an isomorphism. The latter follows from the following observation. We take suitable cofinal systems for $D^{2n}(r)$ and $D^{2n}((1-\delta)r)$ respectively. Then Id^* is the monotonicity map. This monotonicity map can be factored through small distance isomorphisms and hence is an isomorphism, see Proposition 35 in [7]. Therefore

$$d_r(a) \leq d_{r'}(a) \ .$$

Similarly $d_{r'}(a) \leq d_r(a)$. This proves $r = r'$.

3.3 Non isotopic symplectic embeddings

The aim of this section is to prove Theorem 4. Pick a, b such that

$$0 < a < \pi r_1^2 \leq \pi r_2^2 < \pi < b < \pi(r_1^2 + r_2^2) \ .$$

With this choice we have by Theorem 5

$$\begin{aligned} S_3^{[a,b)}(D^4(1,1)) &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ S_3^{[a,b)}(D^4(r_1,r_2)) &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 \ . \end{aligned}$$

The identity map induces with this identification the identity map and is therefore represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, see Proposition 8.

It is easy to construct a sequence $(\Psi_k) \in \mathscr{D}$ such that for $|z| \leq k$ we have

$$\Psi_k(z) = (z_2, z_1)$$

and for $l \geq k$ there exists an isotopy in $\mathscr{D}$ from Ψ_k to Ψ_l which is constant on $\{z \mid |z| \leq k\}$. For $k \geq 2$ the morphism

$$\Psi_k^* : S_3^{[a,b)}(D^4(1,1)) \longrightarrow S_3^{[a,b)}(D^4(r_1,r_2))$$

is independent of k by the isotopy invariance of symplectic homology. Taking a cofinal sequence for $D^4(1,1)$ along the construction outlined before we have for each element (H_l, J_l) an apriori bound for the Morse complex. For k large enough Ψ_k acts like the permutation map on the Morse complex. This gives immediately that Ψ_k^* for k large is given by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since Ψ_k^* is independent of k we have shown that the maps $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are induced by suitable symplectic maps in $\mathscr{D}$. Hence there cannot be an isotopy $(\Psi_s) \subset \mathscr{D}$ with $\Psi_s(D^4(r_1,r_2)) \subset D^4(1,1)$ for $s \in [0,1]$ with $(\Psi_0)^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $(\Psi_1)^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Using the extension after restriction principle and the previous discussion we see that the maps $\Psi_k \ |D^4(r_2,r_2)|$ and Id $\mid D^4(r_1,r_2)$ cannot be isotpic as maps from $D^4(r_1,r_2)$ into $D^4(1,1)$.

3.4 Construction of a symplectic capacity

Let $U \subset \mathbb{C}^n$ be a bounded open set and $0 < \epsilon_1 \leq \epsilon_2 < +\infty$. We have a natural map

$$S_{n+1}^{[\epsilon_1,b)}(U) \longrightarrow S_{n+1}^{[\epsilon_2,b)}(U) \ ,$$

where we assume $b \geq \epsilon_2$. We have an inverse system

$$\left(S_{n+1}^{[\epsilon,b)}(U)\right)_{\epsilon \in (0,b]} \ .$$

We define

$$S_{n+1}^{(0,b)}(U) := \text{inv } \lim \left(S_{n+1}^{[\epsilon,b)}(U)\right)_{\epsilon \in (0,b]} \ .$$

Proposition 9 *For every bounded open set $U \subset \mathbb{C}^n$ there exists a constant $b_0(U) =: b_0$ such that*

$$S_{n+1}^{(0,b)}(U) \neq 0$$

for $b \geq b_0$.

Proof We may assume without loss of generality that $0 \in U$. Hence we find numbers $0 < r < R < +\infty$ such that we have inclusions

$$B^{2n}(r) \subset U \subset B^{2n}(R) \ .$$

For numbers $0 < a < \pi r^2 < \pi R^2 < b \leq +\infty$ we consider the commutative diagram

$$\begin{tikzcd}
S_{n+1}^{[a,b)}(B^{2n}(R)) \arrow[rr, "\sim"] \arrow[dr] & & S_{n+1}^{[a,b)}(B^{2n}(r)) \\
& S_{n+1}^{(a,b)}(U) \arrow[ur] &
\end{tikzcd}$$

where all maps are induced by the identity. In view of Proposition 6 the top arrow is onto. Since however both top groups are isomorphic to $\mathbb{Z}_2$ it is an isomorphism. This implies that $S_{n+1}^{[a,b)}(U)$ is nontrivial for the above choices of a and b. Next we consider for $0 < a' < a$ the natural maps $S^{[a',b)}(*) \longrightarrow S^{[a,b)}(*)$. We obtain the diagram (via Proposition 6)

$$\begin{tikzcd}
S^{[a,b)}(B^{2n}(R)) \arrow[rr, "\sim"] \arrow[dr] \arrow[ddd, "\sim"'] & & S_{n+1}^{[a',b)}(B^{2n}(R)) \arrow[dl] \arrow[ddd, "\sim"] \\
& S_{n+1}^{[a',b')}(U) \arrow[d] & \\
& S_{n+1}^{[a,b)}(U) & \\
S_{n+1}^{[a,b)}(B^{2n}(R)) \arrow[rr, "\sim"] \arrow[ur] & & S_{n+1}^{[a',b)}(B^{2n}(R)) \arrow[ul]
\end{tikzcd}$$

Passing to the inverse limit gives with $\mathbb{Z}_2 \simeq S^{(0,b)}(B^{2n}(R))$, $\mathbb{Z}_2 \simeq S^{(0,b)}(B^{2n}(r))$ for $\pi r^2 < \pi R^2 < b$ the diagram

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[rr, "\sim"] \arrow[dr] & & \mathbb{Z}_2 \\ & S^{(a,b)}(U) \arrow[ur] & \end{tikzcd}$$

proving our assertion. □

Lemma 4 *Consider the group*

$$\Theta := \text{dir} \lim_{\epsilon \longrightarrow 0} S^{(0,+\infty)}(B^{2n}(\epsilon)) \ .$$

Then $\Theta \cong \mathbb{Z}_2$ *and for every bounded open set* $U \subset \mathbb{C}^n$ *there exists a natural map*

$$S^{(0,b)}(U) \xrightarrow{\sigma_\lambda} \Theta$$

for every path component $[U_\lambda]$ *of* U.

Proof Let $U_\lambda \subset U$ be a path component. Pick a $\Psi \in \mathscr{D}$ and $\epsilon_1 > 0$ small such that

$$\Psi(B^{2n}(\epsilon_1)) \subset U_\lambda \subset U \ .$$

We obtain a map $\Psi^* : S_{n+1}^{(0,b)}(U) \longrightarrow S_{n+1}^{(0,b)}(B^{2n}(\epsilon_1))$ and therefore also

$$\Psi^* : S_{n+1}^{(0,b)}(U) \longrightarrow S_{n+1}^{(0,+\infty)}(B^{2n}(\epsilon_1))$$

for $\epsilon_1 < \epsilon_2$ we have a natural map

$$S_{n+1}^{(0,+\infty)}(B^{2n}(\epsilon_2)) \longrightarrow S_{n+1}^{(0,+\infty)}(B^{2n}(\epsilon_1)) \ .$$

In view of an argument used in Proposition 9 this map is an isomorphism. Hence $\Theta \cong \mathbb{Z}_2$. Hence Ψ^* induces a map

$$\Psi^* : S_{n+1}^{(0,b)}(U) \longrightarrow \Theta \ .$$

Now observe that any two such maps Ψ, Φ with $\Psi(B^{2n}(\epsilon)) \subset U_\lambda$, $\Phi(B^{2n}(\epsilon)) \subset U_\lambda$ and $\epsilon > 0$ small enough induce the same map in symplectic homology in view of the isotopy invariance. Hence Ψ^* only depends on the chosen path component of U. Let $(U_\lambda)_{\lambda \in \Lambda}$, $\Lambda = \Lambda_U$ be the collection of path components of U. Let σ_λ be the natural maps $\sigma_\lambda : S_{n+1}^{(0,b)}(U) \longrightarrow \mathbb{Z}_2$ for $\lambda \in \Lambda_U$.

If $\Psi \in \mathscr{D}$ we have by construction a natural diagram if $\Psi(U) \subset V$

$$\begin{tikzcd} S_{n+1}^{[0,b)}(V) \arrow[r, "\Psi^*"] \arrow[dr, "\sigma_\mu^V"'] & S_{n+1}^{(0,b)}(U) \arrow[d, "\sigma_\lambda^U"] \\ & \mathbb{Z}_2 \end{tikzcd}$$

with $\Psi(U_\lambda) \subset V_\mu$.

We define

$$c(U) = \sup_{\lambda \in \Lambda_U} \inf \left\{ b \mid \sigma_\lambda^U : S_{n+1}^{(0,b)}(U) \to \mathbb{Z}_2 \text{ is onto} \right\} .$$

As a by–product of the proof of Proposition 9 we know that for a bounded open set U , $c(U) < \infty$. Moreover the computations for the symplectic homology of an ellipsoid give easily

$$c(E(r)) = \pi r_1^2 .$$

The following properties are obvious: If $\Psi(U) \subset V$ for some $\Psi \in \mathscr{D}$ then $c(U) \leq c(V)$. If we define for an arbitrary open set $U \subset \mathbb{C}^n$

$$c(U) = \sup\{c(V) \mid V \text{ open bounded } V \subset U\} ,$$

then it follows immediately that

$$c(B^2(1) \times \mathbb{C}^{n-1}) = \pi .$$

Also we have

$$c(B^{2n}(1)) = \pi .$$

If U is an open bounded set of $\mathbb{C}^n$ and $\alpha \in \mathbb{R} \setminus (0)$ consider the set αU. For $H \in \mathscr{N}_{\text{reg}}(U)$ we define $H_\alpha \in \mathscr{N}_{\text{reg}}(\alpha U)$ by

$$H_\alpha(t, z) = \alpha^2 H(t, \alpha^{-1} z) .$$

We observe that $dH_\alpha(t, *) = \alpha\omega(X_{H_t}, (\alpha^{-1}z), *)$. Hence

$$X_{H_{\alpha,t}}(z) = \alpha X_{H_t}(\alpha^{-1}z) .$$

If z is a 1–periodic solution of

$$\dot{z} = X_{H_t}(z)$$

then $y := \alpha z$ is a 1–periodic solution of

$$\dot{y} = X_{H_{\alpha,t}}(y) .$$

Multiplying the Morse complex associated to H by α we obtain the Morse complex associated to H_α (for a suitable J_α). Hence

$$c(\alpha U) = \alpha^2 c(U) .$$

Consider the group $\mathscr{D}_{\text{conf}}$ consisting of all diffeomorphisms Ψ of $\mathbb{C}^n$ such that

$$\Psi(z) = \alpha z \text{ for all } |z| \text{ large}$$

for a suitable $\alpha \in \mathbb{R} \setminus \{0\}$ such that $\Psi^*\omega = \alpha^2\omega$.

Hence summing up we have

Theorem 7 *The map c has the following properties. For every nonempty open set U we have $c(U) \in (0, +\infty]$. If $\Psi(U) \subset V$ and $\Psi \in \mathscr{D}_{\text{conf}}$ with $\Psi^*\omega = \alpha^2\omega$ then*

- $c(V) \geq \alpha^2 c(U)$
- $c(B^{2n}(1) = c(B^{2n}(1) \times \mathbb{C}^{n-1}) = \pi$.

Hence c is a symplectic capacity for open sets in U in the sense of [3].

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