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# Perturbation by analytic discs along maximal real submanifolds of $C^N$

**Josip Globevnik**

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Ljubljana, Slovenia

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## 1 Introduction

Let $M$ be a real submanifold of $C^N$, $N \geqq 2$. $M$ is called **totally real** if for no $p \in M$ the tangent space $T_p M$ contains a complex line. If the dimension of $M$ equals $N$ then a totally real manifold $M$ is called **maximal real**.

We denote by $\Delta$ the open unit disc in $C$. A continuous map $f: \bar{\Delta} \to C^N$ which is holomorphic on $\Delta$ and which maps the unit circle $b\Delta$ to $M$ is called an **analytic disc with boundary in** $M$.

In a beautiful paper [Fo] Forstnerič studied the conditions on an immersed analytic disc $f$ with boundary in a maximal real submanifold $M \subset C^2$ which imply the existence of nearby analytic discs with boundaries in $M$. He introduced the **index of** $M$ **along** $f|b\Delta$ which can be calculated from a parametrization of $M$ along $f(b\Delta)$ and showed that if this index $m$ is at least one then there is a $2m+2$ parameter family of nearby analytic discs with boundaries in $M$. On the other hand, if $m \leqq 0$ then the only nearby discs are the ones of the form $f \circ \omega$ where $\omega$ is a conformal automorphism of $\Delta$.

In the present paper we try to understand the situation in general, i.e. when one drops the requirement about $f$ being an immersion and when one replaces $C^2$ by $C^N$. As one would expect the situation becomes considerably more complicated and it turns out that looking at a single integer is no more enough to tell whether there are nearby analytic discs with boundaries in $M$. The same is true in $C^2$ if one drops the requirement that $f$ is an immersion as already indicated by Forstnerič.

In a main step towards the result on the existence of nearby analytic discs Forstnerič found a normal form for the tangent bundle of a maximal real submanifold $M$ of $C^2$ along the boundary of an immersed analytic disc in terms of the index of $M$ along its boundary. In the present paper we find a normal form for the tangent bundle of a maximal real submanifold $M \subset C^N$ along a smooth map $p$: $b\Delta \to M$, which depends on $N$ integers. To get this normal form we do not need to assume that $p$ is the boundary map of an analytic disc and thus we are able to consider a slightly more general

*Problem 1.1* Given a maximal real submanifold $M$ of $C^N$ and a smooth map $p$: $b\Delta \to M$ find smooth maps $\varphi$: $\bar{\Delta} \to C^N$, holomorphic on $\Delta$ which are close to zero map and satisfy $(p+\varphi)\ (b\Delta) \subset M$.

In the special case when $p$ is the boundary map of an analytic disc Problem 1.1 reduces to finding nearby analytic discs with boundaries in $M$.

The required inclusion $(p+\varphi)(b\Delta) \subset M$ means that $(p+\varphi)(\zeta) \in M$ for each $\zeta \in b\Delta$. If we let $M$ vary with $\zeta$ this becomes a selection problem:

*Problem 1.2* Given a smooth map $\zeta \mapsto M(\zeta)$ where for each $\zeta \in b\Delta$, $M(\zeta)$ is a maximal real submanifold of $C^N$, and a smooth map $p$: $b\Delta \to C^N$ such that $p(\zeta) \in M(\zeta)$ for each $\zeta \in b\Delta$, find smooth maps $\varphi$: $\bar{\Delta} \to C^N$, holomorphic on $\Delta$ which are close to zero map and satisfy $(p+\varphi)(\zeta) \in M(\zeta)$ for each $\zeta \in b\Delta$.

In the present paper we study Problem 1.2. In the case when $M$ is a constant map Problem 1.2 reduces to Problem 1.1.

For each $\zeta \in b\Delta$, let $T(\zeta)$ be the tangent space to $M(\zeta)$ at $p(\zeta)$. In all cases that we will study the bundle $\{T(\zeta)$: $\zeta \in b\Delta\}$ will be trivial. We shall see that there is a vector Hilbert boundary value problem naturally associated with the bundle $\{T(\zeta)$: $\zeta \in b\Delta\}$ The partial indices $\kappa_1, \kappa_2, \ldots, \kappa_N$ of this Hilbert problem we call the **partial indices of $M$ along** $p$. The sum $\kappa = \kappa_1 + \cdots + \kappa_N$ is even and is called **the total index of $M$ along** $p$. In the special case of Problem 1.1 $\kappa/2$ equals the Forstnerič index of $M$ along $p$.

Generalizing a result of Forstnerič we shall prove that if all partial indices of $M$ along $p$ are nonnegative then the family of maps $\varphi$ solving Problem 1.2 depends on $\kappa + N$ real parameters. This is still true for small perturbations of $M$. Thus, if all partial indices are nonnegative the number of parameters in the general solution $\varphi$ depends only on the bundle $\{T(\zeta)$: $\zeta \in b\Delta\}$ and the seemingly simpler nature of Problem 1.1 compared to Problem 1.2 plays no role.

After finding our normal form for the tangent bundle we use the implicit mapping theorem in Banach spaces in the way Forstnerič does in his paper, suitably generalized to our situation. A reader who studied the paper of Forstnerič carefully will probably find Sects. 6, 7 and 8 of the present paper too long. However, since in some sense the present paper provides rather definitive solutions to certain problems the author found suitable to include detailed proofs.

The author is grateful to Salah Baouendi who, by asking a clever question about matrix factorization in the context of analytic discs provided the initial motivation for the work whose results are presented here.

## 2 The defining functions for $M$

Given an open set $V \subset C^N$ we shall denote by $\mathscr{C}^k(V)$ the Banach space of all real valued functions of class $\mathscr{C}^k$ on $V$ with the standard norm

$$\|r\|_k = \sum_{|\nu| \leqq k} \sup\{|D^\nu r(w)| : w \in V\}$$

where the derivatives are with respect to the real coordinates on $C^N$.

Let $0<\alpha<1$. Given a Banach space $X$ we denote by $\mathscr{C}^\alpha(b\Delta, X)$ the space of all functions $f: b\Delta \to X$ such that

$$\|f\|_\alpha = \sup_{\zeta\in b\Delta} \|f(\zeta)\| + \sup_{\xi,\eta\in b\Delta,\xi\neq\eta} \frac{\|f(\xi)-f(\eta)\|}{|\xi-\eta|^\alpha} < \infty .$$

The space $\mathscr{C}^\alpha(b\Delta, X)$ with the norm $\| \ \|_\alpha$ and with the usual operations is a Banach space. The spaces $\mathscr{C}^\alpha(b\Delta)=\mathscr{C}^\alpha(b\Delta, R)$ and $\mathscr{C}^\alpha_C(b\Delta)=\mathscr{C}^\alpha(b\Delta, C)$ are Banach algebras and $f\in\mathscr{C}^\alpha_C(b\Delta)$ if and only if $\Re f, \Im f\in\mathscr{C}^\alpha(b\Delta)$. Also, $f\in\mathscr{C}^\alpha(b\Delta, C^N)$ (or $f\in\mathscr{C}^\alpha(b\Delta, R^N)$)) if and only if the component functions belong to $\mathscr{C}^\alpha_C(b\Delta)$ (to $\mathscr{C}^\alpha(b\Delta)$, respectively) – in this case we say that $f$ is of class $\mathscr{C}^\alpha$. Given $f\in\mathscr{C}^\alpha(b\Delta)$ we denote by $\tilde{f}\in\mathscr{C}^\alpha(b\Delta)$ the conjugate function which satisfies $\tilde{f}(0)=0$. It is known that $f\mapsto\tilde{f}$ is a bounded linear map from $\mathscr{C}^\alpha(b\Delta)$ to itself. Finally, by $\mathscr{A}^\alpha$ we denote the subalgebra of $\mathscr{C}^\alpha_C(b\Delta)$ of those functions which extend holomorphically to $\Delta$.

Fix $\alpha$, $0<\alpha<1$. The smoothness in our Problem 1.2 will be $\mathscr{C}^\alpha$-smoothness: we assume that $p$ is of class $\mathscr{C}^\alpha$, that the manifolds $M(\zeta)$ change $\mathscr{C}^\alpha$-smoothly with $\zeta\in b\Delta$ and the maps we seek are of class $\mathscr{C}^\alpha$ on $b\Delta$ and of small $\mathscr{C}^\alpha$-norm. To say that the manifolds $M(\zeta)$ change $\mathscr{C}^\alpha$-smoothly with $\zeta$ means that the defining functions for $M(\zeta)$ near $p(\zeta)$ change $\mathscr{C}^\alpha$-smoothly with $\zeta$. To state this precisely we need functions belonging to $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))$ where $B\subset C^N$ is a small open ball centered at the origin. By definition these are maps $\zeta\mapsto h(\zeta)$ from $b\Delta$ to $\mathscr{C}^2(B)$ which vary $\mathscr{C}^\alpha$-smoothly with $\zeta\in b\Delta$. Often we will identify such an $h$ with the map $H: b\Delta\times B\to R$ defined by $H(\zeta, w)=h(\zeta)(w)$.

Problem 1.2 is a local problem in the sense that the maps $\varphi$ we seek have small $\mathscr{C}^\alpha$-norm. Thus it matters only how the manifolds $M(\zeta)$ change with $\zeta$ near $p(\zeta)$, that is, how the defining functions for $M(\zeta)$ near $p(\zeta)$ change with $\zeta$. Thus we assume that

$$\left\{\begin{array}{l} p: b\Delta\to C^N \text{ is a map of class } \mathscr{C}^\alpha \text{ and there are an open ball } B\subset C^N \\ \text{centered at the origin and maps } r_j\in\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B)),\ 1\leqq j\leqq N, \\ \text{such that for each } \zeta\in b\Delta \\ \text{(i) } M(\zeta)=\{w\in p(\zeta)+B:\ r_j(\zeta)(w-p(\zeta))=0\,(1\leqq j\leqq N)\} \\ \text{(ii) } r_j(\zeta)(0)=0\,(1\leqq j\leqq N), \text{ that is, } p(\zeta)\in M(\zeta) \\ \text{(iii) } dr_1(\zeta)\wedge\cdots\wedge dr_N(\zeta)\neq 0 \text{ on } B\,. \end{array}\right\} \tag{2.1}$$

In addition, we assume that

$$\text{for each } \zeta\in b\Delta \text{ the tangent space to } M(\zeta) \text{ at } p(\zeta) \text{ is totally real}\,. \tag{2.2}$$

To express this in two ways suitable for our purpose we need a simple proposition. Denote by $\langle\,|\,\rangle$ the Hermitian inner product on $R^{2N}\equiv C^N$. We use the term R-orthogonality for the orthogonality with respect to $\Re\langle\,|\,\rangle$ and use $\perp$ to denote R-orthogonal complement.

**Proposition 2.1** *Let $X$ be a totally real,* R*-linear subspace of $C^N$ of dimension $N$. Then $X^{\perp}$ is totally real.*

*Proof.* Suppose that $y, iy \in X^{\perp}$. Then for each $x \in X$, $\Re\langle y|x\rangle = \Re\langle iy|x\rangle = 0$ which implies that $\langle y|x\rangle = 0$ $(x \in X)$. It follows that $\langle y|ix\rangle = -i\langle y|x\rangle = 0$ $(x \in X)$ so $\langle y|z+iw\rangle = 0$ $(z, w \in X)$. Since $X$ is totally real, $X \cap iX = \{0\}$. Since $X$ has dimension $N$ it follows that $X \oplus iX = C^N$ so $\langle y|w\rangle = 0$ $(w \in C^N)$ which implies that $y = 0$. This completes the proof.

For each $\zeta \in b\Delta$, let $T(\zeta)$ be the tangent space to $M(\zeta)$ at $p(\zeta)$. By (2.1) (iii) for each $w \in B$ and for each $\zeta \in b\Delta$ the differentials $dr_j(\zeta)(0)$ are R-linearly independent. In particular, for each $\zeta \in b\Delta$ the normal vectors $v_j(\zeta) = (\text{grad}\, r_j(\zeta))(0)$ to $\{w \in B : r_j(\zeta)(w) = 0\}$ at the origin are R-linearly independent. Since $T(\zeta)$ is totally real Proposition 2.1 implies that $T(\zeta)^{\perp}$, the real span (i.e. the set of all linear combinations with real coefficients) of $v_1(\zeta), \ldots, v_N(\zeta)$, is totally real, that is, $v_1(\zeta), \ldots, v_N(\zeta)$ are linearly independent. Since the real span of a linearly independent set of vectors is totally real Proposition 2.1 implies that $T(\zeta)$ is totally real if and only if the vectors $v_j(\zeta)$, $1 \leqq j \leqq N$, are linearly independent.

Note that the maps $\zeta \mapsto v_j(\zeta)$, $1 \leqq j \leqq N$, are of class $\mathscr{C}^{\alpha}$. Perform the Gram–Schmidt R-orthogonalization with the vectors $v_1(\zeta), \ldots, v_N(\zeta)$, $iv_1(\zeta), \ldots, iv_N(\zeta)$ and call the last $N$ vectors obtained $A_1(\zeta), \ldots, A_N(\zeta)$. Then the maps $\zeta \mapsto A_j(\zeta)$, $1 \leqq j \leqq N$, are of class $\mathscr{C}^{\alpha}$ and for each $\zeta \in b\Delta$ the real span of $A_1(\zeta), \ldots, A_N(\zeta)$ equals $T(\zeta)$.

## 3 Partial indices of Hilbert boundary problem for vector functions

Denote by $Gl(N, C)$ the group of all invertible $N \times N$ matrices with complex entries and by $Gl(N, R)$ the group of all such matrices with real entries.

Given a continuous map $B : b\Delta \mapsto Gl(N, C)$ the Hilbert boundary value problem consists of finding a continuous map $\Psi^+ : \bar{\Delta} \to C^N$, holomorphic on $\Delta$, and a continuous map $\Psi^- : C \backslash \Delta \to C^N$, holomorphic on $C \backslash \bar{\Delta}$ and having at most a pole at infinity such that

$$\Psi^+(\zeta) = B(\zeta)\Psi^-(\zeta) \quad (\zeta \in b\Delta)\,. \tag{3.1}$$

If $\Psi : U \to C^N$ is holomorphic in a neighbourhood $U$ of $\infty$ and has at most a pole at $\infty$ call the **order** of (zero of) $\Psi$ at $\infty$ the integer $k$ such that in a neighbourhood of $\infty$ we have $\Psi(z) = (1/z)^k \Psi_0(z)$ where $\Psi_0$ is holomorphic at $\infty$ and $\Psi_0(\infty) \neq 0$.

It is known that for any map $B : b\Delta \to Gl(N, C)$ of class $\mathscr{C}^{\alpha}$, $0 < \alpha < 1$, there are systems $\Phi_j = (\Phi_j^+, \Phi_j^-)$, $1 \leqq j \leqq N$, of solutions of (3.1) with the following properties

(i) the matrix $\Phi^+(\zeta) = [\Phi_1^+(\zeta), \ldots, \Phi_N^+(\zeta)]$ is invertible for each $\zeta \in \bar{\Delta}$

(ii) the matrix $\Phi^-(\zeta) = [\Phi_1^-(\zeta), \ldots, \Phi_N^-(\zeta)]$ is invertible for each $\zeta \in C \backslash \Delta$

(iii) the order $\kappa$ of $\det[\Phi^-(\zeta)]$ at $\infty$ is equal to the sum of the orders $\kappa_j$ of the columns $\Phi_j^-$ at $\infty$, $1 \leqq j \leqq N$.

Any system of solutions $\Phi_j$ of Problem 3.1 with the properties (i)–(iii) is called a **canonical system** and the matrix with columns $\Phi_j$ is called a **canonical matrix** of Problem 3.1. It is a fundamental property of the canonical systems that the integers $\kappa_j$ ordered so that $\kappa_1 \geqq \kappa_2 \geqq \cdots \geqq \kappa_N$ are the same for all canonical systems. They are called **the partial indices** of Problem 3.1 or **the partial indices of the map** $\zeta \mapsto B(\zeta)$. The sum $\kappa = \kappa_1 + \cdots \kappa_N$ is called **the total index** of Problem 3.1 or **the total index of the map** $B$. It is known that if $0<\alpha<1$ and if $B$ is of class $\mathscr{C}^\alpha$ then the solutions of (3.1) are of class $\mathscr{C}^\alpha$ (see [Ve] or [Ve1] as a general reference).

Let $\Phi$ be a canonical matrix of Problem 3.1 and let

$$\Lambda(\zeta) = \begin{pmatrix} \zeta^{\kappa_1} & 0 & \cdots & \cdots & 0 \\ 0 & \zeta^{\kappa_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \zeta^{\kappa_N} \end{pmatrix}. \tag{3.2}$$

By the property (iii) of $\Phi$ the map $\zeta \mapsto \Phi^-(\zeta)\Lambda(\zeta)$ is holomorphic at $\infty$ and its value at $\infty$ is invertible. Since $\Phi^+(\zeta) = B(\zeta)\Phi^-(\zeta)\Lambda(\zeta)\Lambda(\zeta)^{-1}$ $(\zeta \in b\Delta)$ properties (i) and (ii) imply that $B$ admits a factorization

$$\left\{ \begin{array}{l} B(\zeta) = F^+(\zeta)\Lambda(\zeta)F^-(\zeta) \ (\zeta \in b\Delta) \text{ where } F^+ : \bar{\Delta} \to Gl(N, C) \text{ is} \\ \text{continuous and holomorphic on } \Delta \text{ and } F^- : [C \cup \{\infty\}] \setminus \Delta \to \\ Gl(N, C) \text{ is continuous and holomorphic on } [C \cup \{\infty\}] \setminus \bar{\Delta}\,. \end{array} \right\} \tag{3.3}$$

It can be directly verified [CG, p.9] that if $\Lambda$ in (3.3) is of the form (3.2) with $\kappa_1 \geqq \kappa_2 \geqq \cdots \geqq \kappa_N$ then $\Lambda$ does not depend on a particular factorization, that is, it is the same for all factorizations of $B$ of the form (3.3). Thus we can equivalently define the partial indices of $B$ as the exponents $\kappa_j$, $\kappa_1 \geqq \kappa_2 \geqq \cdots \geqq \kappa_N$, in (3.2) in (any) factorization of $B$ of the form (3.3) provided they exist – and we know they do if $B$ is of class $\mathscr{C}^\alpha$, $0<\alpha<1$.

## 4 Partial indices of $M$ along $p$

Given a matrix $Q$ we denote by $\bar{Q}$ the matrix obtained from $Q$ by replacing each entry with its complex conjugate and we write $\Re Q = \frac{1}{2}(Q + \bar{Q})$.

Let $M$ and $p$ in Problem 1.2 satisfy (2.1) and (2.2). For each $\zeta \in b\Delta$ the tangent space $T(\zeta)$ to $M(\zeta)$ at $p(\zeta)$ is totally real which is equivalent to the gradients $v_j(\zeta) = (\text{grad } r_j(\zeta))(0)$, $1 \leqq j \leqq N$, being linearly independent.

To obtain our normal form of the bundle $\{T(\zeta), \zeta \in b\Delta\}$ we will have to solve the following

*Problem 4.1* Find continuous maps $H_j : \bar{\Delta} \setminus \{0\} \to C^N$, $1 \leqq j \leqq N$, holomorphic on $\Delta \setminus \{0\}$ and having at most poles at 0 such that for each $\zeta \in b\Delta$ the real span of $H_1(\zeta), \ldots, H_N(\zeta)$ is $T(\zeta)$.

For each $\zeta\in b\Delta$, $T(\zeta)$ is the R-orthogonal complement of the real span of $v_1(\zeta),\dots,v_N(\zeta)$. Denote by $G(\zeta)\in Gl(N,C)$ the matrix with rows $v_j(\zeta)$, $1\leqq j\leqq N$. To say that the real span of the columns $H_1(\zeta),\dots,H_N(\zeta)$ of the matrix $H(\zeta)$ is $T(\zeta)$ is the same as to say that $H(\zeta)\in Gl(N,C)$ and

$$\Re[\overline{G(\zeta)}H(\zeta)]=0 \quad (\zeta\in b\Delta) \tag{4.1}$$

that is,

$$H(\zeta)=-\overline{G(\zeta)^{-1}}G(\zeta)\overline{H(\zeta)} \ (\zeta\in b\Delta)\,. \tag{4.2}$$

Alternatively, we can start with a map $A:b\Delta\to Gl(N,C)$ such that for each $\zeta\in b\Delta$ the real span of the columns of $A$ is $T(\zeta)$. To say that the real span of the columns of $H(\zeta)$ is $T(\zeta)$ is the same as to say that there is a matrix $V(\zeta)\in Gl(N,R)$ such that $H(\zeta)\,V(\zeta)=A(\zeta)$ which is the same as to say that $H(\zeta)\in Gl(N,C)$ and

$$H(\zeta)=A(\zeta)\overline{A(\zeta)^{-1}H(\zeta)} \ (\zeta\in b\Delta)\,. \tag{4.3}$$

Since each row of $G(\zeta)$ is R-orthogonal to each column of $A(\zeta)$ we have $\Re[\overline{G(\zeta)}A(\zeta)]=0$, that is, $G(\zeta)\overline{A(\zeta)}=-\overline{G(\zeta)}A(\zeta)$ so for each $\zeta\in b\Delta$ the matrix $-\overline{G(\zeta)^{-1}}G(\zeta)$ in (4.2) coincides with the matrix $A(\zeta)\overline{A(\zeta)^{-1}}$ in (4.3).

Denote

$$B(\zeta)=A(\zeta)\overline{A(\zeta)^{-1}}=-\overline{G(\zeta)^{-1}}G(\zeta) \ (\zeta\in b\Delta) \tag{4.4}$$

and notice that for each $\zeta$, $B(\zeta)$ depend only on $T(\zeta)$ and not on a particular choice of the basis $A(\zeta)$ of $T(\zeta)$ or on a particular choice of defining functions. A different basis $\mathscr{A}(\zeta)$ will satisfy $\mathscr{A}(\zeta)=A(\zeta)V(\zeta)$ with $V$ real so $\mathscr{A}(\zeta)\overline{\mathscr{A}(\zeta)^{-1}}$ $=A(\zeta)V(\zeta)\overline{V(\zeta)^{-1}A(\zeta)^{-1}}=A(\zeta)\overline{A(\zeta)^{-1}}$.

Let $\kappa_1\geqq\kappa_2\geqq\cdots\geqq\kappa_N$ be the partial indices of the map $B$ in (4.5). Since $B$ is of class $\mathscr{C}^\alpha$ we know that $B(\zeta)=F^+(\zeta)\Lambda(\zeta)F^-(\zeta)$ $(\zeta\in b\Delta)$ where $F^+$, $F^-$ are as in (3.3) and $\Lambda(\zeta)$ is the matrix (3.2). Suppose that we choose a different coordinate system in $C^N$. Let $e_1,\dots,e_n$ be the standard basis, let $f_1,\dots,f_n$ be the new basis and let $e_j=\sum_{k=1}^N u_{kj}f_k (1\leqq j\leqq N)$. Then $U=[u_{k_j}]\in Gl(N,C)$ and in the new coordinate system the basis matrix $A(\zeta)$ is replaced by $UA(\zeta)$. Since $A(\zeta)\overline{A(\zeta)^{-1}}=F^+(\zeta)\Lambda(\zeta)F^-(\zeta)$ $(\zeta\in b\Delta)$ it follows that $[U{\cdot}A(\zeta)]\,\overline{[UA(\zeta)]^{-1}}=$ $[UF^+(\zeta)]\Lambda(\zeta)[F^-(\zeta)\overline{U^{-1}}]$ $(\zeta\in b\Delta)$ which, by the uniqueness of $\Lambda(\zeta)$ in such a factorization implies that the partial indices of $\zeta\mapsto[UA(\zeta)]\,\overline{[UA(\zeta)]^{-1}}$ coincide with the partial indices of $\zeta\mapsto A(\zeta)\overline{A(\zeta)^{-1}}$. This shows that the partial indices of the map $B$ depend only on the bundle $\{T(\zeta)\colon\zeta\in b\Delta\}$ and thus we can make the following

**Definition 4.1** *Let $M$, $p$ and $r_j$, $1\leqq j\leqq N$, be as in* (2.1) *and assume that* (2.2) *holds. For each $\zeta\in b\Delta$ let $T(\zeta)$ be the tangent space to $M(\zeta)$ at $p(\zeta)$. For each $\zeta\in b\Delta$ let $G(\zeta)\in Gl(N,C)$ be the matrix whose rows are gradients* $(\operatorname{grad} r_j(\zeta))(0)$, $1\leqq j\leqq N$, *and let $A(\zeta)$ be a matrix such that the real span of its columns is $T(\zeta)$.*

*The partial indices of the map* $B: b\Delta \to Gl(N, C)$ *given by*

$$B(\zeta) = A(\zeta)\overline{A(\zeta)^{-1}} = -\overline{G(\zeta)^{-1}}G(\zeta) \quad (\zeta \in b\Delta)$$

*are called* **the partial indices of** $M$ **along** $p$ *and their sum is called* **the total index of** $M$ **along** $p$.

Let $B$ be as above and let $\Phi$ be a canonical matrix of the Hilbert problem 3.1. We have $\Phi^+(\zeta) = A(\zeta)\overline{A(\zeta)^{-1}}\Phi^-(\zeta)$ $(\zeta \in b\Delta)$ so $\det \Phi^+(\zeta) = \det A(\zeta)(\overline{\det A(\zeta)})^{-1}$ $\det \Phi^-(\zeta)$ $(\zeta \in b\Delta)$. Denote by $W$ the winding number around the origin. Since $\det \Phi^+$ has no zero on $\Delta$ and $\det \Phi^-$ has no zero on $C \backslash \Delta$ and since in a neighbourhood of $\infty$ we have $\det \Phi^-(\zeta) = (1/\zeta)^\kappa \Psi(\zeta)$ where $\kappa$ is the total index of $B$ and where $\Psi$ is holomorphic at $\infty$ and $\Psi(\infty) \neq 0$ it follows that $0 = W(\det A) + W(\det A) - \kappa$ so $\kappa = 2W(\det A)$ is even. Thus in the special case when $M$ is a constant map $\kappa/2$ equals the Forstnerič index of $M$ along $p$.

The preceding discussion shows that the partial indices of $M$ along $p$ depend only on the bundle $\{T(\zeta): \zeta \in b\Delta\}$ and that in fact we can make the following general

**Definition 4.2** *Let* $\{L(\zeta): \zeta \in b\Delta\}$ *be a bundle where for each* $\zeta \in b\Delta$, $L(\zeta)$ *is a N-dimensional totally real* R*-linear subspace of* $C^N$ *and for which there is a map* $A: b\Delta \to Gl(N, C)$ *of class* $\mathscr{C}^\alpha$, $0 < \alpha < 1$, *such that for each* $\zeta \in b\Delta$ *the columns of* $A(\zeta)$ *form a basis of* $L(\zeta)$. *The partial indices of the map* $\zeta \mapsto A(\zeta)\overline{A(\zeta)^{-1}}$ *are called* **the partial indices of the bundle** $L$ *and their sum is called* **the total index of** $L$.

## 5 A normal form for the bundle $\{T(\zeta): \zeta \in b\Delta\}$

Assume that $\kappa_1 \geqq \kappa_2 \geqq \cdots \kappa_N$, are partial indices of a map $B: b\Delta \to Gl(N, C)$ of class $\mathscr{C}^\alpha$, $0 < \alpha < 1$. We already know that $B(\zeta) = F^+(\zeta)\Lambda(\zeta)F^-(\zeta)$ $(\zeta \in b\Delta)$ where $F^+(\zeta)$, $F^-(\zeta)$ are as in (3.3) and where $\Lambda(\zeta)$ is the matrix (3.2). We now show that if $B$ is of a special form, $B(\zeta) = A(\zeta)\overline{A(\zeta)^{-1}}$ $(\zeta \in b\Delta)$, then there is a factorization of $B$ which reflects this:

**Lemma 5.1** *Let* $A: b\Delta \to Gl(N, C)$ *be a map of class* $\mathscr{C}^\alpha$, $0 < \alpha < 1$, *and let* $\kappa_1 \geqq \kappa_2 \geqq \cdots \geqq \kappa_N$ *be the partial indices of the map* $\zeta \mapsto A(\zeta)\overline{A(\zeta)^{-1}}$. *There is a map* $\Theta: \Delta \to Gl(N, C)$ *of class* $\mathscr{C}^\alpha$, *holomorphic on* $\Delta$ *and such that*

$$A(\zeta)\overline{A(\zeta)^{-1}} = \Theta(\zeta)\Lambda(\zeta)\overline{\Theta(\zeta)^{-1}} \quad (\zeta \in b\Delta)$$

*where* $\Lambda(\zeta)$ *is the matrix* (3.2).

*Proof.* Write $B(\zeta) = A(\zeta)\overline{A(\zeta)^{-1}}$ $(\zeta \in b\Delta)$ and observe that $B$ is of class $\mathscr{C}^\alpha$ and that $B(\zeta) = \overline{B(\zeta)^{-1}}$ $(\zeta \in b\Delta)$. Suppose that $(\Psi^+, \Psi^-)$ is a solution of the Hilbert Problem 3.1. Define

$$\Psi_*^+(\zeta) = \overline{\Psi^-(1/\bar{\zeta})} \quad (\zeta \in \bar{\Delta} \backslash \{0\})$$

$$\Psi_*^-(\zeta) = \overline{\Psi^+(1/\bar{\zeta})} \quad (\zeta \in [C \cup \{\infty\}] \backslash \Delta).$$

Then $\Psi_*^+$ is holomorphic on $\Delta\setminus\{0\}$ and $\Psi_*^-$ is holomorphic on $\overline{[C\cup\{0\}]\setminus\bar{\Delta}}$. Conjugating the equality $\Psi^+(\zeta)=B(\zeta)\,\Psi^-(\zeta)$ $(\zeta\in b\Delta)$ we get $\overline{\Psi^+(\zeta)}=B(\zeta)^{-1}\,\overline{\Psi^-(\zeta)}$ $(\zeta\in b\Delta)$ which implies that

$$\Psi_*^+(\zeta)=B(\zeta)\Psi_*^-(\zeta)\quad(\zeta\in b\Delta)$$

so $(\Psi_*^+, \Psi_*^-)$ is a solution of the Hilbert problem 3.1 in a generalized sense as it is not necessarily holomorphic at 0. The proof is now completed by the following fact, proved by Vekua [Ve, p. 150] to solve the Riemann–Hilbert problem for vector functions

**Lemma 5.2** (N.P. Vekua) *Let* $0<\alpha<1$ *and let* $B: b\Delta\to Gl(N, C)$ *be a map of class* $\mathscr{C}^\alpha$, *such that* $\overline{B(\zeta)}=B(\zeta)^{-1}$ $(\zeta\in b\Delta)$. *Let* $\kappa_1\geqq\kappa_2\geqq\cdots\geqq\kappa_N$ *be the partial indices of the Hilbert problem* 3.1 *and let* $\Lambda(\zeta)$ *be the matrix* (3.2). *There is a canonical matrix* $\Omega$ *of Problem* 3.1 *such that* $\Omega_*^+(\zeta)=\Omega^+(\zeta)\Lambda(\zeta)$ $(\zeta\in b\Delta)$.

To complete the proof of Lemma 5.1 observe that $\Omega^+(\zeta)\Lambda(\zeta)=\Omega_*^+(\zeta)=B(\zeta)\Omega_*^-(\zeta)=B(\zeta)\overline{\Omega^+(\zeta)}$ $(\zeta\in b\Delta)$ and put $\Theta(\zeta)=\Omega^+(\zeta)$ $(\zeta\in\bar{\Delta})$.

If $M$, $p$ and $r_j$, $1\leqq j\leqq N$, are as in (2.1), if (2.2) holds and if for each $\zeta\in b\Delta$ we denote by $T(\zeta)$ the tangent space to $M(\zeta)$ at $p(\zeta)$ we know from the end of Sect. 2 that there is a map $A: b\Delta\to Gl(N, C)$ such that for each $\zeta\in b\Delta$ the real span of the columns of $A(\zeta)$ is $T(\zeta)$. We defined the partial indices of $M$ along $p$ as the partial indices of the map $\zeta\mapsto A(\zeta)\overline{A(\zeta)}^{-1}$. Thus the following theorem will provide a normal form for the bundle $\{T(\zeta)\colon \zeta\in b\Delta\}$.

**Theorem 5.1** *Let* $\{L(\zeta)\colon \zeta\in b\Delta\}$ *be a bundle where for each* $\zeta\in b\Delta$, $L(\zeta)$ *is a* $N$-*dimensional, totally real* R-*linear subspace of* $C^N$ *and for which there is a map* $A: b\Delta\to Gl(N, C)$ *of class* $\mathscr{C}^\alpha$ *such that for each* $\zeta\in b\Delta$ *the real span of the columns of* $A(\zeta)$ *is* $L(\zeta)$. *Renumber the partial indices* $\kappa_1, \kappa_2, \ldots, \kappa_N$ *of the bundle* $L$ *so that the first* $2\ell$ *are odd,* $\kappa_j=2m_j+1$ $(1\leqq j\leqq 2\ell)$ *and the remaining* $N-2\ell$ *are even,* $\kappa_j=2m_j$ $(2\ell+1\leqq j\leqq N)$. *There is a map* $\Theta:\bar{\Delta}\to Gl(N, C)$ *of class* $\mathscr{C}^\alpha$, *holomorphic on* $\Delta$ *and such that for every* $\zeta\in b\Delta$ *the real span of the columns of the matrix*

$$\Theta(\zeta)\begin{pmatrix}\zeta^{m_1} & 0 & 0 & \cdots & 0\\ 0 & \zeta^{m_2} & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 0 & \zeta^{m_N}\end{pmatrix}\begin{pmatrix}K(\zeta) & 0 & \cdots & 0 & O\\ \cdots & \cdots & \cdots & \cdots & O\\ 0 & \cdots & 0 & K(\zeta) & O\\ O & O & O & O & I\end{pmatrix}\tag{5.1}$$

*is* $L(\zeta)$. *In the third factor all entries vanish except the ones in the first* $\ell$ *diagonal* $2\times 2$ *matrix entries* $K(\zeta)$ *where*

$$K(\zeta)=\begin{pmatrix}1+\zeta & i(1-\zeta)\\ -i(1-\zeta) & 1+\zeta\end{pmatrix}$$

*and the last* $N-2\ell$ *diagonal entries which all equal* 1.

*Remark 5.1* Note that $K(\zeta)\in Gl(2, C)$ $(\zeta\in b\Delta)$ and

$$K(\zeta)\overline{K(\zeta)^{-1}}=\begin{pmatrix}\zeta & 0\\ 0 & \zeta\end{pmatrix}\quad(\zeta\in b\Delta)\,.$$

*Proof of Theorem 5.1* Observe first that since the total index of $L$ is even the number of odd partial indices is even so one can renumber the partial indices of $L$ as required in the theorem. Choose a map $A: b\Delta\to Gl(N, C)$ of class $\mathscr{C}^\alpha$ such that for each $\zeta\in b\Delta$ the real span of the columns of $A(\zeta)$ is $L(\zeta)$. By Lemma 5.1 there is a map $\Theta:\bar{\Delta}\to Gl(N, C)$ of class $\mathscr{C}^\alpha$, holomorphic on $\Delta$ and such that

$$A(\zeta)\overline{A(\zeta)^{-1}}=\Theta(\zeta)\Lambda(\zeta)\overline{\Theta(\zeta)^{-1}}\quad(\zeta\in b\Delta)\tag{5.2}$$

where $\Lambda(\zeta)$ is the matrix (3.2).

Interchanging $j$'th and $k$'th element on the diagonal of $\Lambda(\zeta)$ is the same as to replace $\Lambda(\zeta)$ by $E_{jk}\Lambda(\zeta)E_{jk}$ where $E_{jk}$ is the $N\times N$ matrix whose elements $e_{ps}$ satisfy $e_{ss}=1$ $(s\neq j,\ s\neq k)$, $e_{jj}=e_{kk}=0$, $e_{jk}=e_{kj}=1$ and whose all other entries vanish. Since $E_{jk}^2=I$, (5.2) implies that $A(\zeta)\overline{A(\zeta)^{-1}}=[\Theta(\zeta)E_{jk}]$ $[E_{jk}\Lambda(\zeta)E_{jk}]\,\overline{[\Theta(\zeta)E_{jk}]^{-1}}$. Since the map $\zeta\mapsto\Theta(\zeta)E_{jk}$ maps $\bar{\Delta}$ to $Gl(N, C)$, is of class $\mathscr{C}^\alpha$ and holomorphic on $\Delta$ it follows that (5.2) still holds if we reorder the diagonal elements in $\Lambda(\zeta)$ as required in the theorem.

By the properties of $K(\zeta)$ we have

$$\Lambda(\zeta)=Q(\zeta)\overline{Q(\zeta)^{-1}}\quad(\zeta\in b\Delta)\tag{5.3}$$

where $Q(\zeta)$ is the product of the last two factors in (5.1). Let $H(\zeta)=\Theta(\zeta)Q(\zeta)$. Then $H(\zeta)\in Gl(N, C)$ $(\zeta\in b\Delta)$ and (5.2) and (5.3) imply that $H(\zeta)=A(\zeta)\overline{A(\zeta)^{-1}\mathrm{H}(\zeta)}$ $(\zeta\in b\Delta)$, that is, (4.3) holds, which, by the discussion preceding (4.3) implies that for each $(\zeta\in b\Delta)$ the real span of the columns of $H(\zeta)$ is $L(\zeta)$. This completes the proof.

*Remark 5.2* Note that Theorem 5.1 provides a solution of Problem 4.1.

We conclude this section by a remark about matrix factorization which we will not need in the sequel.

*Remark 5.3* Let $A: b\Delta\to Gl(N, C)$ be a map of class $\mathscr{C}^\alpha$. The following are equivalent:

(i) there is a map $V: b\Delta\to Gl(N, R)$ of class $\mathscr{C}^\alpha$ such that the map $\zeta\mapsto H(\zeta)=A(\zeta)V(\zeta)$ $(\zeta\in b\Delta)$ has a holomorphic extension to $\Delta$.

(ii) all partial indices of the map $\zeta\mapsto A(\zeta)\overline{A(\zeta)^{-1}}$ equal 0.

To see this observe first that (i) is equivalent to finding $H: b\Delta\to Gl(N, C)$ of class $\mathscr{C}^\alpha$ which extends holomorphically to $\Delta$ and which satisfies $A(\zeta)\overline{A(\zeta)^{-1}}=H(\zeta)\overline{H(\zeta)^{-1}}$ $(\zeta\in b\Delta)$. By Lemma 5.1, (ii) implies the existence of such an $H$. Conversely, given (i) the factorization can be rewritten in the form $A(\zeta)\overline{A(\zeta)^{-1}}=H(\zeta)I\overline{H(1/\bar{\zeta})^{-1}}$ $(\zeta\in b\Delta)$, where $I$ is the identity matrix, that is, $A\overline{A^{-1}}$ admits a factorization (3.3). Since under the assumption that

$\kappa_1 \geqq \cdots \geqq \kappa_N$ the matrix $\Lambda(\zeta)$ in (3.3) does not depend on the particular factorization (ii) follows.

## 6 Parametrization of perturbed selections

Let $M$ and $p$ be as in (2.1) and let (2.2) hold. $M$ is defined by a map $r \in \mathscr{C}^\alpha(b\Delta, \mathscr{C}^\alpha(B))^N$ and so we write $M = M_r$. Small perturbations of $M$ will be of the form $M_\rho$ where $\rho \in \mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))^N$ is close to $r$. In particular, for each $\zeta \in b\Delta$ we have

$$M_\rho(\zeta) = \{w \in p(\zeta) + B: \rho_j(\zeta)(w - p(\zeta)) = 0 \ (1 \leqq j \leqq N)\} .$$

It is easy to see that (2.1) and (2.2) imply that there are a neighbourhood $V$ of $r$ in $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))^N$ and an open ball $B_1$ centered at the origin such that for each $\rho \in V$ and for each $\zeta \in b\Delta$ the set $M_\rho(\zeta) \cap [p(\zeta) + B_1]$ is a nonempty totally real submanifold of $p(\zeta) + B_1$. For each $\zeta$, $M_\rho(\zeta) \cap [p(\zeta) + B_1]$ is a $\mathscr{C}^2$ perturbation of $M_r(\zeta) \cap [p(\zeta) + B_1]$ and $M_\rho(\zeta)$ change $\mathscr{C}^\alpha$-smoothly with $\zeta$.

For each $\rho \in \mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))^N$ sufficiently close to $r$ we want to describe maps $\varphi : b\Delta \to C^N$ of class $\mathscr{C}^\alpha$, with small $\alpha$-norm, which extend holomorphically to $\Delta$ and which satisfy $(p + \varphi)(\zeta) \in M_\rho(\zeta)$ $(\zeta \in b\Delta)$, which, in terms of the defining functions means that

$$\rho_j(\zeta)(\varphi(\zeta)) = 0 \quad (1 \leqq j \leqq N, \zeta \in b\Delta) . \tag{6.1}$$

To do this we first find all maps $\varphi : b\Delta \to C^N$, of class $\mathscr{C}^\alpha$, with small $\alpha$-norm, which satisfy (6.1), and we want to describe these maps in such a way that it will be easy to check later which of them extend holomorphically to $\Delta$.

Let $X_1, X_2, \ldots, X_N : b\Delta \to C^N$, be any maps of class $\mathscr{C}^\alpha$ such that for each $\zeta \in b\Delta$ the real span of $X_1(\zeta), \ldots, X_N(\zeta)$ is $T(\zeta)$, the tangent space to $M(\zeta)$ at $p(\zeta)$. We know from either Sect. 2 or from Sect. 5 that such $n$-tuples exist. Generalizing approach of Forstnerič [Fo, p. 20], for each $u \in (u_1, \ldots, u_N)$, $f = (f_1, \ldots, f_N)$ in $\mathscr{C}^\alpha(b\Delta)^N$ define

$$G(u,f)(\zeta) = \sum_{j=1}^{N} u_j(\zeta) X_j(\zeta) + i \sum_{j=1}^{N} (f_j(\zeta) + i\tilde{f}_j(\zeta)) X_j(\zeta) \quad (\zeta \in b\Delta) .$$

Since for each $\zeta \in b\Delta$ the vectors $X_j(\zeta)$, $iX_j(\zeta)$, $1 \leqq j \leqq N$, form a real basis of $C^N$ and since $g \mapsto \tilde{g}$ is a bounded linear map of $\mathscr{C}^\alpha(b\Delta)$ to itself it follows that $G: \mathscr{C}^\alpha(b\Delta)^N \times \mathscr{C}^\alpha(b\Delta)^N \to \mathscr{C}^\alpha_C(b\Delta)^N$ is a linear isomorphism (that is, a bounded linear bijection with bounded inverse) which implies that given neighbourhoods $U_1, U_2$ of zero in $\mathscr{C}^\alpha(b\Delta)^N$, $G(U_1, U_2)$ is a neighbourhood of zero in $\mathscr{C}^\alpha_C(b\Delta)^N$. Thus, each of the maps $\varphi$ we want to describe is of the form $\varphi = G(u,f)$ with $u, f$ belonging to sufficiently small neighbourhoods $U_1$, $U_2$ of 0 in $\mathscr{C}^\alpha(b\Delta)^N$. Thus, provided that $U_1$, $U_2$ are sufficiently small we will have to find all $u \in U_1, f \in U_2$ such that

$$\rho_j(\zeta)(G(u,f)(\zeta)) = 0 \quad (1 \leqq j \leqq N, \zeta \in b\Delta) \tag{6.2}$$

that is,

$$\Phi(\rho, u, f)(\zeta) = 0 \quad (\zeta \in b\Delta) \tag{6.3}$$

where $\rho = (\rho_1, \ldots, \rho_N)$ and $\Phi(\rho, u, f)(\zeta) = \rho(\zeta)(G(u, f)(\zeta))$ $(\zeta \in b\Delta)$. Since $p(\zeta) \in M(\zeta)$ $(\zeta \in b\Delta)$ we have

$$\Phi(r, 0, 0)(\zeta) = 0 \quad (\zeta \in b\Delta) . \tag{6.4}$$

**Theorem 6.1** *Let $X_1, X_2, \ldots, X_N : b\Delta \to C^N$ be maps of class $\mathscr{C}^\alpha$ such that for each $\zeta \in b\Delta$ the real span of $X_1(\zeta), \ldots, X_N(\zeta)$ is $T(\zeta)$. There are a neighbourhood $W$ of 0 in $\mathscr{C}^\alpha_C(b\Delta)^N$, a neighbourhood $U_1$ of 0 in $\mathscr{C}^\alpha(b\Delta)^N$, a neighbourhood $V$ of $r$ in $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))^N$ and a map $\varphi : V \times U_1 \to \mathscr{C}^\alpha(b\Delta)^N$ of class $\mathscr{C}^1$ such that a map $\psi \in W$ satisfies*

$$p(\zeta) + \psi(\zeta) \in M_\rho(\zeta) \quad (\zeta \in b\Delta)$$

*for some $\rho \in V$ if and only if*

$$\psi(\zeta) = G(u, \varphi(\rho, u))(\zeta) \quad (\zeta \in b\Delta) \tag{6.5}$$

*for $u \in U_1$ and $\rho \in V$.*

We need the following lemma whose proof we give in Sect. 11.

**Lemma 6.1** *Let $B \subset R^m$ be an open ball centered at the origin and let $\Omega = \{v \in \mathscr{C}^\alpha(b\Delta)^m : v(\zeta) \in B \ (\zeta \in b\Delta)\}$. For each $v \in \Omega$ and each $g \in \mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))$ the map $\zeta \mapsto H(g, v)(\zeta) = g(\zeta)(v(\zeta))$ $(\zeta \in b\Delta)$ belongs to $\mathscr{C}^\alpha(b\Delta)$. The map $H : \mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B)) \times \Omega \to \mathscr{C}^\alpha(b\Delta)$ is of class $\mathscr{C}^1$.*

*Proof of Theorem 6.1* Let $U \subset \mathscr{C}^\alpha(b\Delta)^N$ be a neighbourhood of zero such that $G(u, f)(\zeta) \in B$ $(u \in U, f \in U, \zeta \in b\Delta)$. Since $G$ is a bounded linear map Lemma 11.2 implies that the map $\Phi$ mapping $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))^N \times U \times U$ to $\mathscr{C}^\alpha(b\Delta)^N$ is of class $\mathscr{C}^1$. By (6.4), $\Phi(r, 0, 0) = 0$. Assume for a moment that the partial derivative $(D_f\Phi)(r, 0, 0) : \mathscr{C}^\alpha(b\Delta)^N \to \mathscr{C}^\alpha(b\Delta)$ is a linear isomorphism. By the implicit mapping theorem in Banach spaces [Ca] there are neighbourhoods $U_1$, $U_2$ of 0 in $\mathscr{C}^\alpha(b\Delta)^N$, a neighbourhood $V$ of $r$ in $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))^N$ and a $\mathscr{C}^1$ map $\varphi$: $V \times U_1 \to \mathscr{C}^\alpha(b\Delta)^N$ such that

$$u \in U_1, f \in U_2, \rho \in V \text{ and } \Phi(\rho, u, f) = 0$$

is equivalent to

$$u \in U_1, \rho \in V \text{ and } f = \varphi(\rho, v) .$$

We then put $W = G(U_1 \times U_2)$ to complete the proof of Theorem 6.1.

It remains to show that $(D_f\Phi)(r, 0, 0)$ is a linear isomorphism. We generalize an argument of Forstnerič [Fo, p. 21]. Identify $C^N$ with $R^{2N}$ in the standard way and denote by $X_j^*$ the vectors corresponding to $iX_j$ under this identification, $1 \leqq j \leqq N$.

Let $\Psi$ map $g\in\mathscr{C}^\alpha_C(b\Delta)^N$ to $\zeta\mapsto r(\zeta)(g(\zeta))(\zeta\in b\Delta)$. In real coordinates $v_1,\ldots,v_{2N}$, $(D\Psi)$ (0) maps $\omega=(\omega_1,\ldots,\omega_{2N})\in\mathscr{C}^\alpha(b\Delta)^{2N}\equiv\mathscr{C}^\alpha_C(b\Delta)^N$ to

$$\zeta\mapsto\left(\sum_{k=1}^{2N}\frac{\partial r_1(\zeta)}{\partial v_k}(0)\,\omega_k(\zeta),\ldots,\sum_{k=1}^{2N}\frac{\partial r_N(\zeta)}{\partial v_k}(0)\,\omega_k(\zeta)\right)$$

so that if $\omega$ is written as a column then $(D\Psi)(0)\omega)(\zeta)=(\nabla(r(\zeta)))(0)\bullet\omega(\zeta)$ $(\zeta\in b\Delta)$ where $\bullet$ denotes matrix multiplication. The chain rule implies that for each $h\in\mathscr{C}^\alpha(b\Delta)^N$ we have

$$(D_f\Phi(r,0,0)h)(\zeta)=(\nabla(r(\zeta))(0)\bullet\left(\sum_{j=1}^N h_j(\zeta)X_j^*(\zeta)-\sum_{j=1}^N \bar{h}_j(\zeta)X_j(\zeta)\right)\quad(\zeta\in b\Delta)\,.$$

For each $j$, $1\leqq j\leqq N$, $X_j(\zeta)$ is tangent to $M(\zeta)$ at 0 which means that $[\nabla(r(\zeta))](0)\bullet X_j(\zeta)=0$ $(\zeta\in b\Delta,\ 1\leqq j\leqq N)$ so

$$\begin{aligned}(D_f\Phi(r,0,0)h)(\zeta)&=(\nabla(r(\zeta)))(0)\bullet\sum_{j=1}^N h_j(\zeta)X_j^*(\zeta)\\&=C(\zeta)\bullet\begin{pmatrix}h_1(\zeta)\\ \cdots\\ h_N(\zeta)\end{pmatrix}\quad(\zeta\in b\Delta)\end{aligned}$$

where $\zeta\mapsto\mathscr{C}(\zeta)$ is a real $N\times N$ matrix function with entries in $\mathscr{C}^\alpha(b\Delta)$. Suppose for a moment that $\det C(\zeta)=0$ for some $\zeta\in b\Delta$. This means that there are real numbers $a_1,\ldots,a_N$ such that $(\nabla(r(\zeta)))(0)\bullet\sum_{j=1}^N a_jX_j^*(\zeta)=0$. Since $T(\zeta)$ is the null space of the matrix $(\nabla(r(\zeta)))(0)$ it follows that $\sum_{j=1}^N a_jX_j^*(\zeta)=i\sum_{j=1}^N a_jX(\zeta)$ lies in $T(\zeta)$ which contradicts the assumption that $T(\zeta)$ is totally real. This shows that $\det C(\zeta)\neq 0$ $(\zeta\in b\Delta)$ which implies that $C^{-1}$ exists and has components in $\mathscr{C}^\alpha(b\Delta)$. Thus $D_f\Phi(r,0,0)$ is a linear isomorphism with the inverse $h\mapsto C^{-1}h$. This completes the proof of Theorem 6.1.

Later we shall also need the following

**Lemma 6.2** *Let* $X_j$, $1\leqq j\leqq N$, *and*

$$\psi(\rho,u)(\zeta)=G(u,\varphi(\rho,u))(\zeta)\quad(\zeta\in b\Delta)$$

*be as in Theorem* 6.1. *Then*

$$(D_u\psi(r,0)v)(\zeta)=\sum_{j=1}^N v_j(\zeta)X_j(\zeta)\quad(\zeta\in b\Delta)$$

*for each* $v\in\mathscr{C}^\alpha(b\Delta)^N$.

*Proof.* Since $\Phi(\rho,u,\varphi(\rho,u))\equiv 0$ we have $0=D_f\Phi(r,0,0)\circ(D_u\varphi)(r,0)+D_u\Phi(r,0,0)$. As in the proof of Theorem 6.1 we get $(D_u\Phi(r,0,0)v)(\zeta)=(\nabla(r(\zeta))(0)\bullet\sum_{j=1}^N v_j(\zeta)X_j(\zeta)=0$ since $(\nabla(r(\zeta))(0))\bullet X_j(\zeta)=0$ $(1\leqq j\leqq N,$ $\zeta\in b\Delta)$. From the proof of Theorem 6.1 we know that $D_f\Phi(r,0,0)$ is a linear isomorphism so $(D_u\varphi)(r,0)=0$ and it follows that

$$(D_u\psi(r,0)v)(\zeta)=D_uG(0,0)v=\sum_{j=1}^{N} v_j(\zeta)X_j(\zeta) \quad (\zeta\in b\Delta)\,.$$

This completes the proof.

*Remark 6.1* Since the matrix $X(\zeta)$ above is nondegenerate for each $\zeta\in b\Delta$ it follows that $D_u\psi(r,0)$ extends from $\mathscr{C}^\alpha(b\Delta)^N$ to linear isomorphism $I(r,0)$ of $\mathscr{C}^\alpha_{\mathbb{C}}(b\Delta)^N$ onto itself. For each $\rho\in V$, $s\in U_1$ let $I(\rho,s)$ be the bounded linear extension of $D_u\psi(\rho,s)$ to $\mathscr{C}^\alpha_{\mathbb{C}}(b\Delta)^N$. Since $D_u\psi(\rho,s)$ depends continuously on $(\rho,s)$ it follows that $I(\rho,s)$ depends continuously on $(\rho,s)$. Let $\tau>0$ be so small that if $A$ is a bounded linear map of $\mathscr{C}^\alpha_{\mathbb{C}}(b\Delta)^N$ to itself such that $\|A-I(r,0\|<\tau$ then $A$ is a linear isomorphism. Passing to a smaller $\tau$ if necessary we may assume that there is a constant $\eta>0$ such that $\|Aw\|\geqq\eta\|w\|(w\in\mathscr{C}^\alpha_{\mathbb{C}}(b\Delta)^N)$ for all such $A$. Shrinking $U_1$, $V$ and $W$ if necessary we may assume that $\|I(\rho,s)-I(\rho,0)\|<\tau(\rho\in V,\ s\in U_1)$. We may assume that $U_1$ is convex. For $\rho\in V$ and $u_1,u_2\in U_1$ we then have $\psi(\rho,u_2)-\psi(\rho,u_1)=(\int_0^1 D_u\psi(\rho,u_1+t(u_2-u_1))dt)(u_2-u_1)=P(u_2-u_1)$ where $P=\int_0^1 I(\rho,u_1+t(u_2-u_1))dt$. Clearly $\|P-I(r,0)\|<\tau$ and so

$$\|\psi(\rho,u_1)-\psi(\rho,u_2)\|\geqq\eta\|u_2-u_1\| \quad (\rho\in V,\ u_1,u_2\in U_1)\,.$$

## 7 The main result

**Theorem 7.1** *Let $0<\alpha<1$ and let $p:b\Delta\to C^N$ be a map of class $\mathscr{C}^\alpha$. Let $B\subset C^N$ be an open ball centered at the origin and let $r_j\in\mathscr{C}^\alpha(b\Delta,\mathscr{C}^2(B))$, $1\leqq j\leqq N$, be such that for each $\zeta\in b\Delta$*

(i) $r_j(\zeta)(0)=0$ $(1\leqq j\leqq N)$
(ii) $d(r_1(\zeta))\wedge\cdots\wedge d(r_N(\zeta))\neq 0$ *on* $B$.

*For each $\zeta\in b\Delta$ denote by $M_r(\zeta)$ the manifold $\{w\in p(\zeta)+B\colon r_j(\zeta)(w-p(\zeta))=0\ (1\leqq j\leqq N)\}$ and assume that for each $\zeta\in b\Delta$ the tangent space to $M_r(\zeta)$ at $p(\zeta)$ is totally real. Denote by $\kappa$ the total index of $M_r$ along $p$. For each $\rho=(\rho_1,\ldots,\rho_N)\in\mathscr{C}^\alpha(b\Delta,\mathscr{C}^2(B))^N$ in a neighbourhood of $r=(r_1,\ldots,r_N)$ and for each $\zeta\in b\Delta$ write*

$$M_\rho(\zeta)=\{w\in p(\zeta)+B\colon \rho_j(\zeta)(w-p(\zeta))=0\ (1\leqq j\leqq N)\}\,.$$

*If all partial indices of $M_r$ along $p$ are nonnegative then there are an open neighbourhood $V$ of $r$ in $\mathscr{C}^\alpha(b\Delta,\mathscr{C}^2(B))^N$, an open neighbourhood $U$ of $0$ in $R^{\kappa+N}$, an open neighbourhood $W$ of $p$ in $\mathscr{C}^\alpha_{\mathbb{C}}(b\Delta)^N$ and a map $F\colon V\times U\to\mathscr{C}^\alpha_{\mathbb{C}}(b\Delta)^N$ of class $\mathscr{C}^1$ such that*

(i′) $F(r,0)=p$
(ii′) *for each $(\rho,t)\in V\times U$ the map $\zeta\to F(\rho,t)(\zeta)-p(\zeta)$ extends holomorphically to $\Delta$ and $F(\rho,t)(\zeta)\in M_\rho(\zeta)$ $(\zeta\in b\Delta)$*

(iii′) *there is an* $\eta>0$ *such that* $\|F(\rho,t_1)-F(\rho,t_2)\|\geqq\eta|t_1-t_2|$ *for each* $\rho\in V$ *and each* $t_1,t_2\in U$; *in particular,* $F(\rho,t_1)\neq F(\rho,t_2)$ *if* $t_1\neq t_2$

(iv′) *if* $q\in W$ *satisfies* $q(\zeta)\in M_\rho(\zeta)$ $(\zeta\in b\Delta)$ *for some* $\rho\in V$ *and is such that* $p-q$ *extends holomorphically to* $\Delta$ *then* $q=F(\rho,t)$ *for some* $t\in U$.

*For each* $(\rho,t)\in V\times U$ *denote by* $f(\rho,t)$ *the holomorphic extension to* $\Delta$ *of* $\zeta\mapsto F(\rho,t)(\zeta)-p(\zeta)$. *If each partial index is at least one then for each* $\zeta_0\in\Delta$ *the set* $\{f(\rho,t)(\zeta_0): t\in U\}$ *contains an open subset of* $C^N$.

*Remark 7.1* If $p$ is the boundary map of an analytic disc then Theorem 7.1 describes all nearby analytic discs $q$ such that $Q(\zeta)\in M_\rho(\zeta)$ $(\zeta\in b\Delta)$ for some $\rho$ close to $r$.

*Proof of Theorem 7.1* Let $\kappa_1,\dots,\kappa_N$ be the partial indices of $M$ along $p$, $\kappa_j=2m_j+1$ $(1\leqq j\leqq 2\ell)$, $\kappa_j=m_j$ $(2\ell+1\leqq j\leqq N)$. Let $\Theta$ be as in Theorem 5.1 with $L(\zeta)=T(\zeta)$ where for each $\zeta\in b\Delta$, $T(\zeta)$ is the tangent space to $M_r(\zeta)$ at $p(\zeta)$. Let

$$Y(\zeta)=\begin{pmatrix}\zeta^{m_1} & 0 & 0 & \cdots & 0\\ 0 & \zeta^{m_2} & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 0 & \zeta^{m_N}\end{pmatrix}\begin{pmatrix}K(\zeta) & 0 & \cdots & 0 & O\\ \cdots & \cdots & \cdots & \cdots & O\\ 0 & \cdots & 0 & K(\zeta) & O\\ O & O & O & O & I\end{pmatrix}$$

be the product of the last two factors in (5.1). Recall that $Y(\zeta)\overline{Y(\zeta)^{-1}}=\Lambda(\zeta)$ where $\Lambda(\zeta)$ is as in (3.2). By Theorem 5.1 for each $\zeta\in b\Delta$ the columns $X_1(\zeta),\dots,X_N(\zeta)$ of the matrix $X(\zeta)=\Theta(\zeta)Y(\zeta)$ form a basis of $T(\zeta)$. By Theorem 6.1 there are a neighbourhood $W$ of $p$ in $\mathscr{C}^\alpha_C(b\Delta)^N$, a neighbourhood $U_1$ of 0 in $\mathscr{C}^\alpha(b\Delta)^N$, a neighbourhood $V\subset\mathscr{C}^\alpha(b\Delta,\mathscr{C}^2(B))^N$ of $r$ and a map $\varphi$: $V\times U_1\mapsto\mathscr{C}^\alpha_C(b\Delta)^N$ of class $\mathscr{C}^1$ such that $\psi\in W$ satisfies

$$p(\zeta)+\psi(\zeta)\in M_\rho(\zeta)\quad(\zeta\in b\Delta)$$

for some $\rho\in V$ if and only if

$$\psi(\zeta)=G(u,\varphi(\rho,u))(\zeta)\quad(\zeta\in b\Delta)\tag{7.1}$$

for $u\in U_1$ and $\rho\in V$. We now determine which of the maps $\psi$ in (7.1) extend holomorphically to $\Delta$. We have

$$\begin{aligned}G(u,\varphi(\rho,u))(\zeta)&=\sum_{j=1}^N\{u_j(\zeta)+i(\varphi_j(\rho,u)(\zeta)+i\tilde\varphi_j(\rho,u)(\zeta))\}X_j(\zeta)\\&=\Theta(\zeta)\sum_{j=1}^N\{u_j(\zeta)+i(\varphi_j(\rho,u)(\zeta)+i\tilde\varphi_j(\rho,u)(\zeta))\}Y_j(\theta)\,.\end{aligned}$$

Since $\zeta\mapsto\Theta(\zeta)$ extends holomorphically to $\Delta$ with values in $Gl(N,C)$ it follows that the same holds for $\zeta\mapsto\Theta(\zeta)^{-1}$ and thus $G(u,\varphi(\rho,u))$ extends holomorphically to $\Delta$ if and only if $\zeta\mapsto\Phi(\zeta)^{-1}G(u,\varphi(\rho,u))(\zeta)$ extends holomorphically to $\Delta$, that is, if and only if

$$\zeta \mapsto \sum_{j=1}^{N} \{u_j(\zeta)+i(\varphi_j(\rho, u)(\zeta)+i\tilde{\varphi}_j(\rho, u)(\zeta))\} Y_j(\zeta) \tag{7.2}$$

extends holomorphically to $\Delta$. Assume now that all partial indices are non-negative. This means that $m_j \geqq 0 (1 \leqq j \leqq N)$ so $Y_j$ extends holomorphically to $\Delta$ for each $j$, $1 \leqq j \leqq N$. Since for each $j$, $1 \leqq j \leqq N$, the map $\varphi_j(r, u)+i\tilde{\varphi}_j(r, u)$ extends holomorphically to $\Delta$ it follows that (7.2) extends holomorphically to $\Delta$ if and only if

$$\zeta \mapsto \sum_{j=1}^{N} u_j(\zeta) Y_j(\zeta) \text{ extends holomorphically to } \Delta\,. \tag{7.3}$$

Writing $u$ as a column this is equivalent to finding all $u$ such that $Y(\zeta)u(\zeta)=h(\zeta)$ $(\zeta \in b\Delta)$ where the column $h$ extends holomorphically to $\Delta$. Since $u$ is real this is equivalent to finding all $h \in (\mathscr{A}^\alpha)^N$ such that $Y(\zeta)^{-1}h(\zeta)=\overline{Y(\zeta)^{-1}h(\zeta)}(\zeta \in b\Delta)$, that is, $h(\zeta)=Y(\zeta)\overline{Y(\zeta)^{-1}h(\zeta)}(\zeta \in b\Delta)$ which, by the properties of $Y$ is equivalent to

$$h_j(\zeta)=\zeta^{\kappa_j}\overline{h_j(\zeta)} \quad (\zeta \in b\Delta,\ 1 \leqq j \leqq N)\,. \tag{7.4}$$

If $1 \leqq j \leqq N$ and if (7.4) holds then the map

$$\zeta \mapsto \begin{Bmatrix} h_j(\zeta) & (\zeta \in \bar{\Delta}) \\ \zeta^{\kappa_j}\overline{h_j(1/\bar{\zeta})} & (\zeta \in C \backslash \Delta) \end{Bmatrix}$$

is holomorphic on $C$ and at infinity does not grow faster than $\zeta^{\kappa_j}$. By Liouville's theorem $h_j$ must be a polynomial of degree not exceeding $\kappa_j$. Thus, the most general form of $h_j$ satisfying (7.4) is

$$h_j(\zeta)=(t_1^j+it_2^j)+(t_3^j+it_4^j)\zeta+\cdots+(t_3^j-it_4^j)\zeta^{\kappa_j-1}+(t_1^j-it_2^j)\zeta^{\kappa_j} \tag{7.5}$$

where all $t_k^j$ are real. So for each $j$, $1 \leqq j \leqq N$, $h_j$ satisfying (7.4) depends on $\kappa_j+1$ real parameters and thus possible functions $h$ depend on $\kappa+N$ real parameters. In other words, (7.1) extends holomorphically to $\Delta$ if and only if $u=(u_1, \ldots, u_N) \in U_1$ is such that for each $j$, $u_j(\zeta)=Y(\zeta)^{-1}h_j(\zeta)$ $(\zeta \in b\Delta)$ where $h_j$ has the form (7.5). The above map $t \mapsto u(t)$ is a linear map from $R^{\kappa+N}$ to $\mathscr{C}^\alpha(b\Delta)^N$. If we put $U=u^{-1}(U_1)$ and $F(\rho, t)(\zeta)=p(\zeta)+G(u(t), \varphi(\rho, u(t)))(\zeta)$ $(\zeta \in b\Delta)$, then $F$ satisfies (i'), (ii') and (iv') in Theorem 7.1. Since the functions $1, \zeta, \ldots, \zeta^\mu$ are linearly independent there is an $\eta'$ such that $\|u(t)\| \geqq \eta'|t|$ $(t \in R^{\kappa+N})$. Together with Remark 6.1 this proves (iii') in Theorem 7.1.

To prove the last statement of Theorem 7.1 we show that if each partial index is at least one then for each $\zeta_0 \in \Delta$ the derivative of $t \mapsto \psi(r, u(t))(\zeta_0)=G(u(\mathrm{t}), \varphi(r, u(t))(\zeta_0)$ at 0 has maximal real rank, $2N$, and the last statement then follows by the implicit mapping theorem provided that the neighbourhood $U$ is chosen small enough. By Lemma 6.2, $(D_u\psi(r, 0)v)(\zeta)=\Theta(\zeta)Y(\zeta)v(\zeta)$ $(\zeta \in b\Delta)$. Since $u(t)(\zeta)=Y(\zeta)^{-1}h(t)(\zeta)$ it follows that $(Du)(0)t)(\zeta)=Y(\zeta)^{-1}(dh(0)t)(\zeta)$ $(\zeta \in b\Delta)$ which implies that the derivative at $t=0$ of $t \mapsto \psi(r, u(t))$, a $\mathscr{C}^1$ map from $U$ to $(\mathscr{A}^\alpha)^N$ is $t \mapsto \Theta(\zeta)(dh(0)t)(\zeta)$. Given $\zeta_0 \in \Delta$

the evaluation map $\omega \mapsto \omega(\zeta_0)$ is a bounded linear map on $(\mathscr{A}^\alpha)^N$ so the derivative of $t \mapsto \psi(\rho, u(t))(\zeta_0)$ at 0 is $t \mapsto \Theta(\zeta_0)(dh(0)t)(\zeta_0)$. Suppose that $\kappa_j \geqq 1$ $(1 \leqq j \leqq N)$. The previous discussion shows that for each $j$ a possible choice for $h_j$ is $h_j(\zeta) = (t_1^j + it_2^j) + (t_1^j - it_2^j)\zeta^{\kappa_j}$. For each $\zeta_0 \in \Delta$ the map $(s, t) \mapsto (t + is) + (t - is)\zeta_0^{\kappa_j}$ has real rank 2 at 0 and thus already the derivative of $t \mapsto h(t)(\zeta_0)$, restricted to this $2N$ dimensional subspace of $t$'s in $R^{N+\kappa}$, at 0, has maximal rank $2N$. This completes the proof.

The following theorem gives a partial answer to Problem 1.1.

**Theorem 7.2** *Let $V \subset C^N$ be an open set and let $M$ be a maximal real submanifold of $V$ of the form $M = \{w \in V\colon r_j(w) = 0 (1 \leqq j \leqq N)\}$ where $r_j \in \mathscr{C}^2(V)$ $(1 \leqq j \leqq N)$ satisfy $dr_1 \wedge \cdots \wedge dr_N \neq 0$ on $V$. Let $p\colon b\Delta \to C^N$ be a map of class $\mathscr{C}^\alpha$, $0 < \alpha < 1$, such that $p(b\Delta) \subset M$. For each $\zeta \in b\Delta$ let $T(\zeta)$ be the tangent space to $M$ at $p(\zeta)$. Assume that all partial indices of the bundle $\{T(\zeta)\colon \zeta \in b\Delta\}$ are nonnegative and denote by $\kappa$ the total index of $T$. For each $\rho = (\rho_1, \ldots, \rho_N) \in \mathscr{C}^2(V)^N$ in a neighbourhood of $r$ write $M_\rho = \{w \in V\colon \rho_j(w) = 0$ $(1 \leqq j \leqq N)\}$.*

*There are an open neighbourhood $V$ of $r$ in $\mathscr{C}^2(V)^N$, an open neighbourhood $U$ of 0 in $R^{\kappa+N}$, an open neighbourhood $W$ of $p$ in $\mathscr{C}^\alpha_C(b\Delta)^N$ and a map $F \in V \times U \to \mathscr{C}^\alpha_C(b\Delta)^N$ of class $\mathscr{C}^1$ such that* (i')–(iv') *in Theorem* 7.1 *are satisfied with $M_\rho(\zeta)$ replaced with $M_\rho$. Also, the last statement of Theorem* 7.1 *holds.*

*Proof.* For each $\zeta \in b\Delta$, $T(\zeta)$ is the R-orthogonal complement of the space spanned by $v_j(\zeta) = (\operatorname{grad} r_j)(p(\zeta))$, $1 \leqq j \leqq N$. Since $p$ is of class $\mathscr{C}^\alpha$ and $\operatorname{grad} r_j$ are of class $\mathscr{C}^1$ it follows that $\zeta \mapsto v_j(\zeta)$ is of class $\mathscr{C}^\alpha$, $1 \leqq j \leqq N$. By the last paragraph of Sect. 2 there is a map $A\colon b\Delta \mapsto Gl(N, C)$ of class $\mathscr{C}^\alpha$ such that for each $\zeta \in b\Delta$ the columns of $A(\zeta)$ form a basis of $T(\zeta)$. Thus the partial indices of $T$ are well defined and the assumption about the indices makes sense.

For each $\rho \in \mathscr{C}^2(V)^N$ let $M_\rho = \{w \in V\colon \rho(w) = 0\}$. In a way similar to the proof of Theorem 6.1 and using the known result [HT, Lemma 5.1, p. 341] rather than Lemma 6.1 we prove

**Sublemma 7.1** *If $X_1, \ldots, X_N\colon b\Delta \to C^N$ are maps of class $\mathscr{C}^\alpha$ such that for each $\zeta \in b\Delta$ the real span of $X_1(\zeta), \ldots, X_N(\zeta)$ is $T(\zeta)$ and if $G$ is as in Sect.* 6 *then there are a neighbourhood $W$ of* 0 *in $\mathscr{C}^\alpha_C(b\Delta)^N$, a neighbourhood $U_1$ of* 0 *in $\mathscr{C}^\alpha(b\Delta)^N$, a neighbourhood $V$ of $r$ in $\mathscr{C}^2(V)^N$ and a map $\varphi\colon V \times U_1 \to \mathscr{C}^\alpha(b\Delta)^N$ of class $\mathscr{C}^1$ such that a map $\psi \in W$ satisfies $(p + \psi)(b\Delta) \subset M_\rho$ for some $\rho \in V$ if and only if $\psi(\zeta) = G(u, \varphi(\rho, u))(\zeta)$ $(\zeta \in b\Delta)$ for $u \in U$ and $\rho \in V$.*

With this in hand the proof of Theorem 7.2 is a modification of the proof of Theorem 7.1.

*Remark 7.2* Theorem 7.2 generalizes a result of Forstnerič [Fo, p. 6] from $C^2$ to $C^N$. In the case when $p$ is the boundary map of an analytic disc it provides sufficient conditions for the existence of nearby analytic discs with boundaries in manifolds obtained by small perturbations of the original maximal real submanifold. A special case of the problem of existence of such discs was studied by Lempert [Le] to prove the existence of extremal discs in domains obtained by perturbing a given domain. He used the implicit mapping theorem in Banach spaces in a way different from the one above. In the case he studied he simplified computations by proving explicitly the existence of certain real matrix multipliers as in our Remark 5.3.

Note that in the case when $p$ is the boundary map of an analytic disc the last statement in Theorem 7.2 gives some information about the polynomial hull of $M$ and its stability under small perturbations of $M$.

*Remark 7.3* A weaker version of Theorem 7.2 where $\mathscr{C}^\alpha(V)$ is everywhere replaced by $\mathscr{C}^3(V)$ follows from Theorem 7.1. To see this choose a ball $B$ centered at the origin so small that $p(b\Delta)+B \subset V$ and then replace $V$ by $p(b\Delta)+B$. For each $\rho \in \mathscr{C}^3(V)$ define the function $T_\rho$: $b\Delta \times B \to R$ by $(T\rho)(\zeta, w) = \rho(p(\zeta)+w)$. The definition implies that $T$ is a one to one bounded linear map from $\mathscr{C}^3(V)$ to $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))$. A straightforward application of Theorem 7.1 now gives the weaker version of Theorem 7.2.

*Remark 7.4* If we assume more smoothness of the defining maps in Theorem 7.1 and if we want only to describe the maps $F(\rho, t)$ sufficiently close to $p$ with $\rho$ sufficiently close to $r$ we can use Theorem 7.2. Replace $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))^N$ by $\mathscr{C}^2(b\Delta \times B)^N$. Writing $S_\rho(\zeta) = \{w \in B: \rho(\zeta, w) = 0\}$ $(\zeta \in b\Delta)$ there is a neighbourhood $\Omega$ of $b\Delta \times \{0\} \subset C^{N+1}$ such that $\tilde{S}_\rho = \Omega \cap (\bigcup_{\zeta \in b\Delta} \{\zeta\} \times S_\rho(\zeta))$ is a maximal real submanifold of $\Omega$ obtained by intersecting $\Omega$ with the common zero set of functions $\rho_0(\zeta, w) = |\zeta|^2 - 1$ and $\rho_1, \ldots, \rho_N$. The new tangent spaces are $\tilde{T}(\zeta) = Ri\zeta \oplus T(\zeta)$ $(\zeta \in b\Delta)$ so if $\kappa_1, \ldots, \kappa_N$ are the partial indices of $T$ then 2, $\kappa_1, \ldots, \kappa_N$ are the partial indices of $\tilde{T}$. Theorem 7.2 gives the family $\mathscr{F}$ of all maps $\tilde{\varphi} = (\psi, \varphi) \in \mathscr{A}^\alpha \times (\mathscr{A}^\alpha)^N$ of sufficiently small norm such that $\tilde{\varphi}(b\Delta) \subset \tilde{S}_\rho$, that is, $(\psi(\zeta), \varphi(\zeta)) \in \tilde{S}_\rho$ $(\zeta \in b\Delta)$. Since $|\psi(\zeta)| = 1$ $(\zeta \in b\Delta)$ and since $\psi$ is close to the identity it must be an automorphism of $\Delta$. Since $(\psi(\zeta), \varphi(\zeta)) \in \tilde{S}_\rho$ is equivalent to $(\zeta, \varphi \circ \psi^{-1}(\zeta)) \in S_\rho)$ it follows that the set $\{\varphi \circ \psi^{-1}: (\psi, \varphi) \in \mathscr{F}\}$ is precisely the set of all maps $\omega \in (\mathscr{A}^\alpha)^N$ of sufficiently small norm such that $p(\zeta) + \omega(\zeta) \in M_\rho(\zeta)$ $(\zeta \in b\Delta)$.

## 8 A rigidity result

In this section we show that under appropriate assumptions on partial indices there are no nearby selections which are analytic perturbations of the original map.

**Theorem 8.1** *Let $M$ be as in Theorem 7.2 where the defining functions $r_j$ are of class $\mathscr{C}^3$. Let $h$: $\bar{\Delta} \to C^N$ be an analytic disc with boundary in $M$. For each $\zeta \in b\Delta$ let $T(\zeta)$ be the tangent space to $M$ at $h(\zeta)$. Assume that one partial index of the bundle $\{T(\zeta)$: $\zeta \in b\Delta\}$ equals 2 while all other partial indices are negative. Then there is an open neighbourhood $\Omega$ of $h$ in $(\mathscr{A}^\alpha)$ such that if $g(b\Delta) \subset M$ then $g = h \circ \omega$ where $\omega$ is a conformal automorphism of $\Delta$.*

*Remark 8.1* Note that by a result of Chirka [Ch], $h$ is of class $\mathscr{C}^{3-0}$. A reasoning similar to the one in Remark 7.3 and the one at the end of Sect. 2

shows that there is a map $A: b\Delta \mapsto Gl(N, C)$ of class $\mathscr{C}^\alpha$ such that for each $\zeta$, the real span of the columns of $A(\zeta)$ is $T(\zeta)$ and thus the partial indices are well defined.

*Remark 8.2* If $N=2$, *if* $h:\bar{\Delta} \to C^2$ is an immersion and if the Forstnerič index of $M$ along $h|b\Delta$ is nonpositive then (see Proposition 10.1 below) one partial index equals 2 and the other partial index is negative and Theorem 8.1 reduces to a result proved by Forstnerič [Fo, p. 7]. In the proof of Theorem 8.1 below we use the idea from the proof given by Forstnerič. Our proof is more complicated since in our case some partial indices can be odd. We need

**Proposition 8.1** *Let $X$ be a Banach space, $U \subset R^n$ a neighbourhood of 0 and let $F, G: U \mapsto X$ be $\mathscr{C}^1$ maps such that $F(0)=G(0)$ and such that $(DF)(0)$, $(DG)(0)$ both have rank n. Suppose that for every neighbourhood $V \subset U$ of 0 there is a neighbourhood $V_1 \subset U$ of 0 such that $F(V_1) \subset G(V)$. Then there are neighbourhoods $V_1, V_2$ of 0 such that $F(V_1)=G(V_2)$.*

*Proof of Theorem 8.1* Let $A: b\Delta \to Gl(N, C)$ be a map of class $\mathscr{C}^\alpha$ such that for each $\zeta \in b\Delta$ the real span of the columns of $A(\zeta)$ is $T(\zeta)$. We know that whenever $H(\zeta)\overline{H(\zeta)}^{-1}=A(\zeta)\overline{A(\zeta)}^{-1}$ the real span of the columns of $H(\zeta)$ is $T(\zeta)$. By Theorem 5.1 there is a map $\Theta: b\Delta \to Gl(N, C)$ of class $\mathscr{C}^\alpha$ such that $A(\zeta)\overline{A(\zeta)}^{-1}=\Theta(\zeta)\Lambda(\zeta)\overline{\Theta(\zeta)}^{-1}$ $(\zeta \in b\Delta)$ where $\Lambda(\zeta)$ is as in (3.2). Thus, $A(\zeta^2)\overline{A(\zeta^2)}^{-1}=(\Theta(\zeta^2)\Lambda(\zeta))\overline{(\Theta(\zeta^2)\Lambda(\zeta))}^{-1}$ $(\zeta \in b\Delta)$ which implies that if we put $p(\zeta)=h(\zeta^2)$ and if $\tilde{T}(\zeta)=T(\zeta^2)$ is the tangent space to $M$ at $p(\zeta)$ then for each $\zeta \in b\Delta$ the real span of the columns of $X(\zeta)=\Theta(\zeta^2)\Lambda(\zeta)$ is $\tilde{T}(\zeta)$. To find all maps $\gamma$ with small $\alpha$-norm such that $h(\zeta)+\gamma(\zeta) \in M$ $(\zeta \in b\Delta)$ we find all maps $\psi$ with sufficiently small $\alpha$-norm such that $h(\zeta^2)+\psi(\zeta) \in M$ $(\zeta \in b\Delta)$ and then choose those which satisfy $\psi(\zeta) \equiv \psi(-\zeta)$. In this case $\psi(\zeta)$ has the form $\gamma(\zeta^2)$ and the norm of $\gamma$ is small provided that the norm of $\psi$ is small. Sublemma 7.1 implies

**Sublemma 8.1** *There are neighbourhoods $W$ and $U_1$ of 0 in $\mathscr{C}^\alpha_{\mathbb{C}}(b\Delta)^N$ and a map $\varphi: U_1 \to \mathscr{C}^\alpha(b\Delta)^N$ of class $\mathscr{C}^1$ such that a map $\psi \in W$ satisfies $(p+\psi)(b\Delta) \subset M$ if and only if $\psi(\zeta)=G(u, \varphi(u))(\zeta)$ $(\zeta \in b\Delta)$ with $u \in U$ where*

$$G(u,f)(\zeta)=\sum_{j=1}^{N} u_j(\zeta)X_j(\zeta)+i\sum_{j=1}^{N}(f_j(\zeta)+i\tilde{f}_j(\zeta))X_j(\zeta)$$

$$=\Theta(\zeta^2)\left(\sum_{j=1}^{N} u_j(\zeta)\Lambda_j(\zeta)+i\sum_{j+1}^{N}(f_j(\zeta)+i\tilde{f}_j(\zeta))\Lambda_j(\zeta)\right) \quad (\zeta \in b\Delta)\,.$$

*Proof of Theorem 8.1 continued.* Let $\psi \in W$. If $\|\psi\|$ is small enough then $\psi_- \in W$ where $\psi_-(\zeta)=\psi(-\zeta)$ $(\zeta \in b\Delta)$. Suppose that $p(\zeta)+\psi(\zeta) \in M$ $(\zeta \in b\Delta)$. This implies that $p(-\zeta)+\psi(-\zeta)=p(\zeta)+\psi_-(\zeta) \in M$ $(\zeta \in b\Delta)$. Thus for some $u, w \in U_1$

we have $\psi(\zeta)=\Theta(\zeta^2)G(u,\varphi(u))(\zeta)$ $(\zeta\in b\Delta)$ and $\psi_-(\zeta)=\Theta(\zeta^2)G(w,\varphi(w))(\zeta)$ $(\zeta\in b\Delta)$. If

$$\psi(\zeta)=\Theta(\zeta^2)\left(\sum_{j=1}^N u_j(\zeta)\Lambda_j(\zeta)+i\sum_{j=1}^N(\varphi_j(u)+i\tilde{\varphi}_j(u))(\zeta)\Lambda_j(\zeta)\right)(\zeta\in b\Delta)$$

then

$$\psi_-(\zeta)=\Theta(\zeta^2)\left(\sum_{j=1}^N u_j(-\zeta)(-1)^{\kappa_j}\Lambda_j(\zeta)+i\sum_{j=1}^N(\varphi_j(u)+i\tilde{\varphi}_j(u))(-\zeta)(-1)^{\kappa_j}\Lambda_j(\zeta)\right).$$

As any map in $\mathscr{C}_C^\alpha(b\Delta)^N$ is uniquely expressible in the form on the right it follows that if for $\eta\in\mathscr{C}^\alpha(b\Delta)^N$ we set $(Q\eta)(\zeta)=((-1)^{\kappa_j}\eta_j(-\zeta))_{j=1}^N$ we see that $G(u,\varphi(u))(-\zeta)=G(Q(u),\varphi(Q(u))(\zeta)$ $(\zeta\in b\Delta)$. In particular,

$$\text{if } u=Qu \text{ then } G(u,\varphi(u))(\zeta)=G(u,\varphi(u))(-\zeta))\quad(\zeta\in b\Delta)\tag{8.1}$$

provided that the norm of $u$ is small enough. Conversely, let $G(u,\varphi(u))(\zeta)=G(u,\varphi(u))\,(-\zeta)$ $(\zeta\in b\Delta)$. It follows that for each $j$, $1\leqq j\leqq N$, and for each $\zeta\in b\Delta$

$$u_j(-\zeta)(-1)^{\kappa_j}+i(\varphi_j(u)+i\tilde{\varphi}_j(u))(-\zeta)(-1)^{\kappa_j}=u_j(\zeta)+i(\varphi_j(u)+i\tilde{\varphi}_j(u)(\zeta))\tag{8.2}$$

which implies that $(-1)^{\kappa_j}\varphi_j(u)(-\zeta)=\varphi_j(u)(\zeta)$ so $(-1)^{\kappa_j}\tilde{\varphi}_j(u)(-\zeta)=\tilde{\varphi}_j(u)(\zeta)$ and (8.2) gives $u_j(-\zeta)(-1)^{\kappa_j}=u_j(\zeta)$ $(\zeta\in b\Delta)$. Thus $u=Qu$. We proved that

$$\left\{\begin{array}{l}\text{if } \|u\| \text{ is small enough then } u=Qu \text{ if and}\\ \text{only if } G(u,\varphi(u))(-\zeta)=G(u,\varphi(u))(\zeta)\,(\zeta\in b\Delta).\end{array}\right\}\tag{8.3}$$

We now check which of the maps $\psi$ extend holomorphically to $\Delta$. This happens if and only if for each $j$, $1\leqq j\leqq N$

$$\zeta\mapsto(u_j(\zeta)+i(\varphi_j(u)(\zeta)+i\tilde{\varphi}_j(u)(\zeta)))\zeta^{\kappa_j} \text{ extends holomorphically to } \Delta\,.\tag{8.4}$$

With no loss of generality assume that $\kappa_1=2$ and $\kappa_j<0$ $(2\leqq j\leqq N)$. Let $2\leqq j\leqq N$. Since $\kappa_j<0$, since $\varphi(u)+i\tilde{\varphi}_j(u)$ extends holomorphically to $\Delta$ and since $\tilde{\varphi}_j(u)(0)=0$ it follows that $u_j\equiv 0$. If $j=1$ then (8.4) holds if and only if $\zeta\mapsto u_1(\zeta)\zeta^2$ has a holomorphic extension to $\Delta$ which happens if and only if there are complex constants $a_0, a_1$ and a real constant $a_2$ such that $u_1(\zeta)=a_0/\zeta^2+a_1/\zeta+a_2+\overline{a_1}\zeta+\overline{a_0}\zeta^2$ $(\zeta\in b\Delta.)$ Since in this case $Qu=u$ if and only if $u_1(\zeta)\equiv u_1(-\zeta)$ it follows that $\psi$ extends holomorphically to $\Delta$ and satisfies $\psi(\zeta)=\psi(-\zeta)\,(\zeta\in b\Delta)$ if and only if $u_1(\zeta)=a_0/\zeta^2+a_2+\overline{a_0}\zeta^2$ where $a_0$ is a complex constant and $a_2$ is a real constant.

Let $u$: $R^3\to\mathscr{C}^\alpha(b\Delta)^N$ be the map $u(t_1,t_2,t_3)=((t_1+it_2)\zeta^2+t_3+(t_1-it_2)\zeta^{-2},0,\ldots,0)$. The derivative of $u$ at 0 has rank 3 and Lemma 6.2 implies that $t\mapsto\psi(t)=G(u(t),\varphi(u(t)))$ is a $\mathscr{C}^1$-map from a neighbourhood of 0 in $R^3$ to a neighbourhood of 0 in the subspace $E\subset\mathscr{C}_C^\alpha(b\Delta)^N$ of those maps $\omega$ which satisfy $\omega(\zeta)\equiv\omega(-\zeta)$. Via the map $T$, $T(\omega)(\zeta)=\omega(\zeta^2)\,(\zeta\in b\Delta)$, $E$ is isomorphic to $\mathscr{C}^{\alpha}_{C}(b\Delta)^N$. Define the map $H$ by $H(t)=h+T(G(u(t), \varphi(u(t)))$. It is a $\mathscr{C}^1$ map from a neighbourhood of the origin in $R^3$ to a neighbourhood of $h$ in $\mathscr{C}^{\alpha}_{C}(b\Delta)^N$. Clearly $H(0)=h$ and $(DH)(0)$ has rank 3. By construction, for every neighbourhood $V\subset R^3$ of 0, $H(V)$ contains all maps $\psi$ in a sufficiently small neighbourhood of $h$ in $\mathscr{C}^{\alpha}_{C}(b\Delta)^N$ which extend holomorphically to $\Delta$ and satisfy $\psi(\Delta)\subset M$.

Further, since $h\in\mathscr{C}^2(b\Delta)$ the map $J$ mapping $(t_1, t_2, t_3)$ in a neighbourhood of 0 in $R^3$ to the map $\zeta\mapsto h(e^{it_3}(\zeta-(t_1+it_2))/(1-(t_1-it_2)\zeta))$ belonging to $\mathscr{C}^{\alpha}_{C}(b\Delta)^N$ is of class $\mathscr{C}^1$ [HT, Lemma 5.1, p. 341] and it is easy to check that $(DJ)(0)$ has rank 3. Since for each $(t_1, t_2, t_3)$ in a neighbourhood of 0 the map $J(t_1, t_2, t_3)$ extends holomorphically to $\Delta$ if follows that for each neighbourhood $P$ of 0 there is a neighbourhood $Q$ of 0 such that $H(Q)\subset J(P)$. By Proposition 8.1 it follows that there are neighbourhoods $P_1$, $P_2$ of 0 such that $J(P_1)=H(P_2)$, which implies that there is a neighbourhood $\Omega$ of $h$ in $\mathscr{A}^{\alpha}(b\Delta)^N$ with the required properties.

**Proposition 8.2** *Let $U\subset R^N$ be a neighbourhood of 0, let $X$ be a Banach space, let $F$: $U\to X$ be a $\mathscr{C}^1$ map, $F(0)=0$, such that $(DF)(0)$ has the maximal rank $n$. Let $Y=(DF)(0)R^n$ and choose a subspace $Z$ such that $X=Y\oplus Z$. There are a neighbourhood $U_1\subset U$ of 0, a $\mathscr{C}^1$ diffeomorphism $\varphi$: $U_1\to\varphi(U_1)$ where $\varphi(U_1)$ is a neighbourhood of 0 in $Y$, $\varphi(0)=0$, and a $\mathscr{C}^1$ map $\Phi$: $\varphi(U_1)\to Z$, $\Phi(0)=0$, $D\Phi(0)=0$ such that $F(z)=\varphi(z)+\Phi(\varphi(z))$ $(z\in U_1)$.*

*Proof.* By the assumption $L=(DF)(0)\colon R^n\to Y$ is an isomorphism. Let $P\colon X\to Y$ be a linear projection along $Z$. Since $F(x)=L(x)+o(\|x\|)$ we have $(P\circ F)(x)=(P\circ L)(x)+P(o(\|x\|)=L(x)+o_1(\|x\|)$. Thus $D(P\circ F)(0)$: $R^m\to Y$ is an isomorphism so there is a neighbourhood $U_1\subset U$ of 0 such that $\varphi=(P\circ F)|U_1$: $U_1\to\varphi(U_1)$ is a $\mathscr{C}^1$ diffeomorphism. For $w\in\varphi(U_1)$ let $\phi(w)=F(\varphi^{-1}(w))-w$. It is easy to see that $\Phi$ and $\varphi$ have the required properties. This completes the proof.

*Remark 8.3* Let $B$ be the open unit ball of $X$. If $E\subset X$ is a subspace and if $\varepsilon>0$ call the set $\bigcup_{t\geq 0}t((E\cap bB)+\varepsilon B)$ a conic neighbourhood of $E$. By Proposition 8.2 for any conic neighbourhood $\Omega$ of $Y$ there is a neighbourhood $U'\subset U_1$ of 0 such that $F(U')\subset\Omega$. Also, if $\Lambda\subset Y$ is a line passing through 0 then given a conic neighbourhood $P$ of $\Lambda$ there are points in $F(U')\cap(P\setminus\{0\})$ for each neighbourhood $U'\subset U_1$ of 0.

*Proof of Proposition 8.1* With no loss of generality assume that $F(0)=G(0)=0$. Let $L_1=(DF)(0)(R^N)$, $L_2=(DG)(0)(R^N)$ and assume that $L_1\neq L_2$. Since $\dim L_1=\dim L_2=n$ there is a line $\Lambda\subset L_1$, $\Lambda\cap L_2=\{0\}$. There are conical neighbourhoods $\Omega_1$ of $\Lambda$ and $\Omega_2$ of $L_2$ such that $\Omega_1\cap\Omega_2=\{0\}$. By Remark 8.4 there is a neighbourhood $U'\subset U$ of 0 such that $G(U')\subset\Omega_2$. By the assumption there is a neighbourhood $U''\subset U$ of 0 such that $F(U'')\subset G(U')\subset\Omega_2$. Since $\Omega_1\cap\Omega_2=\emptyset$ this contradicts the fact that there are points in $F(U'')\cap(\Omega_1\setminus\{0\})$. This proves that $L_1=L_2=Y$. Choose a subspace

$Z$ so that $X = Y \oplus Z$. By Proposition 8.2 there are a neighbourhood $U_1 \subset U$, diffeomorphisms $\varphi_F: U_1 \to \varphi_F(U_1) \subset Y$, $\varphi_G$: $U_1 \to \varphi_G(U_1) \subset Y$, $\varphi_F(0) = \varphi_G(0) = 0$, and $\mathscr{C}^1$ maps $\Phi_F: \varphi_F(U_1) \to Z$, $\Phi_G: \varphi_G(U_1) \to Z$, such that for every $z \in U_1$ we have $F(z) = \varphi_F(z) + \Phi_F(\varphi_F(z))$ and $G(z) = \varphi_G(z) + \Phi_G(\varphi_G(z))$. By the assumption there is a neighbourhood $V_1 \subset U_1$ such that $F(V_1) \subset G(U_1)$. This implies that $\{\varphi_F(z) + \Phi_F(\varphi_F(z)) : z \in V_1\} \subset \{\varphi_G(z) + \Phi_G(\varphi_G(z)) : z \in U_1\}$. It follows that $\{\varphi_F(V_1) \subset \varphi_G(U_1)$ and, that if $y \in \varphi_F(V_1)$ then $\Phi_F(y) = \Phi_G(y)$. So, if $V_2 = \varphi_G^{-1}(\varphi_F(V_1))$ then $F(V_1) = G(V_2)$. This completes the proof.

## 9 A different look at the main result – a solution of a nonlinear Riemann–Hilbert problem for vector functions

Another, less geometric look at Theorem 7.1 shows that we had to

$$\left\{\begin{array}{l}\text{find maps } \psi \in (\mathscr{A}^\alpha)^N \text{ of sufficiently small norm such that } r_j(\zeta, \psi(\zeta)) \\ = 0 \ (\zeta \in b\Delta,\ 1 \leqq j \leqq N) \text{ where } r_j(\zeta, 0) = 0 \ (\zeta \in b\Delta, 1 \leqq j \leqq N) \text{ and} \\ \text{for each } \zeta \in b\Delta \text{ the tangent space to } M_r(\zeta) \text{ at } p(\zeta) \text{ is totally real.}\end{array}\right\} \tag{9.1}$$

As we know from Sect. 2 the last condition is equivalent to the vectors $v_j(\zeta) = \mathrm{grad}_\omega r_j(\zeta, 0)$, $(1 \leqq j \leqq N)$ being linearly independent which implies (ii) in Theorem 7.1 if $B$ is small enough. As in Sect. 4 denote by $G(\zeta) \in Gl(N, C)$ the matrix with rows $v_j(\zeta)$, $1 \leqq j \leqq N$, $\zeta \in b\Delta$. If each map $(\zeta, w) \mapsto r_i(\zeta, w)$ above is $R$-linear in $w$ then (9.1) reduces to the homogeneous Riemann–Hilbert problem for vector functions:

$$\text{Find all } \psi \in (\mathscr{A}^\alpha)^N \text{ such that } \Re(\overline{G(\zeta)} \psi(\zeta)) = 0 \ (\zeta \in b\Delta)\,. \tag{9.2}$$

This problem was solved by Vekua [Ve] and we used the main ingredients of his solution in Sect. 5. The solution depends on partial indices of the map $\zeta \mapsto \overline{G(\zeta)^{-1}} G(\zeta)$. If there are $p$ nonnegative partial indices whose sum is $l$ then the general solution of (9.2) depends on $p + l$ real parameters. If all partial indices are negative when $\psi = 0$ is the only solution.

We now rewrite Theorem 7.1 as a solution of a nonlinear version of the Riemann–Hilbert problem 9.2.

**Theorem 9.1** *Let $0 < \alpha < 1$, let $B \subset C^N$ be an open ball centered at the origin and let $r_j \in \mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))$, $1 \leqq j \leqq N$, be such that $r_j(\zeta, 0) = 0$ $(\zeta \in b\Delta, 1 \leqq j \leqq N)$ and such that for each $\zeta \in b\Delta$ the vectors $v_j(\zeta) = \mathrm{grad}_w r_j(\zeta, 0)$ are linearly independent. Let $G(\zeta)$ be the matrix with rows $v_j(\zeta)$, $1 \leqq j \leqq N$, and let $\kappa$ be the total index of the map $\zeta \mapsto M(\zeta) = \overline{G(\zeta)^{-1}} G(\zeta)$*

*If all partial indices of $M$ are nonnegative then there are an open neighbourhood $V$ of $r = (r_1, \ldots, r_N)$ in $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))^N$, an open neighbourhood $U$ of $0$ in*

$R^{\kappa+N}$, *an open neighbourhood* $W$ *of* $0$ *in* $(\mathscr{A}^\alpha)^N$ *and a map* $\Phi$: $V\times U \to (\mathscr{A}^\alpha)^N$ *of class* $\mathscr{C}^1$ *such that*

(i) $\Phi(r,0)=0$

(ii) $\rho_j(\zeta, \Phi(\rho,t)(\zeta))=0$ $(\zeta\in b\Delta,\ 1\leqq j\leqq N)$ *for each* $(\rho,t)\in V\times U$

(iii) *there is an* $\eta>0$ *such that* $\|\Phi(\rho,t_1)-\Phi(\rho,t_2)\|\geqq\eta|t_1-t_2|$ *for each* $\rho\in V$ *and each* $t_1,t_2\in U$; *in particular,* $\Phi(\rho,t_1)\neq\Phi(\rho,t_2)$ *if* $t_1\neq t_2$

(iv) *if* $\psi\in W$ *satisfies* $\rho_j(\zeta,\psi(\zeta))=0$ $(\zeta\in b\Delta,\ 1\leqq j\leqq N)$ *for some* $\rho\in V$ *then* $\psi=\Phi(\rho,t)$ *for some* $t\in U$.

*Remark 9.1* Theorem 9.1 describes the structure of solutions of the nonlinear Riemann–Hilbert problem and their smooth dependence on parameters and on functions $r_j$. It seems that the nonlinear Riemann–Hilbert problem in the general form above has not been studied yet. A special case where $G$ is a small perturbation of a diagonal matrix was solved by Pogorzelski [Po. pp. 591–601]. Observe the general fact which in the mentioned special case was found by Pogorzelski: If all partial indices are nonnegative then the number of parameters in the general solution of the nonlinear problem is the same as the number of parameters in the general solution of the corresponding linear problem 9.2. When some, but not all partial indices are negative the nonlinear problem may have no other solution than $\psi=0$ while the corresponding linear problem has nontrivial solutions. To illustrate this we modify an example given by Forstnerič [Fo, p. 41]. Let $B\subset C^2$ be a small ball centered at the origin and for each $\zeta\in b\Delta$ let $M(\zeta)=\{t(1,0)+(s+it^2)(0,\zeta^{-1})\colon (t,s)\in R^2\}\cap B$. We have $0\in M(\zeta)$ $(\zeta\in b\Delta)$ and for each $\zeta\in b\Delta$, $T(\zeta)$ is the real span of $(1,0)$ and $(0,\zeta^{-1})$ and so the partial indices are $0$ and $-2$. Suppose that $\psi\in(\mathscr{A}^\alpha)^2$ satisfies $\psi(\zeta)\in M(\zeta)$ $(\zeta\in b\Delta)$. This implies that for each $\zeta\in b\Delta$ there are $t(\zeta)\in R$, $s(\zeta)\in R$ such that $\psi_1(\zeta)=t(\zeta)$, $\psi_2(\zeta)=(s(\zeta)+it(\zeta)^2\zeta^{-1}$ $(\zeta\in b\Delta)$. The first equality implies that $\psi_1$ is a real constant $t_0=t(\zeta)$ $(\zeta\in b\Delta)$ and since $\psi_2$ is holomorphic we have $\int_0^{2\pi}s(e^{i\theta})d\theta=\int_0^{2\pi}t_0^2d\theta=0$ so $t_0$ and $\zeta\mapsto s(\zeta)=\zeta\psi_2(\zeta)$ is a holomorphic function which vanishes at 0 an which is real on $b\Delta$. Thus $s\equiv 0$ which implies that $\psi\equiv 0$.

## 10 Remarks

Let $M$ be a maximal real submanifold of $C^N$ of class $\mathscr{C}^2$ as in Theorem 7.2 and let $f\colon\bar{\Delta}\to C^N$ be a nonconstant analytic disc with boundary in $M$. We know that by a result Chirka $f\in\mathscr{C}^{2-0}(\bar{\Delta})$. For each $\zeta\in b\Delta$, let $T(\zeta)$ be the tangent space to $M$ at $f(\zeta)$. We want to know to what extent $f$ must be an immersion (that is, has a nonvanishing derivative on $\bar{\Delta}$) and how $f$ being an immersion affects the partial indices of the bundle $\{T(\zeta)\colon\zeta\in b\Delta\}$.

**Proposition 10.1** *Let* $M$, $f$ *and* $T$ *be as above. Then*

(i) *$f'$ vanishes on at most a finite set of points in $\bar{\Delta}$*

(ii) *at least one of the partial indices of $T$ is greater than one*

(iii) *if $f'$ has no zero on $\bar{\Delta}$ and if the partial indices of $T$ satisfy $\kappa_j \leqq 1 (1 \leqq j \leqq N-1)$ then $\kappa_N = 2$*

(iv) *if $f'$ has no zero in a neighbourhood of $b\Delta$, if $f'$ has $v$ zeros on $\Delta$ (counting multiplicity) and if the partial indices of $T$ satisfy $\kappa_j \leqq 2v+1 (1 \leqq j \leqq N-1)$ then $\kappa_N = 2v+2$ .*

*Proof.* Observe first that $i\zeta f'(\zeta) \in T(\zeta)$ $(\zeta \in b\Delta)$. Let $\Theta(\zeta)$ be as in Theorem 5.1 with $L$ replaced by $T$ and let $Q(\zeta)$ be the product of the last two factors in (5.1). Note that $Q(\zeta)\overline{Q(\zeta)}^{-1} = \Lambda(\zeta)$ $(\zeta \in b\Delta)$ where $\Lambda(\zeta)$ is as in (3.2). The properties of $\Theta$ imply that $\zeta \mapsto \Psi(\zeta) = \Theta(\zeta)^{-1} i\zeta f'(\zeta)$ is a map of class $\mathscr{C}^\alpha$, holomorphic on $\Delta$ which vanishes at the origin and has the property that for each $\zeta \in b\Delta$, $\Psi(\zeta)$ is contained in the real span of the columns of $Q(\zeta)$ which implies that $\Psi(\zeta) = \Lambda(\zeta)\overline{\Psi(\zeta)}$ $(\zeta \in b\Delta)$, that is, $\Psi_j(\zeta) = \zeta^{\kappa_j}\overline{\Psi_j(\zeta)}$ $(\zeta \in b\Delta,\ 1 \leqq j \leqq N)$. Writing $\Psi_j(\zeta) = \zeta\Omega_j(\zeta)$ $(\zeta \in b\Delta)$ with $\Omega_j$ holomorphic on $\Delta$, we get

$$\Omega_j(\zeta) = \zeta^{\kappa_j - 2}\overline{\Omega_j(\zeta)} \quad (\zeta \in b\Delta,\ 1 \leqq j \leqq N)\,. \tag{10.1}$$

Since $f$ is not a constant at least one of $\Omega_j$ does not vanish identically and (10.1) implies (ii). To prove (iii) assume that $f'$ has no zero on $\bar{\Delta}$ and that $\kappa_j \leqq 1$ $(1 \leqq j \leqq N-1)$. By (10.1) this implies that $\Omega_j \equiv 0$ $(1 \leqq j \leqq N-1)$. Since $\Omega_N$ is holomorphic and satisfies (10.1) $\Omega_N$ has the form $\Omega_N(\zeta) = a_0 + a_1\zeta + \cdots + \overline{a_1}\zeta^{\kappa_N - 3} + \overline{a_0}\zeta^{\kappa_N - 2}$ where $a_j$ are constants. Note that if $\Omega_N(\zeta) = 0$ then $\Omega_N(1/\bar{\zeta}) = 0$. Since for every $\zeta \in C$ at least one of $\zeta$, $1/\bar{\zeta}$ is contained in $\bar{\Delta}$ the assumption that $f'$ has no zero on $\bar{\Delta}$, which, by the properties of $\Theta$ implies that $\Omega_N$ has no zero on $\bar{\Delta}$ implies that $\kappa_N = 2$ and that $\Omega_N$ is a nonzero real constant. This proves (iii). In the same way we prove (iv). To prove (i) observe that since $f$ is not a constant there is a $j$, $1 \leqq j \leqq N$, such that $\Omega_j$ does not vanish identically. (10.1) implies that $\Omega_j$ has the form $\Omega_j(\zeta) = b_0 + b_1\zeta + \cdots \overline{b_1}\zeta^{\kappa_j - 3} + \overline{b_0}\zeta^{\kappa_j - 2}$ where $b_j$ are constants which shows that $\Omega_j$ vanishes at the most finitely many points of $\bar{\Delta}$ and thus the same holds for $if'(\zeta)\zeta = \Theta(\zeta)\zeta\Omega(\zeta)$. This completes the proof.

In [Fo, p. 41] Forstnerič presents an example showing that in the study in $C^N$, $N \geqq 3$ the total index of the bundle $\{T(\zeta)\colon \zeta \in b\Delta\}$ itself is not enough to describe whether there are any nontrivial solutions of Problem 1.1 and mentions that if the maximal number of linearly independent holomorphic sections of $T$ equals $N$ then Problem 1.1 has nontrivial solutions. This is precisely the case studied in the present paper as shown by

**Proposition 10.2** *Let $\{L(\zeta)\colon \zeta \in b\Delta\}$ be a bundle where each $L(\zeta)$, $\zeta \in b\Delta$, is a $N$-dimensional, totally real $R$-linear subspace of $C^N$ and for which there is a map $A\colon b\Delta \to Gl(N, C)$ of class $\mathscr{C}^\alpha$, $0 < \alpha < 1$, such that for each $\zeta \in b\Delta$ the real span of the columns of $A(\zeta)$ is $L(\zeta)$. The number of nonnegative partial indices of $L$ equals the maximal number $k$ of maps $\Phi_1, \ldots, \Phi_k\colon \bar{\Delta} \to C^N$ of class $\mathscr{C}^\alpha$, holomorphic on $\Delta$ and such that for each $\zeta \in b\Delta$, $\Phi_1(\zeta), \ldots, \Phi_k(\zeta)$ are linearly independent and contained in $L(\zeta)$.*

*Proof.* Suppose that $L$ has $k$ nonnegative partial indices. By Theorem 5.1 there are at least $k$ maps $\Phi_j$ with the above properties. Assume that $k \leqq N-1$ and that there are $k+1$ maps $\Phi_j$ with the above properties. Let $\Theta(\zeta)$ be as in Theorem 5.1 and denote the product of the last two factors in (5.1) by $Q(\zeta)$. The maps $\zeta \mapsto H_j(\zeta) = \Theta(\zeta)^{-1}\Phi_j(\zeta)$ are holomorphic and for each $\zeta \in b\Delta$, $H_j(\zeta)$, $1 \leqq j \leqq N$ are linearly independent and contained in the real span of the columns of the matrix $Q(\zeta)$. Thus, if we denote by $H(\zeta)$ the matrix with columns $H_j(\zeta)$, $1 \leqq j \leqq k+1$, there is a real $N \times (k+1)$ matrix $V(\zeta)$ such that $Q(\zeta)V(\zeta) = H(\zeta)$ $(\zeta \in b\Delta)$. Since $Q(\zeta)\overline{Q(\zeta)}^{-1} = \Lambda(\zeta)$ where $\Lambda(\zeta)$ is as in (3.2) we have $H(\zeta) = \Lambda(\zeta)\overline{H(\zeta)}$ $(\zeta \in b\Delta)$. Denoting by $R_j(\zeta)$ the rows of $H(\zeta)$ we get $R_j(\zeta) = \zeta^{\kappa_j}\overline{R_j(\zeta)}$ $(\zeta \in b\Delta)$ which implies that $R_j \equiv 0$ whenever $\kappa_j < 0$. Thus the number of rows in $H$ which do not vanish identically does not exceed $k$ which implies that the columns of $H$ are linearly dependent, a contradiction. This completes the proof.

## 11 Appendix: Smoothness of composition maps

Let $0 < \alpha < 1$. Fix an open ball $B \subset R^m$ centered at the origin and let $\Omega = \{\varphi \in \mathscr{C}^\alpha(b\Delta, R^m)\colon \varphi(b\Delta) \subset B\}$. $\Omega$ is an open subset of $\mathscr{C}^\alpha(b\Delta, R^m)$. The crucial fact that we need to prove Lemma 6.1 is the continuity of composition maps on spaces $\mathscr{C}^\alpha$. In particular, we have to prove that if $g \in \mathscr{C}^\alpha(b\Delta, \mathscr{C}^1(B))$ then the map sending $v \in \Omega$ to $\zeta \mapsto g(\zeta)(v(\zeta))$ maps $\Omega$ continuously to $\mathscr{C}^\alpha(b\Delta)$. This has been known for a long time in the special case when $g$ does not depend on $\zeta$ [HT, Lemma 5.1, p. 341]. The proof in [HT] does not seem to generalize to the more general case treated here. A different proof of the special case is in [Go]. We use the ideas from this proof and from [AZ, pp. 191–194] to prove the general result.

**Lemma 11.1** *Let $g\colon b\Delta \times B \to R$ be a continuous function such that $\zeta \mapsto g(\zeta, v(\zeta))$ belongs to $\mathscr{C}^\alpha(b\Delta)$ for each $v \in \Omega$. Denote by $G$ the map which maps $v \in \Omega$ to the map $\zeta \mapsto g(\zeta, v(\zeta))$. Let $v \in \Omega$ and assume that for each $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\left\{\begin{array}{l} |g(\xi, v(\xi)+h) - g(\xi, v(\xi)) - g(\eta, v(\eta)+k) + g(\eta, v(\eta))| \\ \quad \leqq \varepsilon\left(|\xi-\eta|^\alpha + \dfrac{|h-k|}{\delta}\right) \\ \text{for every } \xi, \eta \in b\Delta, \text{ every } h, k \in R^m, |h| < \delta, |k| < \delta\,. \end{array}\right\} \tag{11.1}$$

*Then $G$ is continuous at $v$.*

*Proof.* Suppose that $v_n \to v$ *in* $\mathscr{C}^\alpha(b\Delta, R^m)$. With no loss of generality assume that there is an $r < 1$ such that $v_n(b\Delta) \subset rB (n \in N)$. By the uniform continuity of $g$ on $b\Delta \times rB$ we have $g(\zeta, v_n(\zeta)) \to g(\zeta, v(\zeta))$ uniformly for $\zeta \in b\Delta$ since $v_n \to v$ implies that $\sup_{\zeta \in b\Delta}|v_n(\zeta) - v(\zeta)| \to 0$. Thus, $\sup_{\zeta \in b\Delta}|G(v_n)(\zeta) - G(v)(\zeta)| \to 0$.

Let $\varepsilon>0$ and let $\delta>0$ satisfy (11.1). There is some $n_0$ such that for all $n\geqq n_0$ we have

$$\sup_{\zeta\in b\Delta}|v_n(\zeta)-v(\zeta)|<\delta,\quad \sup_{\xi,\eta\in b\Delta,\xi\neq\eta}\frac{|v_n(\xi)-v(\xi)-v_n(\eta)+v(\eta)|}{|\xi-\eta|^\alpha}<\delta\,. \qquad (11.2)$$

Thus, if $n\geqq n_0$ (11.1) and (11.2) imply that

$$\begin{aligned}&|g(\xi,v_n(\xi))-g(\xi,v(\xi))-g(\eta,v_n(\eta))+g(\eta,v(\eta))|\\&\qquad\leqq\varepsilon\left(|\xi-\eta|^\alpha+\frac{|v_n(\xi)-v(\xi)-v_n(\eta)+v(\eta)|}{\delta}\right)\\&\qquad\leqq\varepsilon(|\xi-\eta|^\alpha+|\xi-\eta|^\alpha)\quad(\xi,\eta\in b\Delta)\,.\end{aligned}$$

It follows that

$$\sup_{\xi,\eta\in b\Delta,\xi\neq\eta}\frac{|G(v_n)(\xi)-G(v)(\xi)-G(v_n)(\eta)+G(v)(\eta)|}{|\xi-\eta|^\alpha}\to 0 \text{ as } n\to\infty\,.$$

This implies that $\|G(v_n)-G(v)\|_\alpha\to 0$ which completes the proof.

**Lemma 11.2** *For each $g\in\mathscr{C}^\alpha(b\Delta,\mathscr{C}^1(B))$ and for each $v\in\Omega$ the map $\zeta\mapsto G(v)(\zeta)=g(\zeta,v(\zeta))$ belongs to $\mathscr{C}^\alpha(b\Delta)$ and we have $\|G(v)\|\leqq\|g\|\cdot(2+m^{1/2}\|v\|)$. Moreover, the map $G\colon\Omega\to\mathscr{C}^\alpha(b\Delta)$ is continuous.*

*Proof.* Note first that the assumptions imply that the functions $(\zeta,w)\mapsto g(\zeta,w)$ and $(\zeta,w)\mapsto\frac{\partial g}{\partial w_j}(\zeta,w)$, $1\leqq j\leqq m$, are continuous on $b\Delta\times B$ and bounded by $\|g\|$, the norm of $g$ in $\mathscr{C}^\alpha(b\Delta,\mathscr{C}^1(B))$. Also, for each $w\in B$ the functions $\zeta\mapsto g(\zeta,w)$, $\zeta\mapsto\frac{\partial g}{\partial w_j}(\zeta,w)$, $1\leqq j\leqq m$, are in $\mathscr{C}^\alpha(b\Delta)$ and their norms do not exceed $\|g\|$.

If $v\in\Omega$ then

$$\begin{aligned}&|g(\xi,v(\xi))-g(\eta,v(\eta))|\\&\qquad\leqq|g(\xi,v(\xi))-g(\eta,v(\xi))|+|g(\eta,v(\xi))-g(\eta,v(\eta))|\\&\qquad\leqq\|g\|\cdot|\xi-\eta|^\alpha+\sum_{j=1}^m|v_j(\xi)-v_j(\eta)|\cdot\left|\int_0^1\frac{\partial g}{\partial w_j}(\eta,v(\eta)+\lambda(v(\xi)-v(\eta)))d\lambda\right|\\&\qquad\leqq\|g\|\cdot|\xi-\eta|^\alpha+\sum_{j=1}^m|v_j(\xi)-v_j(\eta)|\cdot\|g\|\,,\end{aligned}$$

so if $\xi\neq\eta$ we have

$$\frac{|g(\xi,v(\xi))-g(\eta,v(\eta))|}{|\xi-\eta|^\alpha}\leqq\|g\|+\|g\|\sum_{j=1}^m\frac{|v_j(\xi)-v_j(\eta)|}{|\xi-\eta|^\alpha}\,.$$

Since $v_j\in\mathscr{C}^\alpha(b\Delta)$, $1\leqq j\leqq m$, this proves the first statement.

To prove the continuity of $G$ we use Lemma 11.1. Let

$$I = g(\xi, v(\xi)+h) - g(\xi, v(\xi)) - g(\eta, v(\eta)+k) + g(\eta, v(\eta))$$

and write $I = I_1 + I_2$ where

$$I_1 = g(\xi, v(\xi)+h) - g(\eta, v(\xi)+h) - g(\xi, v(\xi)) + g(\eta, v(\xi))$$

$$I_2 = g(\eta, v(\xi)+h) - g(\eta, v(\eta)+k) - g(\eta, v(\xi)) + g(\eta, v(\eta)) .$$

We have

$$I_1 = \sum_{j=1}^{m} h_j \int_0^1 \left( \frac{\partial g}{\partial w_j}(\xi, v(\xi)+\lambda h) - \frac{\partial g}{\partial w_j}(\eta, v(\xi)+\lambda h) \right) d\lambda$$

and thus

$$|I_1| \leqq m \cdot \max_{1\leqq j\leqq m} |h_j| \cdot \max_{1\leqq j\leqq m} \sup_{w\in B} \left| \frac{\partial g}{\partial w_j}(\xi, w) - \frac{\partial g}{\partial w_j}(\eta, w) \right|$$

$$\leqq m \cdot |h| \cdot \|g\| \cdot |\xi - \eta|^\alpha .$$

Further

$$\begin{aligned} I_2 &= \sum_{j=1}^{m} (v_j(\xi)+h_j-v_j(\eta)-k_j) \int_0^1 \frac{\partial g}{\partial w_j}(\eta, v(\eta)+k+\lambda(v(\xi)+h-v(\eta)-k)) d\lambda \\ &\quad - \sum_{j=1}^{m} (v_j(\xi)-v_j(\eta)) \int_0^1 \frac{\partial g}{\partial w_j}(\eta, v(\eta)+\lambda(v(\xi)-v(\eta))) d\lambda \\ &= I_2' + I_2'' \end{aligned}$$

where

$$\begin{aligned} I_2' = \sum_{j=1}^{m} (v_j(\xi)-v_j(\eta)) \cdot \int_0^1 \Bigg( & \frac{\partial g}{\partial w_j}(\eta, v(\eta)+\lambda(v(\xi)-v(\eta))+k+\lambda(h-k)) \\ & - \frac{\partial g}{\partial w_j}(\eta, v(\eta)+\lambda(v(\xi)-v(\eta))) \Bigg) d\lambda \end{aligned}$$

$$I_2'' = \sum_{j=1}^{M} (h_j - k_j) \cdot \int_0^1 \frac{\partial g}{\partial w_j}(\eta, v(\eta)+\lambda(v(\xi)-v(\eta))+k+\lambda(h-k)) d\lambda .$$

Since $v(b\Delta) \subset B$ if follows that there are $\delta_0 > 0$ .and $r < 1$ such that $v(b\Delta) + \delta_0 B \subset B$ where $B$ is the open unit ball of $R^m$. Since $\frac{\partial g}{\partial w_j}$, $1 \leqq j \leqq m$, are continuous on $b\Delta \times B$ it follows that for each $j$, $1 \leqq j \leqq m$, the map $\Psi_j$ defined by

$$\Psi_j(\xi, \eta, h, k, \lambda) = \frac{\partial g}{\partial w_j}(\eta, v(\eta)+\lambda(v(\xi)-v(\eta))+k+\lambda(h-k))$$

is uniformly continuous on $b\Delta \times b\Delta \times \delta_0 B \times [0, 1]$. Note that the integrands in $I_2'$ are of the form $\Psi_j(\xi, \eta, h, k, \lambda) - \Psi_j(\xi, \eta, 0, 0, \lambda)$. Thus, if $\varepsilon > 0$ is arbitrary, passing to a smaller $\delta_0 > 0$ we may assume that

$$m\delta_0 \|g\| < \varepsilon/2 \tag{11.3}$$

and for each $j$, $1 \leqq j \leqq m$,

$$\max_{1 \leqq j \leqq m} \|v_j\| \cdot |\Psi_j(\xi, \eta, h, k, \lambda) - \Psi_j(\xi, \eta, 0, 0, \lambda)| < \frac{\varepsilon}{2m} \tag{11.4}$$

whenever $|h| < \delta_0$, $|k| < \delta_0$ where $\|v_j\|$ is the norm of $v_j$ in $\mathscr{C}^\alpha(b\Delta)$. Now, (11.3) and (11.4) imply that

$$|I_1| + |I_2'| \leqq m\delta_0 \|g\| \cdot |\xi - \eta|^\alpha + m \cdot |\xi - \eta|^\alpha \cdot \frac{\varepsilon}{2m} \leqq \varepsilon |\xi - \eta|^\alpha . \tag{11.5}$$

Further, there is a universal constant $c$ such that

$$|I_2''| \leqq c|h-k| \cdot \int_0^1 \max_{1 \leqq j \leqq m} |\Psi_j(\xi, \eta, h, k, \lambda)| d\lambda . \tag{11.6}$$

Choose $\delta$, $0 < \delta < \delta_0$, so small that

$$c|\Psi_j(\xi, \eta, h, k, \lambda)| < \varepsilon/\delta \quad \text{if } |h| < \delta_0, |k| < \delta_0 . \tag{11.7}$$

Now, if $|h| < \delta$, $|k| < \delta$ then (11.5), (11.6) and (11.7) imply that

$$|I| \leqq \varepsilon \left( |\xi - \eta|^\alpha + \frac{|h-k|}{\delta} \right)$$

and by Lemma 11.1 it follows that $G$ is continuous at $v$. This completes the proof.

*Proof of Lemma 6.1* Let $g \in \mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))$ and let $v \in \mathscr{C}^\alpha(b\Delta, R^m)$. Lemma 11.2 implies that $H(g, v) \in \mathscr{C}^\alpha(b\Delta)$ for each $(g, v) \in \mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B)) \times \Omega$. We have

$$\begin{aligned}
&H(g+\varphi, v+w)(\zeta) - H(g, v)(\zeta) \\
&\quad = \varphi(\zeta)(v(\zeta) + w(\zeta)) + \{g(\zeta)(v(\zeta) + w(\zeta)) - g(\zeta)(v(\zeta))\} \\
&\quad = \varphi(\zeta)(v(\zeta)) + \sum_{j=1}^m w_j(\zeta) \int_0^1 \frac{\partial g(\zeta)}{\partial w_j} (v(\zeta) + \lambda w(\zeta)) d\lambda \\
&\qquad + \sum_{j=1}^m w_j(\zeta) \int_0^1 \frac{\partial \varphi(\zeta)}{\partial w_j} (v(\zeta) + \lambda w(\zeta)) d\lambda \\
&\quad = \varphi(\zeta)(v(\zeta)) + \sum_{j=1}^m w_j(\zeta) \frac{\partial g(\zeta)}{\partial w_j} (v(\zeta)) + I_1(\zeta) + I_2(\zeta)
\end{aligned}$$

where

$$I_1(\zeta) = \sum_{j=1}^m w_j(\zeta) \int_0^1 \left( \frac{\partial g(\zeta)}{\partial w_j} (v(\zeta) + \lambda w(\zeta)) - \frac{\partial g(\zeta)}{\partial w_j} (v(\zeta)) \right) d\lambda ,$$

$$I_2(\zeta) = \sum_{j=1}^m w_j(\zeta) \int_0^1 \frac{\partial g(\zeta)}{\partial w_j} (v(\zeta) + \lambda w(\zeta)) d\lambda .$$

By Lemma 11.2

$$\left(\zeta \mapsto \frac{\partial g(\zeta)}{\partial w_j}(v(\zeta))\right) \in \mathscr{C}^\alpha(b\Delta) \quad (1 \leqq j \leqq m) \tag{11.8}$$

and since $\mathscr{C}^\alpha(b\Delta)$ is a Banach algebra it follows that

$$w \mapsto \left(\zeta \mapsto \sum_{j=1}^m w_j(\zeta) \frac{\partial g(\zeta)}{\partial w_j}(v(\zeta))\right)$$

is a bounded linear map from $\mathscr{C}^\alpha(b\Delta, R^m)$ to $\mathscr{C}^\alpha(b\Delta)$. Further, by Lemma 11.2, $\varphi \mapsto (\zeta \mapsto \varphi(\zeta)(v(\zeta)))$ is a bounded linear map from $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))$ to $\mathscr{C}^\alpha(b\Delta)$. We now show that $H$ is differentiable at $(g, v)$. For each $j$, $1 \leqq j \leqq n$, we have $\frac{\partial g}{\partial w_j} \in \mathscr{C}^\alpha(b\Delta, \mathscr{C}^1(B))$ so by Lemma 11.2 the maps $u \mapsto \left(\zeta \mapsto \frac{\partial g(\zeta)}{\partial w_j}(u(\zeta)\right)$ are continuous on $\Omega$. Thus, there is a constant $c(g, v)$ such that

$$\begin{aligned} \|I_1\|_\alpha &\leqq \sum_{j=1}^m \|w_j\|_\alpha \cdot \int_0^1 \left\| \frac{\partial g(\bullet)}{\partial w_j}(v(\bullet) + \lambda w(\bullet)) - \frac{\partial g(\bullet)}{\partial w_j}(v(\bullet)) \right\|_\alpha d\lambda \\ &\leqq c(g, v) \cdot \|w\| \cdot o(\|w\|)\,. \end{aligned}$$

Further, since for each $j$, $1 \leqq j \leqq N$, the map $\zeta \mapsto \frac{\partial \varphi(\zeta)}{\partial w_j}$ belongs to $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^1(B))$, Lemma 11.2 and the definition of the norm in $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))$ imply that for each $\lambda$, $0 \leqq \lambda \leqq 1$,

$$\begin{aligned} \left\| \frac{\partial \varphi(\bullet)}{\partial w_j}(v(\bullet) + \lambda w(\bullet)) \right\|_\alpha &\leqq \left\| \frac{\partial \varphi(\bullet)}{\partial w_j}(v(\bullet)) \right\|_\alpha + \eta_j(\|w\|) \\ &\leqq (2 + m^{1/2}\|v\|) \cdot \left\| \frac{\partial \varphi}{\partial w_j} \right\|_{\mathscr{C}^\alpha(b\Delta, \mathscr{C}^1(B))} + \eta_j(\|w\|) \\ &\leqq (2 + m^{1/2}\|v\|) \cdot \|\varphi\| + \eta_j(\|w\|) \end{aligned}$$

where $\|\varphi\|$ is the norm of $\varphi$ in $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))$ and where $\eta_j(\|w\|) \to 0$ as $w \to 0$. It follows that there are a constant $C(v)$ and a function $\eta$, $\eta(\|w\|) \to 0$ as $w \to 0$, such that for $\|w\|$ small we have

$$\|I_2\|_\alpha \leqq C(v) \cdot \|w\| \cdot (\|\varphi\| + \eta(\|w\|))\,.$$

This proves that $H$ is differentiable at $(g, v)$ and that

$$(DH)(g, v)(\varphi, w)(\zeta) = \sum_{j=1}^m w_j(\zeta) \frac{\partial g(\zeta)}{\partial w_j}(v(\zeta)) + \varphi(\zeta)(v(\zeta))\,.$$

To prove that $H$ is of class $\mathscr{C}^1$ we have to show that $w \mapsto \left(\zeta \mapsto \sum_{j=1}^m w_j(\zeta) \frac{\partial g(\zeta)}{\partial w_j}(v(\zeta))\right)$, bounded linear map from $\mathscr{C}^\alpha(b\Delta, R^m)$ to $\mathscr{C}^\alpha(b\Delta)$ depends continuously on $(g, v)$, and that $\varphi \mapsto (\zeta \mapsto \varphi(\zeta)(v(\zeta)))$, a bounded linear map from $\mathscr{C}^\alpha(b\Delta, \mathscr{C}^2(B))$ to $\mathscr{C}^\alpha(b\Delta)$ depends continuously on $v$. To prove the first statement, since $\mathscr{C}^\alpha(b\Delta)$ is a Banach algebra, it is enough to show that given $g\in\mathscr{C}^\alpha(b\Delta,\mathscr{C}^2(B))$ and $v\in\Omega$, for each $j$, $1\leqq j\leqq m$, $J_j=\left\|\frac{\partial g(\bullet)}{\partial w_j}(v(\bullet))-\frac{\partial\tilde{g}(\bullet)}{\partial w_j}(\tilde{v}(\bullet))\right\|_\alpha$ is arbitrarily small provided that $g\in\mathscr{C}^\alpha(b\Delta,\mathscr{C}^2(B))$, $\tilde{v}\in\Omega$ and that $\|g-\tilde{g}\|$ and $\|v-\tilde{v}\|$ are small enough. We have

$$J_j\leqq\left\|\frac{\partial g(\bullet)}{\partial w_j}(v(\bullet))-\frac{\partial g(\bullet)}{\partial w_j}(\tilde{v}(\bullet))\right\|_\alpha+\left\|\frac{\partial(g-\tilde{g})}{\partial w_j}(\tilde{v}(\bullet))\right\|_\alpha. \tag{11.9}$$

Since $\frac{\partial g}{\partial w_j}\in\mathscr{C}^\alpha(b\Delta,\mathscr{C}^1(B))$ Lemma 11.2 implies that the first term is arbitrarily small provided that $\|v-\tilde{v}\|$ is small enough. By Lemma 11.2 and by the definition of the norm in $\mathscr{C}^\alpha(b\Delta,\mathscr{C}^2(B))$ we have

$$\begin{aligned}\left\|\frac{\partial(g-\tilde{g})(\bullet)}{\partial w_j}(\tilde{v}(\bullet))\right\|_\alpha&\leqq(2+m^{1/2}\|\tilde{v}\|)\cdot\left\|\frac{\partial(g-\tilde{g})(\bullet)}{\partial w_j}\right\|_{\mathscr{C}^\alpha(b\Delta,\mathscr{C}^1(B)))}\\&\leqq(2+m^{1/2}\|\tilde{v}\|)\cdot\|g-\tilde{g}\|\end{aligned}$$

which shows that the second term in (11.9) is arbitrarily small provided that $\|g-\tilde{g}\|$ and $\|v-\tilde{v}\|$ are small enough. This proves the first statement. To prove the second statement we have to show that given $v\in\Omega$ and $\varepsilon>0$ there is a $\delta>0$ such that $\|\varphi(\bullet)(\tilde{v}(\bullet))-\varphi(\bullet)(v(\bullet))\|_\alpha<\varepsilon$ for all $\varphi$, $\|\varphi\|<1$ and all $v$, $\|\tilde{v}-v\|<\delta$. We have

$$\|\varphi(\bullet)(\tilde{v}(\bullet))-\varphi(\bullet)(v(\bullet))\|_\alpha\leqq\sum_{j=1}^m\|\tilde{v}_j-v_j\|_\alpha\int_0^1\left\|\frac{\partial\varphi(\bullet)}{\partial w_j}(v(\bullet)+\lambda(\tilde{v}(\bullet)-v(\bullet))\right\|_\alpha d\lambda\,.$$

Since for each $j$, $1\leqq j\leqq N$, we have $\frac{\partial\varphi}{\partial w_j}\in\mathscr{C}^\alpha(b\Delta,\mathscr{C}^1(B))$ Lemma 11.2 applies to show that

$$\begin{aligned}\left\|\frac{\partial\varphi(\bullet)}{\partial w_j}(v(\bullet)+\lambda(\tilde{v}(\bullet)-v(\bullet)))\right\|_\alpha&\leqq\left\|\frac{\partial\varphi}{\partial w_j}\right\|_{\mathscr{C}^\alpha(b\Delta,\mathscr{C}^1(B))}\cdot(2+m^{1/2}\cdot\|v+\lambda(\tilde{v}-v)\|)\\&\leqq\|\varphi\|\cdot(3+m^{1/2}\|v\|)\end{aligned}$$

provided that $\|\tilde{v}-v\|\leqq 1$. Thus, there is a constant $c(v)$ such that

$$\|\varphi(\bullet)(\tilde{v}(\bullet))-\varphi(\bullet)(v(\bullet))\|_\alpha\leqq c(v)\cdot\|\tilde{v}-v\|\cdot\|\varphi\|$$

provided that $\|\tilde{v}-v\|\leqq 1$. This completes the proof.

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