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LIP manifolds: from metric to Finslerian structure★

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0 Introduction

This introduction will be a short survey on the subject and a description of the motivations about the concepts introduced.

Let (M, g) be a smooth Riemannian manifold of dimension n. It is known that, given the metric g, it is possible to define the length of C^1-piecewise curves and to construct an intrinsic geodesic distance δ on M; on the other hand, when δ is given, the original metric can be restored by differentiating the function that gives the distance from a point [B-dR].

In particular, we want to recall that the directional derivative along a vector v gives the Riemannian norm of v; i.e., in a local coordinate chart (U, Φ) at the point $x \in M$, if $\xi = \Phi(x)$ and if v is a vector of $\Phi(U) \subset \mathbb{R}^n$, putting

$$(*) \qquad \varphi(\xi, v) = \lim_{t \to 0} \frac{\delta(\Phi^{-1}(\xi), \Phi^{-1}(\xi + tv))}{t},$$

one has (with abuse of notation)

$$(**) \qquad \varphi(x, v) = \|v\|_{g(x)} = \sqrt{g_{ij}(x) v^i v^j}\,.$$

Let us consider now a Lipschitz manifold M (briefly **LIP manifold**) of dimension n, i.e. a topological manifold (with countable basis) whose change of charts are Lipschitz functions. Following the presentation of Teleman [T], we consider on M a **LIP Riemannian metric** g, i.e. an "*elliptic*" metric generally with measurable coefficients, because it is not possible to ask a greater regularity.

These manifolds, which generalize the *polyhedra*, can have vertices, edges, conical points, even not isolated. Moreover, it is possible to move the singularities of the carrier to singularities of a metric g on a smooth carrier; for example,

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the problem of geodesics with obstacles, or on a manifold with boundary, can be studied as a problem about metrics with singularities.

Metrics with singularities can be found in Physics, in Engineering [B]: in the general case, the carrier is regular and the singularities of the metric have a physical meaning, e.g. they are caused by "irregularities" of the materials.

Given (M, g), starting from g, it is possible to construct a *intrinsic* distance in the following way [DC-P1].

If N is a set of zero measure (with respect to the measure induced by g or by the charts), a LIP curve $\gamma : [0,1] \to M$ is said **transversal** to N (abbreviated $\gamma \overset{\mathsf{T}}{\cap} N$) if

$$\operatorname{meas}\{t \in [0,1]; \gamma(t) \in N\} = 0 .$$

If a LIP atlas $\mathscr{A}$ of M is fixed, we consider the integral

$$L_{\mathscr{A}}(\gamma; g) = \int_0^1 \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}\,dt$$

which we put equal to $+\infty$ if it does not exist or is not well defined; however with a suitable choice of the set $N = N(\mathscr{A})$ the previous integral has the meaning of the *length* of γ (with respect to the atlas $\mathscr{A}$). Then one constructs the intrinsic distance, which depends on g but is independent of $\mathscr{A}$,

$$\rho^g(x,y) = \sup_N \{\inf_\gamma \{L_{\mathscr{A}}(\gamma; g); \gamma(0) = x, \text{approach } \gamma(1) = y, \gamma \overset{\mathsf{T}}{\cap} N\}; \operatorname{meas}(N) = 0\}$$

which turns out to be also given by

$$\begin{aligned}\rho^g(x,y) &= [\inf\{\|du\|_\infty; u \in \mathscr{L}ip(M), u(x) = 0, u(y) = 1\}]^{-1}\\ &= \sup\{|u(x) - u(y)|; u \in \mathscr{L}ip(M), \|du\|_\infty \leqq 1\},\end{aligned}$$

where $\mathscr{L}ip(M)$ is the set of Lipschitz functions on M. This distance coincides, under very general conditions (cf. [DC-P2] and [DC-P3]), with the integral one

$$\delta^g(x,y) = \lim_{p\to\infty} \left[\inf\left\{\left[\int_M |du|^p d\mu\right]^{1/p}; u \in \mathscr{L}ip(M), u(x) = 0, u(y) = 1\right\}\right]^{-1}$$

(where $|\cdot|$ and $d\mu$ depend, as usual, on the metric g on M).

Now, using the distance ρ^g (or δ^g), one can define the length of γ, $\mathscr{L}(\gamma; \rho^g)$, in the usual manner, and prove that [DC-P2]

$$\mathscr{L}(\gamma; \rho^g) = \sup_N \left\{\liminf_{\tau\to\gamma} \mathscr{L}_{\mathscr{A}}(\tau; g); \tau \overset{\mathsf{T}}{\cap} N, \operatorname{meas}(N) = 0\right\}$$

and

$$\rho^g(x,y) = \inf_\gamma \{\mathscr{L}(\gamma; \rho^g); \gamma(0) = x, \gamma(1) = y\},$$

i.e (M, ρ^g) is a *metric space with intrinsic distance* [R] or a *length space* [G].

In this framework, when considering the limit $(*)$, one can ask whether $(**)$ holds.

In [DC-P4] an example ($\mathbb{R}^2$ with a suitable LIP metric) is produced to show that, under our hypotheses, the equality $(**)$ does not hold on a set of *positive* measure; φ cannot be expressed almost everywhere as a square root of a quadratic form. Nevertheless

$$\mathscr{L}(\gamma; \rho^g) = \int_0^1 \varphi(\gamma, \dot{\gamma})dt .$$

In general the function φ, with respect to the second variable, is positive homogeneous of degree 1 and convex, i.e. it is a metric of "*Finsler type*".

Finslerian metrics in the classical case or in weaker hypotheses are studied by Busemann and Mayer [B-M] and with generalization by Pauc [P] in view of calculus of variations on metric spaces.

The previous results lead to the introduction (in [DC-P4]) of the LIP *Finslerian manifolds with* LIP *Finslerian metric* (M, F) which generalize, on one hand, the Finslerian manifolds and, on the other hand, the LIP *Riemannian manifolds* if

$$F(x, v) = \sqrt{g_{ij}(x)v^i v^j} .$$

Also on (M, F), if γ is a curve transversal to a suitable set of zero measure (as in [DC-P1]) it is possible to define the integral

$$L_{\mathscr{A}}(\gamma; F) = \int_0^1 F(\gamma, \dot{\gamma})dt ,$$

and consequently to introduce (geometrically and analytically) a geodesic distance, ρ^F, induced by the Finslerian structure F. The length, $\mathscr{L}(\gamma; \rho^F)$, of an absolutely continuous curve γ (briefly $\gamma \in AC(x, y)$) usually defined by the metric structure ρ^F, coincides with one, $L(\gamma; F)$, defined by the Finslerian (or Riemannian) structure F. So the metric space (M, ρ^F) becomes a *length space*.

Moreover, if the function $F(\cdot, v)$ is upper-semicontinuous, then the transversality condition and the supremum on N can be left out in the above definitions.

In the present paper the following problem is considered from a general viewpoint. Let M be a LIP manifold with a distance δ, which is locally equivalent to the Euclidean one; which hypotheses are needed to endow M with a Finsler structure F in such way that F induces the original given distance?

In the first place, when δ is given, the function φ defined by $(*)$, endowes M with a Finsler structure, secondly it is possible to show if γ is an absolutely continuous curve, then the following

$$\mathscr{L}(\gamma; \delta) = \int_0^1 \varphi(\gamma, \dot{\gamma})dt ,$$

holds, i.e. the length can be calculated by an integral.

Evidently (M, δ) must necessarily be a length space in order for the problem to be answered positively. Under such a hypothesis, in general the inequality $\delta \leqq \rho^\varphi$ holds; an example shows that $\delta \neq \rho^\varphi$ can actually be true. Now a necessary and sufficient condition for $\delta(x, y) = \rho^\varphi(x, y)$ is that

$$\delta(x,y) = \sup_N \left\{ \inf_\gamma \{\mathscr{L}(\gamma;\delta); \gamma \in AC(x,y), \gamma \overset{\mathrm{T}}{\cap} N\} \ \mathrm{meas}(N) = 0 \right\} .$$

Finally we observe that, if one starts from a Finsler structure F, and considers the distance ρ^F and the function φ that corresponds to it through $(*)$, then the example given above yields $\varphi \neq F$. However if one repeats the procedure starting from φ nothing new is obtained, viz. the construction given above is "stable" and the metric φ becomes LIP *Finslerian stable* (or *quasi-Finslerian* according to [DG3]). Moreover if φ is the "*derivative*" of ρ^F, it follows that $\rho^F = \rho^\varphi$ even when $\varphi \neq F$.

Therefore a LIP manifold M with a Finslerian structure F may not be stable with respect to F, but it is isometric, as a metric space, to the LIP Finslerian *stable* manifold (M, φ). On the other hand there exist LIP Riemannian manifold (M, g) that are not isometric to LIP *Riemannian stable manifolds.* Obviously a smooth Finslerian (resp. Riemannian) manifold in the classical sense, is even a LIP Finslerian stable (resp. LIP Riemannian stable); the result still holds for continuous metrics on manifolds of class C^1 at least.

The paper answers, at least partially, conjectures, in the context of quasi-Riemannian and quasi-Finslerian metric spaces, formulated by De Giorgi (cfr. [DG1, DG2, DG3]).

Venturini, in the framework of the questions here considered, studied systematically the relationships between the class $\mathscr{D}(V)$ of distance functions locally equivalent to the Euclidean distance and the class $\mathscr{M}(V)$ of Borel metrics locally equivalent to the Euclidean metric on an open connected subset $V \subseteq \mathbb{R}^n$. The operators he introduced (in [V])

$$\Delta : \mathscr{D}(V) \to \mathscr{M}(V) \quad I : \mathscr{M}(V) \to \mathscr{D}(V)$$

respectively of derivation and integration are similar to the ones defined by us.

1 LIP Finslerian manifold

(1.1) A **Lipschitz manifold (LIP manifold)** of dimension n is a pair consisting of a topological manifold M and an equivalence class of LIP atlases [L-V, T]. A LIP atlas $\mathscr{A}$ on M is a family of charts $\{(U_\alpha, \Phi_\alpha)\}(\alpha \in \Lambda)$, where $\{U_\alpha\}$ forms an open cover of $M, \Phi_\alpha : U_\alpha \to V_\alpha$ maps homeomorphically U_α onto a set V_α which is open either in $\mathbb{R}^n$ or in $\mathbb{R}^n_+$ and $\forall \alpha, \beta$

$$\Phi_{\alpha\beta} = \Phi_\beta \circ \Phi_\alpha^{-1} : \Phi_\alpha(U_\alpha \cap U_\beta) \to \Phi_\beta(U_\alpha \cap U_\beta)$$

defines a Lipschitz homeomorphism.

(1.2) A **(weakly) Finslerian structure** on $V_\alpha \subset \mathbb{R}^n$ is a function $F_\alpha : V_\alpha \times \mathbb{R}^n \to \mathbb{R}$ such that

1) $F_\alpha(\,\cdot\,, v)$ is measurable $\forall v \in \mathbb{R}^n$ and $F_\alpha(\xi, \,\cdot\,)$ is continuous for a.e. $\xi \in V_\alpha$;
2) $F_\alpha(\xi, v) > 0$ for a.e. ξ if $v \neq 0$;
3) $F_\alpha(\xi, tv) = tF_\alpha(\xi, v)$ if $t \geqq 0$.

A **Finslerian structure** on M is a collection $F = \{F_\alpha\}$ of (weakly) Finslerian structure F_α on $V_\alpha = \Phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$, such that $\forall \alpha, \beta$ the following compatibility condition

(1.3) $$F_\alpha(\xi, v) = F_\beta(\Phi_{\alpha\beta}(\xi), (d\Phi_{\alpha\beta})(\xi)(v)) \quad \text{a.e. } \xi \in V_\alpha, \quad \forall v \in \mathbb{R}^n,$$

holds, where $\Phi_\alpha^{-1}(\xi) \in U_\alpha \cap U_\beta$.

(1.4) A Finsler structure F will be called a **LIP Finslerian metric** on M if $\forall \alpha$ two (strictly) positive constants h_α and k_α exist such that

(1.5) $$h_\alpha |v| \leqq F_\alpha(\xi, v) \leqq k_\alpha |v| \quad \text{a.e. } \xi \in V_\alpha$$

where $|v|$ is the standard Euclidean norm.

A LIP manifold M with a LIP Finslerian metric F will be called simply a **LIP Finslerian manifold** (M, F). If $x = \Phi_\alpha^{-1}(\xi) \in U_\alpha$, sometimes we write $F(x)$ instead $F_\alpha(\xi, \cdot)$ and expressively put

$$\|v\|_{F(x)} = F_\alpha(\xi, v).$$

If (M, g) is a LIP *Riemannian manifold* [T] and

$$F_\alpha(\xi, v) = (g_\alpha(\xi)(v, v))^{1/2} = \|v\|_{g(x)},$$

(M, g) is a LIP Finslerian manifold too.

In the following we shall always assume M to be a LIP Finslerian manifold unless otherwise indicated.

2 The "derivative" of distance function

(2.1) Let M be a LIP manifold and δ a distance which is locally equivalent to the Euclidean one in any chart (U_α, Φ_α). Moreover if

$$\sigma_\alpha(\xi, \eta) = \delta(\Phi_\alpha^{-1}(\xi), \Phi_\alpha^{-1}(\eta)) \quad \forall \xi, \eta \in V_\alpha$$

there exist two positive constants h_α and k_α such that

(2.2) $$h_\alpha |\xi - \eta| \leqq \sigma_\alpha(\xi, \eta) \leqq k_\alpha |\xi - \eta| \quad \forall \xi, \eta \in V_\alpha.$$

We put

$$\limsup_{t \to 0^+} \frac{\sigma_\alpha(\xi, \xi + t\eta)}{t} = \varphi_\alpha(\xi, \eta), \quad \liminf_{t \to 0^+} \frac{\sigma_\alpha(\xi, \xi + t\eta)}{t} = \underline{\varphi}_\alpha(\xi, \eta)$$

then

(2.2)′ $$h_\alpha |\eta| \leqq \varphi_\alpha(\xi, \eta) \leqq k_\alpha |\eta| \quad \forall \xi, \eta \in V_\alpha,$$

and analogously for $\underline{\varphi}_\alpha$.

(2.3) Lemma. *The functions φ_α and $\underline{\varphi}_\alpha$, which depend on the charts, are*

(i) *positive homogeneous of degree* 1 *with respect to* η;
(ii) LIP *with respect to* η *and Borel-measurable with respect to* ξ;
(iii) *compatible with the change of charts.*

Proof. For simplicity's sake we omit the index α when there is no risk of ambiguity.

(i) For $c > 0$

$$\varphi(\xi, c\eta) = \limsup_{t\to 0^+} \frac{\sigma(\xi, \xi + tc\eta)}{t} = c \limsup_{t\to 0^+} \frac{\sigma(\xi, \xi + tc\eta)}{tc} = c\varphi(\xi, \eta) .$$

Since the arguments we are going to use hold for φ and for $\underline{\varphi}$ we give them only for φ.

(ii) Taking into account that δ is a distance, by (2.2), one has

$$\sigma(\xi, \xi + t\eta_1) \leqq \sigma(\xi, \xi + t\eta_2) + \sigma(\xi + t\eta_2, \xi + t\eta_1) \leqq \sigma(\xi, \xi + t\eta_2) + tk|\eta_2 - \eta_1|$$

whence

$$\varphi(\xi, \eta_1) \leqq \varphi(\xi, \eta_2) + k|\eta_2 - \eta_1|$$

and interchanging the rôle of η_2 and η_1

$$|\varphi(\xi, \eta_1) - \varphi(\xi, \eta_2)| \leqq k|\eta_2 - \eta_1| ,$$

from which the LIP-nature of φ for every ξ follows.

By definition, $\xi \to \varphi(\xi, \eta)$ may be regarded as the limit of lower semi-continuous functions and, as consequence, as Borel-measurable with respect to ξ.

(iii) If $\Phi_\alpha^{-1}(\xi) \in U_\alpha \cap U_\beta$, then there exists a $\bar{t}$ such that for $0 \leqq t \leqq \bar{t}$ one has $\Phi_\alpha^{-1}(\xi + t\eta) \in U_\alpha \cap U_\beta$. If $\Phi_{\alpha\beta} = \Phi_\beta \circ \Phi_\alpha^{-1}$ is differentiable in ξ and consequently for a.e. $\Phi_\alpha^{-1}(\xi) \in U_\alpha \cap U_\beta$, then

$$\begin{aligned}\sigma_\alpha(\xi, \xi + t\eta) &= \delta(\Phi_\alpha^{-1}(\xi), \Phi_\alpha^{-1}(\xi + t\eta)) = \sigma_\beta(\Phi_{\alpha\beta}(\xi), \Phi_{\alpha\beta}(\xi + t\eta)) \\ &= \sigma_\beta(\Phi_{\alpha\beta}(\xi), \Phi_{\alpha\beta}(\xi) + td\Phi_{\alpha\beta}(\xi)\eta + o(t)) .\end{aligned}$$

It follows for a.e. x

$$\varphi_\alpha(\xi, \eta) = \varphi_\beta(\Phi_{\alpha\beta}(\xi), d\Phi_{\alpha\beta}(\xi)\eta) . \quad \square$$

By the previous lemma, the functions φ_α and $\underline{\varphi}_\alpha$ satisfy the conditions (1.2), then they define a **Finslerian structure** on M, which becomes a LIP *Finslerian manifold* (M, φ).

Now for a generic Finslerian structure F, the integral

$$L_{\mathscr{A}}(\gamma, F) = \int_0^1 F(\gamma, \dot{\gamma})dt , \tag{2.4}$$

is not well defined for every curve γ and, in general case, the value depends on the chosen atlas. In the particular case $F = \varphi$, it is indipendent of the atlas, because the following theorem holds.

(2.5) Theorem. *For every absolutely continuous curve* $\gamma : [0, 1] \to M$

$$\mathscr{L}(\gamma; \delta) = \int_0^1 \varphi(\gamma, \dot{\gamma})dt = \int_0^1 \underline{\varphi}(\gamma, \dot{\gamma})dt$$

where $\mathscr{L}(\gamma; \delta)$ *is the usual length of* γ *in* (M, δ).

Proof. Since the functions φ_α are compatible with the changes of chart, it suffices to prove the theorem in a chart (U_α, Φ_α) and, for convenience, we shall omit the suffix α.

Let T be a partition of [0, 1] and $\tilde{\gamma} = \Phi(\gamma)$. Since γ is AC, then $\tilde{\gamma}$ and $\sigma(\tilde{\gamma}(t_i), \tilde{\gamma}(t))$ are differentiable a.e. and whence at the points at which it is differentiable one has

$$\frac{d}{dt}\sigma(\tilde{\gamma}(t_i), \tilde{\gamma}(t)) = \lim_{h\to 0} \frac{\sigma(\tilde{\gamma}(t_i), \tilde{\gamma}(t) + \dot{\tilde{\gamma}}(t)h + o(h)) - \sigma(\tilde{\gamma}(t_i), \tilde{\gamma}(t))}{h}$$
$$\leqq \liminf_{h\to 0} \sigma(\tilde{\gamma}(t), \tilde{\gamma}(t) + \dot{\tilde{\gamma}}(t)h)/h = \underline{\varphi}(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)),$$

then

$$\delta(\gamma(t_i), \gamma(t_{i+1})) = \sigma(\tilde{\gamma}(t_i), \tilde{\gamma}(t_{i+1})) \leqq \int_{t_i}^{t_{i+1}} \underline{\varphi}(\tilde{\gamma}, \dot{\tilde{\gamma}})dt$$

from which

$$\mathscr{L}(\gamma; \delta) \leqq \int_0^1 \underline{\varphi}(\gamma, \dot{\gamma})dt. \tag{2.6}$$

But the function $\mathscr{L}(\gamma(t)) = \mathscr{L}(\gamma|[0, t])$ is increasing, so by a corollary of Fatou lemma

$$\int_0^1 \frac{d}{dt}\mathscr{L}(\gamma(t))dt \leqq \mathscr{L}(\gamma).$$

If t is a derivability point of $\mathscr{L}(\gamma(t))$ and of $\tilde{\gamma}(t)$, then

$$\frac{d}{dt}\mathscr{L}(\gamma(t)) = \lim_{h\to 0} \frac{\mathscr{L}(\gamma(t+h)) - \mathscr{L}(\gamma(t))}{h}$$
$$\geqq \limsup_{h\to 0} \frac{\sigma(\tilde{\gamma}(t), \tilde{\gamma}(t+h))}{h} = \varphi(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)).$$

Now by (2.6) too,

$$\int_0^1 \varphi(\gamma, \dot{\gamma})dt \leqq \mathscr{L}(\gamma) \leqq \int_0^1 \underline{\varphi}(\gamma, \dot{\gamma})dt$$

but $\varphi \geqq \underline{\varphi}$, from which the assertion follows. □

It can be remarked that every AC curve γ is transversal to the set in which $\varphi > \underline{\varphi}$.

(2.7) Corollary. *Under our hypotheses, for* a.e. $\xi, \varphi(\xi, \eta) = \underline{\varphi}(\xi, \eta)$ *holds and whence*

$$\varphi(\xi, \eta) = \lim_{t\to 0} \frac{\sigma(\xi, \xi + t\eta)}{t} \quad \text{a.e.}$$

Moreover for a.e. ξ *the function* $\varphi(\xi, .)$ *is symmetric* (*i.e.* $\varphi(\xi, \eta) = \varphi(\xi, -\eta)$).

Proof. One has to prove that $\varphi = \underline{\varphi}$ a.e. By the linearity and continuity of φ e $\underline{\varphi}$ (with respect to η) it suffices to prove that for every direction η and for a.e. ξ one has $\varphi(\xi,\eta) = \underline{\varphi}(\xi,\eta)$ from which the assertion follows by the separability of $\mathbb{R}^n$.

Let us assume that there exist $\bar{\eta}$ and a set $C \subset V_\alpha$ of positive measure such that

$$\underline{\varphi}(\xi,\bar{\eta}) < \varphi(\xi,\bar{\eta}) \quad \forall \xi \in C\,.$$

One can then find a segment of a straight line contained in V_α that has end points $\bar{\xi}\,\mathrm{e}\,\bar{\xi}+k\eta$ and which cuts C in a set of positive 1-dimensional measure. Then by (2.5)

$$0 < \int_0^1 [\varphi(\bar{\xi}+tk\bar{\eta},\bar{\eta}) - \underline{\varphi}(\bar{\xi}+tk\bar{\eta},\bar{\eta})]\,dt = 0$$

which is impossible.

Analogously one proves that $\varphi(\xi,\eta) = \varphi(\xi,-\eta)$ for a.e. ξ. □

3 Recalls on intrinsic distances

In order that this paper may be as self-contained as possible, we recall some concepts and notions contained in [DC-P4], to which we refer for details.

Let (M,F) be a LIP *Finslerian manifold* and $\mathscr{A}$ its atlas. If γ is an absolutely continuous curve (AC), we introduce the integral (2.4), with the understanding that $L_{\mathscr{A}}(\gamma;F) = +\infty$ whenever it does not exist. We put

$$AC(M;x,y) = \{\gamma : I \to M | \gamma \text{ is } AC \text{ and } \gamma(\partial I) = \{x\} \cup \{y\}\}\,,$$
$$\mathscr{N} = \{N \subset M; \mathrm{meas}(N) = 0\}\,.$$

For a fixed set $N \in \mathscr{N}$

$$\rho_N^F(x,y) = \inf_\gamma \{L_{\mathscr{A}}(\gamma;F) | \gamma \in AC(M;x,y), \gamma \overset{\mathsf{T}}{\cap} N\} \tag{3.1}$$

is a pseudodistance dependent on $\mathscr{A}$; on the contrary

$$\rho^F(x,y) = \sup\{\rho_N^F(x,y); N \in \mathscr{N}\} \tag{3.2}$$

is a *distance* on M independent of $\mathscr{A}$.

In an analytic way it is possible to define an intrinsic distance induced by F using the "dual norm".

If $F = \{F_\alpha\}$ is a symmetric Finslerian structure on M, we introduce the "*dual*" function on $F, F^* = \{F_\alpha^*\}$ through

$$F_\alpha^*(\xi,w) = \sup_v \{\langle w,v\rangle : F_\alpha(\xi,v) \leqq 1\} \tag{3.3}$$

where $\langle \cdot\,,\,\cdot \rangle$ is the duality in $\mathbb{R}^n$.

By the properties of F_α, a.e. one has

$$F_\alpha^*(\xi,w) = \sup_{v \neq 0} \left\langle w, \frac{v}{F_\alpha(\xi,v)} \right\rangle = \max_{v \neq 0} \left\langle w, \frac{v}{F_\alpha(\xi,v)} \right\rangle\,.$$

(3.4) Theorem. *The function*

$$F_\alpha^* : V_\alpha \times \mathbb{R}^n \to \mathbb{R}$$

verifies the following properties

1) $F_\alpha^*(\xi, \cdot)$ *is a norm, which is locally equivalent to the Euclidean norm*;
2) $\langle w, v\rangle \leqq F_\alpha^*(\xi, w)F_\alpha(\xi, v)$;
3) $F_\alpha^*(\xi, \cdot)$ *is Lipschitz and* $F_\alpha^*(\cdot, \eta)$ *is measurable*;
4) $(F_\alpha^*)^*(\xi, w) \leqq F_\alpha(\xi, w)$ a.e. *and equality holds if and only if* $F(\xi, \cdot)$ *is convex*;
5) $F_\alpha^*(\xi, w) = F_\beta^*(\Phi_{\alpha\beta}(\xi), {}^t(d\Phi_{\alpha\beta})(\xi)^{-1}(w))$ *for every change of charts.*

(3.5) If $f : M \to \mathbb{R}$ is a LIP function and $z \in U_\alpha$, we put

$$(F^*(df))(z) = F_\alpha^*(\zeta, (d\tilde{f}(\zeta))$$

where $\tilde{f} = f \circ \Phi_\alpha^{-1}$ and $\zeta = \Phi_\alpha(z)$.

Moreover $\forall x, y \in M$ we introduce in way similar to [DC-P3] the intrinsic distance δ^F defined by

$$(3.6) \qquad \delta^F(x, y)^{-1} = \inf\{\|F^*(df)\|_\infty; f \in \mathscr{L}ip(M), f(x) = 0, f(y) = 1\}\,.$$

This unusual definition of distance has the advantage of being "naturally" invariant, if F_α is replaced by $\tilde{F}_\alpha$ s.t. $\tilde{F}_\alpha = F_\alpha$ a.e.; namely the distance depends only on the equivalence class to wich F_α belongs.

If M is a manifold of class C^1 and F a Finslerian structure of class C^0, then $\delta^F = \rho^F$, i.e. the intrinsic distance coincides with the usual distance, induced by F (cf. [DC-P4, theorem 3.6]).

Moreover one shows [DC-P4, theorem 4.7]

(3.7) Theorem. *If* ρ^{**} *is the distance function induced by* $(F^*)^*$, *then*

$$\rho^{**}(x, y) = \delta^F(x, y) \leqq \rho^F(x, y)$$

and the equality holds if F *is convex.*

Our goal (see Introduction) is to compare the distances ρ^φ and δ^φ with the original δ, where φ is the LIP Finslerian structure defined by "$\langle$ differentiation" $\rangle$.

4 The main theorem

We premise the following theorem

(4.1) Theorem. *Let* δ *be a distance locally equivalent to a Euclidean one,* φ *the "derivative" of* δ *and* φ^* *its dual. Then*

$$\|\varphi^*(d(\delta(\xi, \cdot)))\|_{L_\infty} \leqq 1, \ \delta^\varphi(\xi, \eta) \geqq \delta(\xi, \eta)\,.$$

Proof. Fixed $\xi \in V_\alpha$, we put for $\eta \in V_\alpha$

$$f(\eta) = \sigma(\xi, \eta) = \delta(\Phi_\alpha^{-1}(\xi), \Phi_\alpha^{-1}(\eta)) .$$

In the differentiability points

$$f(\eta + t\omega) - f(\eta) = t\langle (df)(\eta), \omega\rangle + o(t)$$

from which by definition of f

$$\langle (df)(\eta), \omega\rangle + \frac{o(t)}{t} = \frac{\sigma(\xi, \eta + t\omega) - \sigma(\xi, \eta)}{t} \leqq \frac{\sigma(\eta, \eta + t\omega)}{t}$$

whence a.e.

$$\langle (df)(\eta), \omega\rangle \leqq \varphi_\alpha(\eta, \omega) \Rightarrow \varphi_\alpha^*((df)(\eta)) \leqq 1 .$$

The assertion follows from the definition of δ^φ. □

(4.2) Corollary. *Under our hypotheses the function $\varphi(\xi, \cdot)$ is convex.*

Proof. On account of the convexity of $\varphi^{**}(\xi, \cdot)$, it suffices to prove that $\varphi^{**} = \varphi$. Now by 4) of (3.4) $\varphi^{**} \leqq \varphi$; it remains to show that $\varphi^{**} \geqq \varphi$. If ρ^{**} is the distance induced by φ^{**}, the theorem 3.7 gives $\delta^\varphi = \rho^{**}$ and by the previous theorem $\delta \leqq \delta^\varphi \leqq \rho^{**}$. Then [see Th. 2.6, part 2, DC-P3]

$$\begin{aligned}\varphi_\alpha(\xi, \eta) &= \lim_{t \to 0} \frac{\sigma_\alpha(\xi, \xi + t\eta)}{t} \\ &\leqq \limsup_{t \to 0} \frac{1}{t} \rho^{\varphi^{**}}(\Phi_\alpha^{-1}(\xi), \Phi_\alpha^{-1}(\xi + t\eta)) \leqq \varphi_\alpha^{**}(\xi, \eta) ,\end{aligned}$$

whence the conclusion. □

(4.3) It follows from the definitions that $\rho^\varphi \geqq \delta$. Now we want to check when $\rho^\varphi = \delta$.

Since (M, ρ^{**}) and (M, ρ) are length metric spaces, it is clear that a necessary condition is that (M, δ) is a length metric space. Howerer this condition is not sufficient as can be seen in the following example.

(4.4) Let $M = \mathbb{R}^2$ be described by $x = (x_1, x_2)$. We put

$$g_{ij}(x) = \begin{cases} 4\delta_{ij} & x_1 \neq 0 \\ \delta_{ij} & x_1 = 0 \end{cases}$$

$$\delta(x, y) = \inf\left\{\int_0^1 \sqrt{g_{ij}(\gamma)\dot{\gamma}_i\dot{\gamma}_j}\, dt; \gamma \in AC(x, y)\right\} .$$

Obviously

$$|x - y| \leqq \delta(x, y) \leqq 2|x - y| ,$$

and the geodesics are union of segments of straight lines.

If $x_1 \neq 0, \varphi(x, y) = 2|y|$; if $x_1 = 0$,

$$\varphi(x, y) = \begin{cases} \sqrt{3}|y_1| + |y_2| & |y_2 - x_2| > |y_1|/\sqrt{3} \\ 2|y| & |y_2 - x_2| < |y_1|/\sqrt{3} . \end{cases}$$

Since $\varphi(x, y) = 2|y|$ a.e., then

$$\rho^{\varphi}(x, y) = \rho^{\varphi^{**}}(x, y) = 2|x - y| \geqq \delta(x, y)$$

and there exist couples of points, for example $(0, x_2)$ and $(0, y_2)$, from which the strict inequality holds, but

$$\delta(x, y) = \inf\left\{\int_0^1 \varphi(\gamma, \dot{\gamma}); \gamma \in AC(x, y)\right\} .$$

(4.5) Theorem. *Let $\delta : M \times M \longrightarrow \mathbb{R}$ be a distance locally equivalent to the Euclidean one and let $\varphi = \{\varphi_\alpha\}$ the Finslerian structure so built in any chart (U_α, Φ_α):*

$$\varphi_\alpha(\xi, \eta) = \lim_{t \to 0} \frac{\sigma_\alpha(\xi, \xi + t\eta)}{t}, \quad \sigma_\alpha(\xi, \eta) = \delta(\Phi_\alpha^{-1}(\xi), \Phi_\alpha^{-1}(\eta)) .$$

Then

$$\delta(x, y) = \inf\left\{\int_0^1 \varphi(\gamma, \dot{\gamma}); \gamma \in AC(x, y)\right\} .$$

In order that δ coincides with the distance ρ^{φ}, induced by φ, a necessary and sufficient condition is that

$$\delta(x, y) = \sup_N \left\{\inf_\gamma \left\{\mathscr{L}(\gamma; \delta); \gamma \in AC(x, y), \gamma \overset{T}{\cap} N\right\}, N \in \mathscr{N}\right\} .$$

A sufficient condition is that φ is upper-semicontinuous.

Proof. The condition for δ is necessary and sufficient; namely by definition of ρ^{φ} and by theorem 2.5 one has

$$\begin{aligned}\rho^{\varphi}(x, y) &= \sup_N \left\{\inf_\gamma \left\{L(\gamma); \gamma \in AC(x, y), \gamma \overset{T}{\cap} N\right\}, N \in \mathscr{N}\right\} \\ &= \sup_N \left\{\inf_\gamma \left\{\mathscr{L}(\gamma); \gamma \in AC(x, y), \gamma \overset{T}{\cap} N\right\} N \in \mathscr{N}\right\} .\end{aligned}$$

The condition for φ is sufficient by theorem 3.3 of [DC-P4]. □

(4.6) Main Theorem. *Let M be a LIP manifold, δ a distance locally equivalent to an Euclidean one, φ the Finslerian structure constructed by the "derivative" of δ and ρ the distance induced by φ. If one repeats the procedure (as in* Sect. 2) *starting from ρ, then the function $\tilde{\varphi}$ is* a.e. *equal to φ.*

Moreover if $\tilde{\rho}$ is the distance induced by the Finslerian structure $\tilde{\varphi}$

$$\tilde{\rho} = \rho (= \rho^{**} = \delta^{\varphi}) ,$$

where ρ^{**} is the distance induced by φ^{**} and δ^{φ} the intrinsic one induced by φ.

Proof. By the convexity and the uniform Lipschitzianity of φ, one can prove, by suitable modifications of the argument used in [DC-P3, Theorem 2.6, part 2)] that in any chart

$$\tilde{\varphi}_\alpha(\xi,\eta) = \lim_{t\to 0} \frac{\rho(\Phi_\alpha^{-1}(\xi), \Phi_\alpha^{-1}(\xi + t\eta))}{t} \leqq \varphi_\alpha(\xi,\eta) \quad \text{a.e.} ;$$

because in general $\rho \geqq \delta$ one has even $\tilde{\varphi}_\alpha(\xi,\eta) \geqq \varphi_\alpha(\xi,\eta)$, from which the conclusion follows a.e.

Now ρ verifies the conditions of the previous theorem, hence $\tilde{\rho} = \rho$. By convexity of φ, from the theorems 3.7 and 3.4

$$\delta^\varphi = \rho^{**} = \rho(= \tilde{\rho}). \quad \square$$

(4.7) Remark. The family φ, built through the indicated procedure, has greater "regularity" than what we originally requested from a Finslerian structure. In particular φ is Borel-measurable and it makes sense to evaluate, for *every* absolutely continuous curve γ the following integral

$$L(\gamma) = \int_0^1 \varphi(\gamma, \dot{\gamma})\, dt \,.$$

Moreover $\varphi(\xi,\ \cdot\ )$ is a family of *norms*, dependent on ξ and locally equivalent to the Euclidean norm.

5 Stable Finslerian manifolds

Let (M,F) be a LIP Finslerian manifold and ρ^F the intrinsic distance induced by F. Then if φ is the "derivative" of ρ^F, in general, $\varphi \neq F$, as the following example (studied in [DC-P4]) shows.

(5.1) Example. Let the sequence $\{\alpha_h\}(h \in \mathbb{N})$ be dense in $\mathbb{R}$ and A the following open subset of $\mathbb{R}^2$

$$A = \left\{ x = (x_1,x_2) \in \mathbb{R}^2 |\, \min \left\{ \inf_h |x_1 - \alpha_h|2^h, \inf_h |x_2 - \alpha_h|2^h \right\} < 1 \right\} .$$

On $\mathbb{R}^2$ we consider the LIP Riemannian metric g defined by

$$g_{ij}(z) = \delta_{ij} \quad z \in A, \quad g_{ij}(z) = 4\delta_{ij} \quad z \notin A \,,$$

(where δ_{ij} is the Kronecker symbol). The "derivative" φ of ρ^g is

$$\varphi(x,v) = \begin{cases} \sqrt{v_1^2 + v_2^2} & z \in A \\ |v_1| + |v_2| & a.e. z \notin A \,. \end{cases}$$

Hence $\varphi(x,v) \neq \|v\|_{g(x)}$; moreover φ is a norm, that is not induced by inner product.

It is interesting to see when $\varphi = F$.

(5.2) A LIP Finslerian manifold (M,F) is said to be **Finslerian stable** if $\varphi = F$, i.e.

$$\lim_{t\to 0}\frac{\rho^F(\Phi_\alpha^{-1}(\xi),\Phi_\alpha^{-1}(\xi+t\eta))}{t} = F(\xi,\eta)\,.$$

Analogously a LIP Riemannian manifold (M,g) is called **Riemannian stable**, if the "derivative" of ρ^g coincides with g.

(5.3) Theorem. *If (M,F) is a* LIP *Finslerian manifold, ρ^F the intrinsic distance induced by F and φ its "derivative", then $\rho^F = \rho^\varphi$ and (M,φ) is a* LIP *Finslerian stable manifold.*

Proof. By theorem 2.5, $\rho^F = \rho^\varphi$. Moreover ρ^F verifies the necessary and sufficient condition of theorem 4.5, then the "derivative" of $\rho^F(=\rho^\varphi)$ is φ. □

Remarks

(5.4) The example given in (5.1) proves that, by starting from a LIP *Riemannian manifold* (M,F) and by "differentiating" ρ^F, one obtains a stable Finslerian manifold that is *not* a Riemannian manifold.

(5.5) Now we shall compare our definition of stability with the notion of *quasi-Finslerian space* introduced in [DG1] and [DG3]. To this end, we give a new definition of intrinsic distance.

Following the idea in [DC-P3], we put

$$\bar{\delta}^F(x,y)^{-1} = \lim_{p\to+\infty}\left[\inf\left\{\int_M |F^*(df)|^p d\mu;\, f\in\mathscr{L}ip(M), f(x)=0, f(y)=1\right\}\right]^{1/p}$$

where μ is a measure compatible with the changes of chart and equivalent to the Lebesgue measure on any V_α, e.g. the Hausdorff n-dimensional measure if $n = \dim M$.

On account of the formal properties of F^* (cf.(3.4)), and by repeating the arguments in [DC-P3], one can conclude that also $\bar{\delta}^F$ is a distance, which we shall call the **integral distance.**

Of course, the cases where $\bar{\delta}^F = \delta^F$ are of particular interest, when δ^F is the intrinsic distance defined (3.6). One sees that equality holds if the sets

$$\mathbb{B}(\xi,r) = \{\eta\in M;\rho^\varphi(\xi,\eta)\leqq r\}$$

have finite measure. In particular, this assumption is verified if M has finite measure or if $\tilde{M}$ is a complete manifold [G].

(5.6) Theorem. *Under our hypotheses on M, let δ be a distance locally equivalent to the Euclidean one and let φ be its "derivative". Then*

$$\bar{\delta}^\varphi = \delta^\varphi(=\rho^\varphi)\,.$$

Proof. Because $\varphi^*(\xi,\,\cdot\,)$ is LIP, one can repeat with suitable modifications the proofs in [DC-P3, (5.5) and (5.11)]. Then

$$\|\varphi^*(d\bar{\delta}(\xi,\,\cdot\,))\|_\infty \leqq 1 . \tag{5.7}$$

Now let $N \subset M$ be the subset on which (5.7) is not verified or $d\bar{\delta}$ is not well defined. Then, if $\gamma \in AC(x, y)$ is a curve transversal to N,

$$\bar{\delta}(x,y) \leqq \int_0^1 \varphi^*(d\bar{\delta})\varphi(\gamma,\dot{\gamma})dt \leqq \int_0^1 \varphi(\gamma,\dot{\gamma})dt ,$$

which leads to $\bar{\delta} \leqq \rho$ by definition of ρ.

Since the "derivative" of ρ $(= \rho^\varphi)$ is a.e. equal to φ (cf.(4.6)), then, by (4.1) $\varphi^*(d\varrho) \leqq 1$ a.e.

Now we consider (for $x \neq y$) the function

$$\bar{f}(z) = \begin{cases} \frac{\rho(x,z)}{\rho(x,y)} & \rho(x,z) \leqq \rho(x,y) \\ 1 & \rho(x,z) \geqq \rho(x,y) ; \end{cases}$$

then $\bar{f}(x) = 0, \bar{f}(y) = 1$ and $(d\bar{f})(z) = 0$ for $z \in M - \mathbb{B}(x, \rho(x, y))$.

Moreover $\forall p > 1$

$$\begin{aligned} \inf_f \left\{ \left(\int_M [\varphi^*(df)]^p dv \right)^{1/p} \right\} &\leqq \left(\int_{\mathbb{B}} [\varphi^*(d\bar{f})]^p dv \right)^{1/p} \\ &\leqq \rho(x,y)^{-1}(\text{meas } \mathbb{B}(x, \rho(x,y)))^{1/p} , \end{aligned}$$

from which $\bar{\delta}(x,y)^{-1} \leqq \rho(x,y)^{-1}$ follows by taking the limit. □

By Theorem 4.6, the following corollary ensues.

(5.8) Corollary. *The equality*

$$\lim_{t\to 0} \frac{\bar{\delta}^\varphi(\Phi_\alpha^{-1}(\xi), \Phi_\alpha^{-1}(\xi + t\eta))}{t} = \varphi(\xi,\eta) \quad \text{a.e.}$$

holds.

Hence one concludes that, when $\bar{\delta}^F = \delta^F$, a stable Finslerian manifold is a quasi-Finslerian space.

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