Math. Z. 218, 603–609 (1995)

Mathematische Zeitschrift


# Weak lower semicontinuity of polyconvex integrals: a borderline case

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Received 27 April 1993; in final form 23 December 1993

## 1 Introduction

In this paper we prove the weak lower semicontinuity of a functional of the form

$$\int_\Omega g(\det Du)\,dx\,,$$

where $Du$ is the gradient matrix of a function $u \in W^{1,n}(\Omega;\mathbf{R}^n)$, and $\Omega$ is a bounded open set in $\mathbf{R}^n$. More precisely, we prove that, if $g$ is convex and satisfies

$$g(t) \geq a|t| - b \tag{1.1}$$

for suitable constants $a > 0$ and $b \geq 0$, then

$$\int_\Omega g(\det Du)\,dx \;\leq\; \liminf_{h\to\infty} \int_\Omega g(\det Du_h)\,dx \tag{1.2}$$

for every sequence $(u_h)$ in $W^{1,n}(\Omega;\mathbf{R}^n)$ converging to $u \in W^{1,n}(\Omega;\mathbf{R}^n)$ in the weak topology of $W^{1,p}(\Omega;\mathbf{R}^n)$, with $p \geq n-1$. In the case $n = 2$ the hypothesis that $(u_h)$ converges weakly in $W^{1,1}(\Omega;\mathbf{R}^2)$ can be replaced by the weaker assumption that $(u_h)$ converges to $u$ in $L^1(\Omega;\mathbf{R}^2)$ and is bounded in $W^{1,1}(\Omega;\mathbf{R}^2)$.

In the case $p > n-1$ this result was proved by B. Dacorogna and P. Marcellini in [4] without the coerciveness hypothesis (1.1). In the case $n = 2$ their method gives (1.2) under the assumption that $(u_h)$ converges weakly in $W^{1,1}(\Omega;\mathbf{R}^2)$, but can not be applied to sequences converging in $L^1(\Omega;\mathbf{R}^2)$ that are only bounded in $W^{1,1}(\Omega;\mathbf{R}^2)$. See also L. Carbone and R. De Arcangelis [3] and J. Malý [10] for related results.

We remark that (1.2) does not hold if $(u_h)$ converges weakly in $W^{1,p}(\Omega;\mathbf{R}^n)$, with $p < n-1$, as shown by a recent counterexample due to J. Malý [9]. In the same paper he proves, without the coerciveness assumption (1.1), that (1.2) holds when $u_h$ and $u$ are orientation preserving diffeomorphisms and $(u_h)$ converges weakly in $W^{1,p}(\Omega;\mathbf{R}^n)$, with $p \geq n-1$.

We note that in all these results the hypothesis $u_h$, $u \in W^{1,n}(\Omega;\mathbf{R}^n)$ can not be replaced by the hypothesis $u_h$, $u \in W^{1,p}(\Omega;\mathbf{R}^n)$ for $n-1 \leq p < n$, as shown by a counterexample due to J.M. Ball and F. Murat [2].

In Section 3 we consider the more general case of a functional of the form

$$\int_\Omega f(x,u,Du)\,dx\,,$$

defined for $u \in W^{1,\nu}(\Omega;\mathbf{R}^m)$, with $\nu = \min\{m,n\}$. We assume that $f$ is polyconvex and satisfies, for an integer $k \leq \nu$, the coerciveness inequality

$$f(x,y,A) \geq a|\mathcal{M}_k^\nu(A)| - b\,,$$

where $\mathcal{M}_k^\nu(A)$ denotes the vector whose components are the determinants of all minors of the matrix $A$ of order greater than or equal to $k$. Under a suitable lower semicontinuity hypothesis on $f$ we prove that

$$\int_\Omega f(x,u,Du)\,dx \leq \liminf_{h\to\infty} \int_\Omega f(x,u_h,Du_h)\,dx$$

for every sequence $(u_h)$ in $W^{1,\nu}(\Omega;\mathbf{R}^m)$ converging to $u \in W^{1,\nu}(\Omega;\mathbf{R}^m)$ in the weak topology of $W^{1,p}(\Omega;\mathbf{R}^m)$, with $p \geq k-1$.

The special case where $f(x,y,A) = \psi(x,y)g(A)$, $m=n$, and $p > n-1$ was studied by W. Gangbo [5] without any coerciveness hypothesis.

The proof of our results relies on a lower semicontinuity theorem with respect to $L^1(\Omega;\mathbf{R}^m)$ convergence proved by E. Acerbi and G. Dal Maso [1], that is based on the results of M. Giaquinta, G. Modica, and J. Souček [6], [7].

## 2 The model case

Let $g\colon \mathbf{R} \to \mathbf{R}$ be a convex function. Assume that there exist $a > 0$ and $b \geq 0$ such that

$$g(t) \geq a|t| - b \tag{2.1}$$

for every $t \in \mathbf{R}$. Let $\Omega$ be a bounded open set in $\mathbf{R}^n$. Under these hypotheses we have the following lower semicontinuity result.

**Theorem 2.1** *Let $u$ and $u_h$, $h \in \mathbf{N}$, be functions in $W^{1,n}(\Omega;\mathbf{R}^n)$. Assume that $(u_h)$ converges to $u$ in $L^1(\Omega;\mathbf{R}^n)$ and that $\|u_h\|_{W^{1,n-1}(\Omega;\mathbf{R}^n)}$ is bounded uniformly with respect to $h$. Then*

$$\int_\Omega g(\det Du)\,dx \leq \liminf_{h\to\infty} \int_\Omega g(\det Du_h)\,dx\,.$$

The proof depends on the following Theorem 2.2, that is based on the results of M. Giaquinta, G. Modica, and J. Souček. Let $\mathbf{M}^{m\times n}$ be the space of all $m\times n$ matrices. For every $A \in \mathbf{M}^{m\times n}$, we denote by $\mathcal{M}(A)$ the vector whose components are the determinants of the minors of the matrix $A$ of arbitrary order. We recall that a function $f:\mathbf{M}^{m\times n} \to \mathbf{R}$ is said to be polyconvex, if there exists a convex function $\varphi:\mathbf{R}^{\tau} \to \mathbf{R}$ such that $f(A) = \varphi(\mathcal{M}(A))$ for every $A \in \mathbf{M}^{m\times n}$, where $\tau$ is the number of all minors of an $m\times n$ matrix.

**Theorem 2.2** *Let $f:\mathbf{M}^{m\times n} \to \mathbf{R}$ be polyconvex. Assume that there exist $a > 0$ and $b \geq 0$ such that*

$$f(A) \geq a|\mathcal{M}(A)| - b \tag{2.2}$$

*for every $A \in \mathbf{M}^{m\times n}$. If $u$ and $u_h$, $h \in \mathbf{N}$, belong to $W^{1,\nu}(\Omega;\mathbf{R}^m)$, where $\nu = \min\{m,n\}$, and if $(u_h)$ converges to $u$ in $L^1(\Omega;\mathbf{R}^m)$, then*

$$\int_\Omega f(Du)\,dx \leq \liminf_{h\to\infty} \int_\Omega f(Du_h)\,dx\,. \tag{2.3}$$

*Proof.* The deduction of this result from [6] and [7] can be found in Corollary 3.13 of [1] under the additional assumption that there exist $c > 0$ and $d \geq 0$ such that

$$f(A) \leq c|A|^{\nu} + d$$

for every $A \in \mathbf{M}^{m\times n}$. If $f$ does not satisfy this assumption, let us write $f(A) = \varphi(\mathcal{M}(A))$, where $\varphi:\mathbf{R}^{\tau} \to \mathbf{R}$ is a convex function such that

$$\varphi(\xi) \geq a|\xi| - b \tag{2.4}$$

for every $\xi \in \mathbf{R}^{\tau}$. Then there exists an increasing sequence $(\varphi_k)$ of convex functions converging to $\varphi$ such that each function $\varphi_k$ is Lipschitz continuous on $\mathbf{R}^{\tau}$ with Lipschitz constant $k$ and satisfies (2.4) for $k \geq a$. Indeed, it is enough to define

$$\varphi_k(\xi) = \sup_{|\xi^*|\leq k} \left(\langle \xi^*, \xi\rangle - \varphi^*(\xi^*)\right),$$

where $\varphi^*$ is the Young-Fenchel conjugate of $\varphi$. Set $f_k(A) = \varphi_k(\mathcal{M}(A))$ and note that

$$f_k(A) \leq \varphi_k(0) + k|\mathcal{M}(A)| \leq \varphi_k(0) + kc|A|^{\nu}\,.$$

From the previous step we get

$$\int_\Omega f_k(Du)\,dx \leq \liminf_{h\to\infty} \int_\Omega f_k(Du_h)\,dx \leq \liminf_{h\to\infty} \int_\Omega f(Du_h)\,dx\,.$$

Passing to the limit as $k \to \infty$ we obtain (2.3).

*Proof of Theorem 2.1.* Given $\varepsilon > 0$, let us consider the function $f_\varepsilon:\mathbf{M}^{n\times n} \to \mathbf{R}$ defined by

$$f_\varepsilon(A) = g(\det A) + \varepsilon|\mathcal{M}_1^{n-1}(A)|\,,$$

where $\mathcal{M}_1^{n-1}(A)$ denotes the vector whose components are the determinants of all minors of the matrix $A$ of order less than or equal to $n-1$. As $g$ is convex and satisfies (2.1), it is clear that $f_\varepsilon$ is polyconvex and satisfies (2.2) for suitable constants $a > 0$ and $b \geq 0$. By Theorem 2.2 we have

$$\int_\Omega f_\varepsilon(Du)\,dx \leq \liminf_{h\to\infty} \int_\Omega f_\varepsilon(Du_h)\,dx\,. \tag{2.5}$$

Since

$$\int_\Omega g(\det Du)\,dx \leq \int_\Omega f_\varepsilon(Du)\,dx$$

and

$$\int_\Omega f_\varepsilon(Du_h)\,dx \leq \int_\Omega g(\det Du_h)\,dx + \varepsilon K \|u_h\|_{W^{1,n-1}(\Omega;\mathbf{R}^n)}$$

for a suitable constant $K$, independent of $\varepsilon$ and $h$, from (2.5) we obtain

$$\int_\Omega g(\det Du)\,dx \leq \liminf_{h\to\infty} \int_\Omega g(\det Du_h)\,dx + \varepsilon KC\,,$$

where $\|u_h\|_{W^{1,n-1}(\Omega;\mathbf{R}^n)} \leq C$ for every $h$. The conclusion follows by passing to the limit as $\varepsilon \to 0$.

## 3 A more general result

More generally, using the same technique, we can prove the following result. Let $f\colon \Omega\times\mathbf{R}^m\times\mathbf{M}^{m\times n} \to [0,+\infty[$ be a function such that:

(i) for every $(x,y) \in \Omega\times\mathbf{R}^m$ the function $A \mapsto f(x,y,A)$ is polyconvex on $\mathbf{M}^{m\times n}$;

(ii) there exist $a > 0$, $b \geq 0$ and $k \in \mathbf{N}$, with $1 \leq k \leq \nu = \min\{m,n\}$, such that

$$f(x,y,A) \geq a|\mathcal{M}_k^\nu(A)| - b$$

for every $x \in \Omega$, $y \in \mathbf{R}^m$, $A \in \mathbf{M}^{m\times n}$, where $\mathcal{M}_k^\nu(A)$ denotes the vector whose components are the determinants of all minors of the matrix $A$ of order greater than or equal to $k$;

(iii) for every $x_0 \in \Omega$, $y_0 \in \mathbf{R}^m$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x,y,A) \geq (1-\varepsilon)f(x_0,y_0,A)$$

for every $x \in \Omega$, $y \in \mathbf{R}^m$, $A \in \mathbf{M}^{m\times n}$, with $|x - x_0| < \delta$ and $|y - y_0| < \delta$.

Note that every function of the form $f(x,y,A) = \psi(x,y)g(A)$ satisfies conditions (i), (ii), (iii), if $\psi\colon \Omega\times\mathbf{R}^m \to [0,+\infty[$ is lower semicontinuous, $\psi(x,y) \geq 1$, $g\colon \mathbf{M}^{m\times n} \to [0,+\infty[$ is polyconvex, and $g(A) \geq a|\mathcal{M}_k^\nu(A)| - b$.

**Theorem 3.1** *Assume that $f\colon \Omega\times\mathbf{R}^m\times\mathbf{M}^{m\times n} \to [0,+\infty[$ satisfies (i), (ii), (iii). Let $u$ and $u_h$, $h \in \mathbf{N}$, be functions in $W^{1,\nu}(\Omega;\mathbf{R}^n)$. Assume that $(u_h)$ converges to*

*$u$ in $L^1(\Omega;\mathbf{R}^m)$ and that $\|u_h\|_{W^{1,k-1}(\Omega;\mathbf{R}^m)}$ is bounded uniformly with respect to $h$. Then*

$$\int_\Omega f(x,u,Du)\,dx \le \liminf_{h\to\infty}\int_\Omega f(x,u_h,Du_h)\,dx\,.$$

The proof depends on the following theorem.

**Theorem 3.2** *Let $f\colon \Omega\times\mathbf{R}^m\times\mathbf{M}^{m\times n}\to[0,+\infty[$ be a function satisfying conditions (i) and (iii). Assume that there exist $a>0$ and $b\ge 0$ such that*

$$f(x,y,A)\ge a|\mathcal{M}(A)|-b$$

*$x\in\Omega$, $y\in\mathbf{R}^m$, $A\in\mathbf{M}^{m\times n}$. If $u$ and $u_h$, $h\in\mathbf{N}$, belong to $W^{1,\nu}(\Omega;\mathbf{R}^m)$ and if $(u_h)$ converges to $u$ in $L^1(\Omega;\mathbf{R}^m)$, then*

$$\int_\Omega f(x,u,Du)\,dx \le \liminf_{h\to\infty}\int_\Omega f(x,u_h,Du_h)\,dx\,.$$

*Proof.* The theorem is proved in Corollary 3.13 of [1] under the additional hypothesis that $f$ is continuous and that $f$ satisfies an estimate from above of the form

$$f(x,y,A)\le c|A|^\nu+d\,.$$

To prove the theorem in the general case, we associate with $f$ the class $\mathcal{G}$ of all functions $g\colon\Omega\times\mathbf{R}^m\times\mathbf{M}^{m\times n}\to[0,+\infty[$ with the following properties:

(a) $g(x,y,A)\le f(x,y,A)$ for every $x\in\Omega$, $y\in\mathbf{R}^m$, $A\in\mathbf{M}^{m\times n}$;

(b) $g$ satisfies all hypotheses of Theorem 3.2 with constants $a$, $b$ depending on $g$;

(c) $g$ is continuous on $\Omega\times\mathbf{R}^m\times\mathbf{M}^{m\times n}$ and satisfies an inequality of the form

$$g(x,y,A)\le c|A|^\nu+d\,,$$

with constants $c$ and $d$ depending on $g$.

First of all let us prove that

$$f(x_0,y_0,A)=\sup_{g\in\mathcal{G}} g(x_0,y_0,A) \tag{3.1}$$

for every $x_0\in\Omega$, $y_0\in\mathbf{R}^m$, $A\in\mathbf{M}^{m\times n}$. By property (iii) of $f$, for every $\varepsilon>0$ there exists $\delta>0$ such that $B(x_0,\delta)\subseteq\Omega$ and

$$f(x,y,A)\ge(1-\varepsilon)f(x_0,y_0,A)$$

for every $x\in\Omega$, $y\in\mathbf{R}^m$, $A\in\mathbf{M}^{m\times n}$, with $|x-x_0|<\delta$ and $|y-y_0|<\delta$. Let $\psi\colon\Omega\times\mathbf{R}^m\to\mathbf{R}$ be a cut-off function between $B(x_0,\frac{\delta}{2})\times B(y_0,\frac{\delta}{2})$ and $B(x_0,\delta)\times B(y_0,\delta)$, i.e., $\psi\in C_0^\infty(\Omega\times\mathbf{R}^m)$, $0\le\psi\le1$ on $\Omega\times\mathbf{R}^m$, $\psi=1$ on $B(x_0,\frac{\delta}{2})\times B(y_0,\frac{\delta}{2})$, and $\psi=0$ out of $B(x_0,\delta)\times B(y_0,\delta)$. Let us represent the polyconvex function $f(x_0,y_0,\cdot)$ as $f(x_0,y_0,A)=\varphi(\mathcal{M}(A))$, where $\varphi\colon\mathbf{R}^\tau\to[0,+\infty[$

is a convex function satisfying the estimate from below (2.4), and let us consider an increasing sequence $(\varphi_j)$ of non-negative convex functions converging to $\varphi$ such that $\varphi_j$ is Lipschitz continuous on $\mathbf{R}^\tau$ with Lipschitz constant $j$ and satisfies (2.4) for $j \geq a$. Define $g_j^\varepsilon : \Omega \times \mathbf{R}^m \times \mathbf{M}^{m\times n} \to [0, +\infty[$ as

$$g_j^\varepsilon(x,y,A) = \psi(x,y)(1-\varepsilon)\varphi_j(\mathscr{M}(A)) + \big(1-\psi(x,y)\big)(a|\mathscr{M}(A)| - b)^+ .$$

It is easy to see that the functions $g_j^\varepsilon$ belong to the class $\mathscr{G}$ for $j \geq a$. Passing to the limit as $j \to \infty$ we get

$$(1-\varepsilon)f(x_0,y_0,A) = \lim_{j\to\infty} g_j^\varepsilon(x_0,y_0,A) \leq \sup_{g\in\mathscr{G}} g(x_0,y_0,A) .$$

As $\varepsilon \to 0$ we obtain (3.1). By Lindelöf Theorem (see, e.g., [8], Chapter 1, Theorem 15) there exists a sequence $(g_i)$ in $\mathscr{G}$ such that

$$f(x,y,A) = \sup_{i\in\mathbf{N}} g_i(x,y,A)$$

for every $x \in \Omega$, $y \in \mathbf{R}^m$, $A \in \mathbf{M}^{m\times n}$. By the stability properties of the class $\mathscr{G}$ it is not restrictive to assume that the sequence $(g_i)$ is increasing. Since each function $g_i$ satisfies all conditions of Corollary 3.13 in [1], we have

$$\int_\Omega g_i(x,u,Du)\,dx \leq \liminf_{h\to\infty} \int_\Omega g_i(x,u_h,Du_h)\,dx \leq \liminf_{h\to\infty} \int_\Omega f(x,u_h,Du_h)\,dx .$$

All that remains is to take the limit as $i \to \infty$.

*Proof of Theorem 3.1.* It is enough to consider, for every $\varepsilon > 0$, the function

$$f_\varepsilon(x,y,A) = f(x,y,A) + \varepsilon|\mathscr{M}_1^{k-1}(A)| ,$$

where $\mathscr{M}_1^{k-1}(A)$ denotes the vector whose components are the determinants of all minors of the matrix $A$ of order less than or equal to $k-1$. The conclusion follows easily as in the proof of Theorem 2.1.

*Acknowledgement.* This work is part of the Project EURHomogenization, Contract SC1-CT91-0732, of the Program SCIENCE of the Commission of the European Communities, and of the Research Project "Irregular Variational Problems" of the Italian National Research Council. The paper was written while the second author was visiting the International School for Advanced Studies (SISSA) and the International Centre for Theoretical Physics (ICTP) in Trieste.

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