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The least solution for the polynomial interpolation problem

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1 Introduction

We consider the following problem: Given a subspace Λ of the algebraic dual Π' of the space Π of s-variate polynomials, find a space $P \subset \Pi$ which is **correct for** Λ. By this we mean that every continuous linear functional F on Λ can be interpolated by exactly one $p \in P$ in the sense that $F\lambda = \lambda p$ for all $\lambda \in \Lambda$. Among the many solutions, we choose a particular one, which we call the **least solution** and denote by $\Lambda_\downarrow$, and which is obtained by a certain map $\Lambda \mapsto \Lambda_\downarrow$ from subspaces of Π' to (homogeneous) subspaces of Π. We call this map the **least map** and give (in Sect. 4) a comprehensive discussion of its properties. With these properties in hand, we provide (in Sect. 3) and verify (in Sects. 5 and 6) a rather striking list of properties that single out $\Lambda_\downarrow$ from the collection $\mathrm{IP}(\Lambda)$ of all possible solutions. We pay special attention to Lagrange interpolation, i.e., to Λ spanned by point-evaluations, as this is the case of most practical interest. It is also what started our interest in this topic (cf. [BR1]).

We use (standard) multivariate notation throughout, in the following disciplined way. We use $x, y, z, \theta, \vartheta$ for points in $\mathbb{R}^s$ (or $\mathbb{C}^s$), with $x(j)$ the jth component of $x \in \mathbb{R}^s$, and use t (resp. ξ) throughout for real (resp. complex) scalars. The letters $\alpha, \beta, \gamma, \kappa$ denote multi-integers, while the letters $j, k, \ldots, n$ denote (simple) integers. For $\alpha \in \mathbb{Z}_+^s$, the power function

$$x \mapsto x^\alpha = \prod_{j=1}^{s} x(j)^{\alpha(j)}$$

is denoted by $(\,)^\alpha$. As usual, D is the differentiation symbol, hence a space closed under differentiation is termed D-invariant, and for a polynomial or power series q, $q(D)$ is its evaluation at D. The scalar product $\sum_{j=1}^{s} x(j)\,y(j)$ for $x, y \in \mathbb{R}^s$

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is simply denoted by xy. In addition, for $\theta\in\mathbb{R}^s$, e_θ stands for the exponential function

$$(1.1) \qquad e_\theta\colon x\mapsto e^{\theta x}.$$

Finally, for $k\in\mathbb{Z}_+$, Π_k (resp., $\Pi_{<k}$) is the subspace of Π of polynomials of total degree at most (resp., less than) k.

Our approach makes essential use of the well-known identification of Π' with the space $\mathbb{R}[\![X]\!]$ of formal power series, via a pairing of the form

$$\mathbb{R}[\![X]\!]\times\Pi\to\mathbb{R}\colon (f,p)\mapsto \sum_{\alpha\in\mathbb{Z}^s_+} w(\alpha)\,\alpha(f)\,D^\alpha p(0),$$

in which $(\alpha(f))_\alpha$ denotes the sequence of coefficients in the power series f, and $(w(\alpha))_\alpha$ are some (positive) weights. We choose here $w(\alpha):=1/\alpha!$, since then the pairing $\langle\cdot,\cdot\rangle$ satisfies

$$\langle f,p\rangle = p(D)f(0),$$

for any polynomial p and any analytic power series f. The pairing $(f,p)\mapsto p(D)f(0)$ was earlier exploited in Sect. 7 of [DR], where the theory of exponential box splines was employed to solve a certain class of polynomial interpolation problems, and was also the pairing used in [BR1]. For completeness, we include in Sect. 2 a short discussion of the space $\mathbb{R}[\![X]\!]$ of formal power series and its identification with the dual Π' of the space of polynomials. We also give in that section a precise statement of the interpolation problem, and define $\Lambda_\downarrow$ to be the linear span of all $\lambda_\downarrow$, with $\lambda_\downarrow$ the unique homogeneous polynomial for which $\lambda-\lambda_\downarrow$ is of higher order than is λ, and $\lambda\in\Lambda$. Here is an outline of the rest of the paper.

Section 3 starts off with three guiding examples: The first is Lagrange interpolation, i.e., interpolation at some finite set $\Theta\subset\mathbb{R}^s$, whose least solution we denote correspondingly by Π_Θ. The second example extends this to a setup which includes Hermite interpolation and even Birkhoff interpolation. These two examples correspond to the material on polynomial interpolation in our paper [BR1]. The third example concerns Radon interpolation, i.e., the use of line integrals as interpolation conditions, as used in tomography and suggested to us by Nira Dyn, a suggestion which led us to the study of arbitrary interpolation conditions from Π' pursued in the present paper. Chief tool for the analysis of Lagrange and Hermite interpolation problems is the fact (already much exploited in [DR] and [BR1]) that, in terms of the above-mentioned pairing (and for $p,q\in\pi$), $\langle qe_\theta, p\rangle = q(D)p(\theta)$, and therefore evaluation at $\theta\in\mathbb{R}^s$ is represented by the simple exponential e_θ. These examples are meant to help with the appreciation of the list of eight particular properties of the 'least solution' $\Lambda_\downarrow\in\mathrm{IP}(\Lambda)$, whose discussion fills out the rest of the section. The properties concern: (A) generality, (B) monotonicity, (C) constructibility, (D) minimal degree, (E) interaction with convolution, (F) interaction with homogeneous maps, (G) annihilation/differentiation, (H) tensor products. For example, the minimal degree property states that, among all $P\in\mathrm{IP}(\Lambda)$, $\Lambda_\downarrow$ is of least degree in the strong sense that $\dim(P\cap\Pi_k)\leq\dim(\Lambda_\downarrow\cap\Pi_k)$ for every k. As another example, the annihilation property concerns associated differential operators and states, for the special case of Lagrange interpolation at the points of Θ, that, for any polynomial p vanishing on Θ, necessarily $p_\uparrow(D)\Pi_\Theta=0$ (with $p_\uparrow$ the leading term of p). It further states that if $p(D)\Pi_\Theta=0$, then necessarily some q with the same leading term as p must vanish on Θ.

Section 4 commences the verification of all these properties. The section starts with the observation that, even when the space Λ of interpolation conditions is infinite-dimensional, the interpolation problem is essentially finite-dimensional since the only w^*-continuous linear functionals on Λ are of the form $\lambda\mapsto\lambda p$ for some polynomial p. There are two main results in this section: one is a proof of the assertion that $(\Lambda_\downarrow)_\perp=\text{span}\{p_\uparrow: p\in\Lambda_\perp\}$, with $\Lambda_\perp$ the annihilator of Λ in Π, i.e., the joint kernel of $\lambda\in\Lambda$. The other is a proof that, for a D-invariant Λ, the vanishing of the constant-coefficient differential operator $p(D)$ on $\Lambda_\downarrow$ implies that $q(D)\Lambda=0$ for some q with the same leading term as p, while the vanishing of $p(D)$ on Λ implies the vanishing of the leading term of $p(D)$ on $\Lambda_\downarrow$.

Section 5 concentrates on the minimal-degree property of the least solution. It contains proofs of the facts that all minimal degree solutions and all homogeneous solutions to the interpolation problem can already be determined by $\Lambda_\downarrow$ (without knowledge of Λ). We also take up there the question under what conditions a polynomial space P might be dual to a polynomial space Q in the sense that the map $p\mapsto\langle\cdot,p\rangle_{|Q}$ provides a w^*-dense embedding of P in the algebraic dual Q' of Q.

Section 6 relates the special case (to which Lagrange and Hermite interpolation belong, while Birkhoff and Radon interpolation do not) of a (finite-dimensional) D-invariant Λ to earlier results of ours (in [BR2] and [BDR]) concerning the connection of polynomial interpolation to polynomial ideals with finite variety, hence to box spline theory: for a finite collection of homogeneous differential operators with constant coefficients, we discuss an approach for estimating from below the dimension of their joint kernel (in $\mathbb{R}[\![X]\!]$) and, at times, identifying this kernel with Π_Θ for a certain $\Theta\subset\mathbb{C}^s$.

In the seventh, and final, section we show how $\Lambda_\downarrow$, for particular cases of Radon interpolation, can be determined as a certain subspace of Π_Θ for a carefully chosen Θ. The discussion there illustrates the difficulties one may have to overcome when Λ is not D-invariant.

The present paper only deals with the theoretical aspects of the least solution to the polynomial interpolation problem. Questions of construction are taken up in the companion paper [BR3], in which an algorithm for obtaining $\Lambda_\downarrow$ from a spanning sequence for Λ is presented and computational details are discussed.

For reasons of convenience, the discussion here is limited by and large to *real* polynomials. Most results extend to the complex case by the appropriate use of complex conjugates, i.e., by changing the pairing to $\langle f,p\rangle:=\sum_\alpha \alpha(f)\,\overline{\alpha(p)}/\alpha!$, and by replacing $\Lambda_\downarrow$ in some places by $\overline{\Lambda_\downarrow}$.

2 The interpolation problem

We are interested in **interpolation**. By this we mean the construction of a function f (the interpolant) which matches given information of the form

$$\lambda f=F(\lambda)$$

for all linear functionals λ in some set Λ. Having assumed the λ to be *linear* functionals, it is no loss of generality to assume that Λ is a linear space of linear functionals. On the other hand, this requires that the information F be consistent, i.e., F is necessarily a linear functional on Λ.

We intend to choose the interpolants from the space Π of polynomials in s variables (over $\mathbb{R}$), and put no restriction on $\lambda \in \Lambda$ other than that they should be defined (at least) on Π. Thus

$$\Lambda \subseteq \Pi'.$$

We heavily use the (standard) representation of Π' as the space $\mathbb{R}[\![X]\!]$ of **formal power series** (of non-negative powers). This representation is based on the pairing

$$\text{(2.1)} \quad \mathbb{R}[\![X]\!] \times \Pi \to \mathbb{R} : (f, p) \mapsto \langle f, p \rangle := \sum_{\alpha \in \mathbb{Z}_+^s} \frac{\alpha(f)\,\alpha(p)}{\alpha!} = \sum_{\alpha \in \mathbb{Z}_+^s} \frac{\alpha(f)\, D^\alpha p(0)}{\alpha!},$$

in which $\alpha(f)$ denotes the αth (normalized) coefficient in the formal power series (for) f, i.e.,

$$f = \sum_{\alpha \in \mathbb{Z}_+^s} X^\alpha \frac{\alpha(f)}{\alpha!}, \qquad f \in \mathbb{R}[\![X]\!],$$

where X^α is the formal power symbol:

$$X^\alpha := \prod_{j=1}^{s} X(j)^{\alpha(j)}.$$

Choosing p in (2.1) to be the power function $(\,)^\alpha$, we get from (2.1) that

$$\text{(2.2)} \qquad \alpha(f) = \langle f, (\,)^\alpha \rangle,$$

hence the representation of Π' by $\mathbb{R}[\![X]\!]$ is given by the invertible linear map

$$\Pi' \to \mathbb{R}[\![X]\!] : \ \lambda \mapsto \sum_\alpha X^\alpha \frac{\lambda(\,)^\alpha}{\alpha!}.$$

In these terms, formal differentiation of $f \in \mathbb{R}[\![X]\!]$ is, in effect, a shift, i.e., $D^\beta f$ is defined by

$$\alpha(D^\beta f) := (\alpha + \beta)(f), \qquad \alpha, \beta \in \mathbb{Z}_+^s.$$

Thus, for α, β in $\mathbb{Z}_+^s$ and $f \in \mathbb{R}[\![X]\!]$,

$$\langle D^\beta f, (\,)^\alpha \rangle = \alpha(D^\beta f) = (\alpha + \beta)(f) = \langle f, (\,)^{\alpha+\beta} \rangle,$$

and, hence,

$$\text{(2.3)} \qquad \langle q(D) f, p \rangle = \langle f, qp \rangle = \langle p(D) f, q \rangle, \qquad f \in \mathbb{R}[\![X]\!],\ p, q \in \Pi.$$

In the sequel, we *identify* Π' *and* $\mathbb{R}[\![X]\!]$, and thus we can think of the elements of Π' simultaneously as sequences indexed by $\alpha \in \mathbb{Z}_+^s$, or else as linear functionals on Π. We choose to topologize $\mathbb{R}[\![X]\!]$ with the topology of pointwise convergence, or equivalently equip Π' with the w^*-topology, making thereby Π' into a Fréchet space, and making Π the w^*-continuous dual of Π':

(2.4) **Fact.** *F is a w^*-continuous linear functional on Π' if and only if $F = \langle \cdot, q\rangle$ for some $q \in \Pi$.*

With this identification of Π' with $\mathbb{R}[[X]]$, Π is naturally embedded in Π'. Thus, $p \in \Pi$ can be (and is) treated as an element of Π, as a linear functional (power series) in Π', and as an analytic function on $\mathbb{R}^s$. Furthermore, many non-polynomial $\lambda \in \Pi'$ of interest to us can also be reasonably interpreted as a function **analytic at** 0, viz. the function to which the power series converges uniformly. If it is important to distinguish between λ and its analytic limit, we write $\lambda^\vee$ for the latter, and refer to it as the **generating function** of λ. We denote by

$$A_0$$

the collection of all $\lambda \in \Pi'$ analytic at the origin.

For us, the most important example of $\lambda \in \Pi$ is point evaluation at θ, i.e., the linear functional

$$\delta_\theta\colon\ p \mapsto p(\theta). \tag{2.5}$$

Since $\delta_\theta(\,)^\alpha = \theta^\alpha$, the formal power series corresponding to δ_θ is $\sum_{\alpha \in \mathbb{Z}_+^s} X^\alpha \dfrac{\theta^\alpha}{\alpha!}$. Hence

$$\delta_\theta{}^\vee = e_\theta.$$

If $\lambda = \mu_{|\Pi}$ for some distribution μ, then it is often possible to determine $\lambda^\vee$ directly from the identity

$$\lambda^\vee(z) = \langle \mu, e_z\rangle. \tag{2.6}$$

For example, $\langle \delta_\theta, e_z\rangle = e^{\theta z} = e_\theta(z) = (\delta_\theta{}^\vee)(z)$. The identity (2.6) is particularly useful when it is hard to determine directly the action of λ on the monomials $(\,)^\alpha$.

Finally, we note the identity

$$\langle \lambda, p\rangle = p(D)\,\lambda^\vee(0) \tag{2.7}$$

valid for any $\lambda \in A_0$ and any $p \in \Pi$.

In these terms, the interpolation problem to be studied in this paper is the following. For a given linear subspace Λ of Π', determine a linear subspace P of Π so that the pair $\langle \Lambda, P\rangle$ is **correct** in the sense that

$$P \to \Lambda^*\colon\ p \mapsto \langle \cdot, p\rangle_{|\Lambda}$$

is 1-1 and onto. We denote by IP(Λ) the interpolation problem induced by Λ as well as the collection of solutions P to this problem.

Here, Λ^* denotes the continuous dual of $\Lambda \subset \Pi'$ with respect to the induced topology. This is an appropriate choice since any $F \in \Lambda^*$ is extendible to Π'^*

(Hahn-Banach), hence is representable as $\langle \cdot, q \rangle_{|\Lambda}$ for some $q \in \Pi$ (Fact 2.4), and also, conversely, the restriction to Λ of every $p \in \Pi = \Pi'^*$ is continuous in this topology of Λ, namely, $\langle \cdot, p \rangle_{|\Lambda} \in \Lambda^*$.

As we will see, $\mathrm{IP}(\Lambda)$ is never empty and is infinite unless Λ is dense in Π'. Among the possibly infinitely many solutions, we single out a particular solution which, in addition to many other nice features to be described, is of the least possible degree (in a strong sense to be made precise). The description of this particular solution makes use of a particular map (which we call the **least map**), from subspaces of Π' to homogeneous subspaces of Π, and which we introduce now.

We use Π_k to denote all polynomials of (total) degree at most k, and

$$\Pi_k^0$$

to denote the space of all **homogeneous** polynomials of degree k (with the 0 polynomial included as usual). Recall that the **order** of the power series $\lambda \in \Pi'$, denoted by ord λ, is defined by

$$\text{ord}\,\lambda := \min\{|\alpha| : \alpha(\lambda) \neq 0\}. \tag{2.8}$$

For a formal power series $\lambda \neq 0$, its **initial form** (or **least term**) $\lambda_\downarrow$ is the unique homogeneous polynomial $\lambda_\downarrow \in \Pi^0_{\text{ord}\,\lambda}$ that satisfies $\text{ord}(\lambda - \lambda_\downarrow) > \text{ord}\,\lambda$. For completeness, we set $0_\downarrow := 0$. This definition can be written in terms of the power series coefficients as follows:

$$\alpha(\lambda_\downarrow) = \begin{cases} \alpha(\lambda), & \text{if } \beta(\lambda) = 0 \text{ for every } |\beta| < |\alpha|; \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

(5.8) **Theorem.** *Let Λ be a subspace of Π', and define*

$$\Lambda_\downarrow := \text{span}\{\lambda_\downarrow : \lambda \in \Lambda\}.$$

Then, for every $F \in \Lambda^*$, there exists a unique $p \in \Lambda_\downarrow$ such that $F = \langle \cdot, p \rangle_{|\Lambda}$. Hence, $\Lambda_\downarrow \in \mathrm{IP}(\Lambda)$.

For a finite-dimensional Λ, Theorem 5.8 implies the following result, which is recorded for subsequent use:

(2.10) **Proposition.** *For any finite-dimensional subspace Λ of Π'*, $\dim \Lambda = \dim \Lambda_\downarrow$.

We refer hereafter to the space $\Lambda_\downarrow$ as "the least solution of the interpolation problem". A discussion of the various aspects of the least map $\Lambda \mapsto \Lambda_\downarrow$ as well as the proof of Theorem 5.8 can be found in Sect. 5.

3 Properties and examples of the least solution

In this section, we present some typical examples of linear functional spaces Λ (i.e., interpolation conditions), and then discuss in detail several attractive properties that the least solution $\Lambda_\downarrow$ possesses, with the initial examples being used to illustrate these properties. Some of the claims made in this section will be proved only in subsequent sections. Our primary aim here is to provide the reader with a reasonable overview and a better insight, which may be helpful in reading the other parts of the paper.

(3.1) *Example.* The basic and most important example in our discussion is the **Lagrange interpolation problem**, i.e., the particular choice $\Lambda := \mathrm{span}\{\delta_\theta\}_{\theta\in\Theta}$, for some finite $\Theta \subset \mathbb{R}^s$, with δ_θ point evaluation at θ (see (2.5)). The corresponding space of generating functions is the exponential space

$$\mathrm{Exp}_\Theta := \mathrm{span}\{e_\theta : \theta \in \Theta\}. \tag{3.2}$$

For this Lagrange interpolation problem, we use $\mathrm{IP}(\Theta)$ rather than $\mathrm{IP}(\Lambda)$ to denote the set of solutions. Also, we use

$$\Pi_\Theta := (\mathrm{Exp}_\Theta)_\downarrow$$

to denote its least solution. Note that, regardless of the choice of Θ, the space Exp_Θ is always translation-invariant, hence also D-invariant. Also, it is easy to characterize here $\mathrm{IP}(\Theta)$ algebraically: $P \in \mathrm{IP}(\Theta)$ if and only if $\Pi = P \oplus I_\Theta$, with $I_\Theta \subset \Pi$ the ideal of all polynomials that vanish on Θ. However, this characterization does not readily provide solutions to problems of interest, e.g., to find the maximal Π_d which is included in some solution $P \in \mathrm{IP}(\Theta)$.

Although the linear functional space Λ is defined here (and in other examples to come) with the aid of a basis (namely, $\{\delta_\theta\}_{\theta\in\Theta}$), one cannot deal in the context of the least map with the basis elements alone, but must treat the whole linear functional space. Indeed, although the set $\{e_\theta\}_{\theta\in\Theta}$ forms a basis for Exp_Θ, we have $\{e_{\theta\downarrow}\}_{\theta\in\Theta} = \{1\}$ (while Π_Θ, as any other solution of $\mathrm{IP}(\Theta)$, must have dimension equal to $\#\Theta = \dim \mathrm{Exp}_\Theta$).

(3.3) *Example.* This example extends the Lagrange interpolation problem above, and also contains the Hermite interpolation problem and the Hermite-Birkhoff interpolation problem (cf. [BR1]). Λ is again finite-dimensional, and a basis for Λ is given by (the restriction to Π of) distributions with one-point support. That is, a typical basis element $\lambda \in \Lambda$ is of the form

$$\lambda : p \mapsto q(D)\, p(\theta),$$

where $q \in \Pi$ and $\theta \in \mathbb{R}^s$ are λ-dependent. With the aid of (2.6), we compute that

$$\lambda^\vee(z) = \langle \lambda, e_z \rangle = q(z)\, e_z(\theta) = q(z)\, e_\theta(z),$$

and therefore the generating function space is now a finite-dimensional subspace of $\sum_{\theta\in\Theta} e_\theta \Pi$ for some finite $\Theta \subset \mathbb{R}^s$. In contrast to the previous example, there is no guarantee here that Λ is D-invariant.

(3.4) *Example.* Λ is finite-dimensional and is spanned by (say, compactly supported) measures. E.g., each basis element ℓ is a *line integral* of the form

$$\ell : p \mapsto \int_0^1 p(a + (b-a)t)\, dt,$$

where $a, b \in \mathbb{R}^s$ are ℓ-dependent. Again, the generating function is easily computed from (2.6):

$$\ell^{\vee}(z) = \langle \ell, e_z \rangle = \frac{e_b(z) - e_a(z)}{(b-a)\,z}.$$

Note that now, in contrast to the Lagrange case, the generating function space is never D-invariant (since the derivatives of the univariate function $t \mapsto (e^t - 1)/t$ are linearly independent). From the standpoint of this paper, the lack of D-invariance here makes this interpolation problem harder than others like the Lagrange interpolation problem. □

With these examples in mind, we start now the discussion of the properties of the least solution $\Lambda_{\downarrow}$ of the interpolation problem IP(Λ).

Property A: Generality

The space Λ of interpolation conditions might be taken to be *any* subspace of the dual of Π. Even when restricting our attention to the Lagrange interpolation problems (in more than one variable), a general method for obtaining a solution does not seem to be a trivial task: given $n \geqq 2$, one cannot make up one subspace $P \subset \Pi$ of dimension n that solves all Lagrange problems associated with some $\Theta \subset \mathbb{R}^s$ of cardinality n. Therefore, the choice of the solution space must depend on the geometry of Θ. However, trying to determine a suitable P by studying these geometrical considerations seems to be painful, and usually results in restrictive assumptions on Θ.

Property B: Monotonicity

For subspaces Λ and M of Π',

$$\Lambda \subset M \;\Rightarrow\; \Lambda_{\downarrow} \subset M_{\downarrow}. \tag{3.5}$$

This (obvious) property is crucial if one wants to construct $\Lambda_{\downarrow}$ inductively. It also makes it possible to provide a *Newton form* for the interpolant.

Property C: Constructibility

From a practical point of view, this is probably the most important property. We proposed in [BR1] an algorithm which constructs, in finitely many arithmetic operations, from a given basis for the finite-dimensional Λ, another basis, say $\{\lambda_j\}_{j=1}^n$, such that $\{\lambda_{j\downarrow}\}_{j=1}^n$ is bi-orthogonal to $\{\lambda_j\}_{j=1}^n$, hence forms a basis for $\Lambda_{\downarrow}$. The construction of the interpolant If to a function f then proceeds in the usual way, i.e.,

$$f \mapsto If := \sum_{j=1}^{n} \lambda_{j\downarrow} \langle \lambda_j, f \rangle,$$

which uses only the data $\{\langle\lambda_j, f\rangle\}_{j=1}^n$ on f. A modified version of the above-mentioned algorithm, its relation to Gauß elimination, algorithmic details and some Lagrange interpolation examples are discussed in [BR3]. These two algorithms can, in turn, be used to construct a basis for the polynomial subspace of a box spline space; cf. Sect. 6.

Property D: Minimal degree

In general, it is desirable to keep the polynomials in the solution space P of the interpolation problem Λ of as small a degree as possible, and, in particular, to make the d for which $\Pi_d \subset P$ as large as possible. There are limits to this, since P must exclude polynomials on which all the interpolation conditions vanish. In the discussion here, we use the "minimal-degree" notion in the following sense.

(3.6) **Definition.** We say that the polynomial space P is **minimally correct for** Λ (or, is a **minimal-degree solution**) if $P \in \mathrm{IP}(\Lambda)$ and

$$\dim(Q \cap \Pi_k) \leqq \dim(P \cap \Pi_k), \qquad \forall Q \in \mathrm{IP}(\Lambda), \qquad k \in \mathbb{Z}_+.$$

We denote by $\mathrm{MIP}(\Lambda)$ the collection of all minimal-degree solutions for Λ. The following theorem implies that $\mathrm{MIP}(\Lambda)$ is never empty.

(5.10) **Theorem.** *The space $\Lambda_\downarrow$ is minimally correct for Λ.*

Thus, $P \in \mathrm{MIP}(\Lambda)$ if and only if $P \in \mathrm{IP}(\Lambda)$ and

$$\dim(P \cap \Pi_k) = \dim(\Lambda_\downarrow \cap \Pi_k), \qquad \forall k \in \mathbb{Z}_+.$$

We show later (in Sect. 5) that $\mathrm{MIP}(\Lambda)$ can be characterized directly by $\Lambda_\downarrow$ (without recourse to Λ), and that, further, $\Lambda_\downarrow$ is the only homogeneous polynomial space that can be used in this characterization.

There are various efforts in the literature to find (primarily Lagrange and Hermite) interpolation conditions which are correct for Π_k (for some $k \in \mathbb{Z}_+$). It is therefore reassuring to conclude, in view of Theorem 5.10, the following.

(3.7) **Corollary.** *Let Λ be a subspace of Π'. If $\Pi_k \in \mathrm{IP}(\Lambda)$ for some $k \in \mathbb{Z}_+$, then $\Lambda_\downarrow = \Pi_k$.*

Finally, we note that, generally speaking, the minimal degree property conflicts with generality and constructibility. E.g., in the Lagrange case, there are "easy-to-implement" schemes which can be used to find spaces in $\mathrm{IP}(\Theta)$ (cf. [GM]), yet these spaces are, in general, far from being of minimal degree, nor are they canonical, for the solution space depends on ordering Θ, as well as on the choice of certain free parameters.

The remaining properties below concern the interaction between the least map and some basic operations on Π', such as convolution, differentiation, homogeneous maps and taking tensor products.

Property E: Interaction with convolution; the translation-invariance of $\Theta \mapsto \Pi_\Theta$

In order to distinguish between the multiplication of $\mu \in \mathbb{R}[\![X]\!] = \Pi'$ with $\lambda \in \mathbb{R}[\![X]\!] = \Pi'$ and the application of $\mu \in \Pi' = \mathbb{R}[\![X]\!]$ to $\lambda \in \Pi \subset \Pi'$, we write

$$\mu * \lambda$$

for the former, as $\{\alpha(\mu*\lambda)\}_\alpha$ is indeed the convolution product of $\{\alpha(\mu)\}_\alpha$ and $\{\alpha(\lambda)\}_\alpha$. Since, for any $\lambda, \mu\in\Pi'$,

$$(\mu*\lambda)_\downarrow=\lambda_\downarrow\,\mu_\downarrow, \tag{3.8}$$

we reach the following conclusion:

(3.9) **Proposition.** *Let Λ be a subspace and μ an element of Π'. Then*

$$(\mu*\Lambda)_\downarrow=\mu_\downarrow\,\Lambda_\downarrow. \tag{3.10}$$

In particular, if $\mu_\downarrow$ is a nonzero constant, then

$$(\mu*\Lambda)_\downarrow=\Lambda_\downarrow.$$

(3.11) *Example.* For the Lagrange interpolation problem IP(Θ), $\Lambda^\vee$ is the exponential space Exp_Θ. If we take $\mu^\vee$ to be any exponential e_r ($r\in\mathbb{R}^s$), then $\mu_\downarrow=1$, hence (by Proposition 3.9)

$$(e_r\,\mathrm{Exp}_\Theta)_\downarrow=(\mathrm{Exp}_\Theta)_\downarrow=\Pi_\Theta.$$

On the other hand, $e_r\,\mathrm{Exp}_\Theta=\mathrm{Exp}_{(r+\Theta)}$, and we thus obtain that *the least solution of the Lagrange problem is invariant under translations of Θ*: for every $r\in\mathbb{R}^s$ and $\Theta\subset\mathbb{R}^s$,

$$\Pi_{(r+\Theta)}=\Pi_\Theta. \tag{3.12}$$

As a matter of fact, the main property of IP(Θ) used for (3.12) is the fact that the basis $\{\delta_\theta\}_{\theta\in\Theta}$ for Λ is obtained by shifting a single linear functional (viz., δ_0). For this reason, we have the following extension of (3.12):

(3.13) **Corollary.** For *$\lambda\in\Pi'$ and finite $\Theta\subset\mathbb{R}^s$, define $\Lambda:=\mathrm{span}\{E^\theta\lambda\colon\ \theta\in\Theta\}$, where $\langle E^\theta\lambda, p\rangle:=\langle\lambda, p(\cdot+\theta)\rangle$. Then*

$$\Lambda_\downarrow=\lambda_\downarrow\,\Pi_\Theta.$$

We exploit this observation in the next example.

(3.14) *Example.* Suppose that X is a matrix in $\mathbb{R}^{s\times n}$ with non-zero columns, and let X stand also for the collection (more precisely, the multiset) of the columns of X. Each $x\in X$ (considered as a vector in $\mathbb{R}^s\backslash 0$) induces a line integral ℓ_x:

$$\ell_x\colon\ p\mapsto\frac{1}{2}\int_{-1}^{1}p(tx)\,dt.$$

We define ℓ_X to be the convolution product of all the line integrals ℓ_x, $x\in X$. The density measure of X is known as a **(centered) box spline** [BH]. The generating function of ℓ_X can easily be computed (compare with Example 3.4):

$$\ell_X{}^\vee(z)=\prod_{x\in X}\frac{\sinh(xz)}{xz}. \tag{3.15}$$

Since the box spline is a unit measure centered at the origin, $\langle \ell_X, p\rangle$ provides an average value of p around the origin. We now generate a family of linear functionals from ℓ_X by translation, and by changing the magnitude (but not the direction) of each $x \in X$. A typical functional ℓ obtained by such a modification is of the form

$$\ell^{\vee}(z) = e_{\theta} \prod_{x \in X} \frac{\sinh(t_x\, xz)}{xz},$$

where $\{t_x\}_{x \in X}$ are some ℓ-dependent non-zero scalars and $\theta \in \mathbb{R}^s$ is ℓ-dependent as well. Suppose that Λ is the span of (say, finitely many) linear functionals, all obtained by modifying the same original box spline. In this case the functionals in Λ provide average values in balls of possibly different diameters around different points.

Note now that the *homogeneous* polynomial $q(z) := \prod_{x \in X} (xz)$ (of degree n) appears in the denominator of (the generating function of) every functional in Λ. In view of Proposition 3.9, we may obtain $\Lambda_{\downarrow}$ in the form $\mathrm{M}_{\downarrow}/q$, with the exponential space $\mathrm{M}^{\vee}$ spanned by the exponentials of the form

$$\mu^{\vee}(z) = e_{\theta} \prod_{x \in X} \sinh(t_x\, xz). \quad \square$$

More specific examples of this nature are discussed in Sect. 7.

Property F: Homogeneous maps

A linear map $A\colon \Pi' \to \Pi'$ is **homogeneous of degree** k if $A(\Pi_j^0) \subset \Pi_{j+k}^0$ for every $j \geqq 0$. If A is such a map, it satisfies

$$(A\lambda)_{\downarrow} = A(\lambda_{\downarrow}),$$

unless $A(\lambda_{\downarrow}) = 0$. This implies that, for any space $\Lambda \subset \Pi'$,

$$A(\Lambda_{\downarrow}) \subset (A\Lambda)_{\downarrow}. \tag{3.16}$$

Since, in particular, any directional differentiation is a homogeneous map, this provides the following result of much use later.

(3.17) **Proposition.** *If a subspace Λ of Π' is D-invariant, then so is $\Lambda_{\downarrow}$.*

Since Exp_{Θ} is D-invariant, we have the following.

(3.18) **Corollary.** *The least space Π_{Θ} associated with the Lagrange interpolation problem* $\mathrm{IP}(\Theta)$ *is D-invariant.*

In particular, there are no "jumps" in the homogeneous grades of Π_{Θ}, i.e.,

$$\Pi_{\Theta} \cap \Pi_k^0 = 0 \;\Rightarrow\; \Pi_{\Theta} \cap \Pi_{k+j}^0 = 0,\ \forall j > 0.$$

Also, the homogeneous dimensions of Π_{Θ} constitute the Hilbert function of some (homogeneous) ideal.

(3.19) *Remark.* It should be clear that $\Lambda_{\downarrow}$ *might* be D-invariant even though Λ is not (take a one-dimensional Λ which does not vanish on the constants and is not an exponential space). On the other hand, not every space of the form $\Lambda_\downarrow$ is D-invariant: on multiplying any Λ by any polynomial that vanishes at the origin, we obtain a space M whose least space $\mathrm{M}_\downarrow$ does not contain constants (cf. Proposition 3.9), hence is not D-invariant. □

If A, in addition to being homogeneous, is also *injective*, then equality must hold in (3.16). A particular case of interest is a **linear change of variables**, i.e., a linear invertible map $A\colon \mathbb{R}^s \to \mathbb{R}^s$, which is lifted to Π by the definition $Ap(x) := p(Ax)$.

(3.20) **Proposition.** *Let A be a linear change of variables. Then, $(A\Lambda)_\downarrow = A(\Lambda_\downarrow)$ for every subspace $\Lambda \subset \Pi'$.*

With A^t being the transposed map of A, this implies that

$$A(\Pi_\Theta) = \Pi_{A^t\Theta},$$

since $A(\mathrm{Exp}_\Theta) = \mathrm{Exp}_{A^t\Theta}$. In particular, rotation and reflection of Θ result in a similar action on Π_Θ, so that symmetries of this type in Θ are preserved in Π_Θ.

(3.21) *Example.* With $s=2$, let Θ consist of the four intersection points of the ellipse $a_1(\,)^{2,0} + a_2(\,)^{0,2} = 1$ with the coordinate axes. Then $\Pi_1(\mathbb{R}^2) \subset \Pi_\Theta$, by the minimal degree property of Π_Θ, since no linear $p \in \Pi(\mathbb{R}^2)$ vanishes on Θ. Furthermore, Θ is invariant under reflection across each of the axes, which means that $\Pi_\Theta \cap \Pi_2^0$ may contain only polynomials of the form $c_1(\,)^{2,0} + c_2(\,)^{0,2}$ (polynomials of the form $c(\,)^{1,1}$, which are also invariant under the above reflections, are excluded since they vanish on Θ). If the ellipse is circular, then Θ is invariant under rotation by 90 degrees, hence so is Π_Θ, which implies that $c_1 + c_2 = 0$ (the other possibility $c_1 = c_2$ is excluded since then the quadratic polynomial assumes a constant value on Θ). If the ellipse is not circular, then $c_1(\,)^{2,0} + c_2(\,)^{0,2} \in \Pi_\Theta \cap \Pi_2^0$ if and only if (c_1, c_2) is perpendicular to the vector (a_1, a_2). This will follow as well from the general discussion concerning annihilation (see Property G below). □

In case we choose the linear map A to be the scaling operator

$$\sigma_h\, \lambda \mapsto \lambda(\cdot/h),$$

we may use the fact that $\Lambda_\downarrow$ is scale-invariant (as is every homogeneous space) to conclude that

$$(\sigma_h \Lambda)_\downarrow = \sigma_h(\Lambda_\downarrow) = \Lambda_\downarrow,$$

which implies in the Lagrange case that

$$\Pi_{\Theta/h} = \Pi_\Theta.$$

Property G: Annihilation

For a D**-invariant** Λ, i.e., a Λ closed under (formal) differentiation, the study of the relation between the actions of differential operators on Λ and $\Lambda_\downarrow$ is very useful. The next theorem summarizes our main results in this direction.

We use here the notation $q_\uparrow$ for the **leading term** of the polynomial q, i.e., $q_\uparrow$ is the unique homogeneous polynomial that satisfies

$$\deg(q-q_\uparrow)<\deg q. \tag{3.22}$$

We also use $q(D)$ for the (formal) differential operator with constant coefficients obtained by evaluating q at D. Note that in general $q(D)$ is neither injective (unlike convolution operators) nor a homogeneous map. However, if q is homogeneous, then $q(D)$ is homogeneous, of order $-\deg q$.

(4.11) **Theorem.** *Let Λ be a D-invariant subspace of Π', and let p be a polynomial.*
(a) *If $p(D)\,\Lambda_\downarrow=0$, then $q(D)\,\Lambda=0$, for some $q\in\Pi$ with $q_\uparrow=p_\uparrow$.*
(b) *If $p(D)\,\Lambda=0$, then $p_\uparrow(D)\,\Lambda_\downarrow=0$.*

This theorem is of particular interest for the Lagrange interpolation problem, to which it applies since Exp_Θ is D-invariant: One has $p(D)\,e_\theta=p(\theta)\,e_\theta$ (for $p\in\Pi$ and $\theta\in\mathbb{R}^s$). This also implies that

$$p(D)(e_\theta)=0 \iff p(\theta)=0. \tag{3.23}$$

Thus, Theorem 4.11 reads in the Lagrange case as follows.

(3.24) **Corollary.** *For a finite $\Theta\subset\mathbb{R}^s$, and $p\in\Pi$:*
(a) *If $p(D)(\Pi_\Theta)=0$, then q vanishes on Θ for some $q\in\Pi$ with $q_\uparrow=p_\uparrow$.*
(b) *If p vanishes on Θ, then $p_\uparrow(D)(\Pi_\Theta)=0$.*

(3.25) *Example: Harmonic polynomials.* Suppose that we want to approximate functions which are harmonic in the open unit disk $U\subset\mathbb{R}^2$ and continuous on its closure U^-, by interpolating their values on the unit circle (say, at the roots of unity). It is obvious (and well-known) that this can be done by using harmonic polynomials of sufficiently high degree. It is therefore very pleasing to see that the least solution provides exactly these harmonic polynomials:

(3.26) **Theorem.** *Let $s=2$. Then Π_Θ consists of harmonic polynomials if and only if Θ lies on some circle in the plane.*

Proof. Assume that Θ lies on the circle given by the quadratic equation $p=0$. In this case, the leading term $p_\uparrow(D)$ of $p(D)$ is the Laplacian, and, by Corollary 3.24 (b), $p_\uparrow(D)(\Pi_\Theta)=0$, hence Π_Θ is a harmonic space.

Conversely, assume that Π_Θ is annihilated by the Laplacian $L(D)$. Since $L(D)$ is homogeneous, we may apply Corollary 3.24 (a) to find a polynomial p such that $p_\uparrow=L$ and p vanishes on Θ. Since $L=(\,)^{2,0}+(\,)^{0,2}$, the equation $p=0$ defines a circle. □

Theorem 4.11 might also be helpful for some types of non-Lagrange interpolation problems. An example is discussed in Sect. 7.

Whether or not Λ is D-invariant can often be decided by the following criterion.

(6.1) **Proposition.** *A closed subspace Λ of Π' is D-invariant if and only if $\Lambda_\perp$ is a polynomial ideal (in Π).*

Here, the **annihilator** or **kernel** $\Lambda_\perp\subset\Pi$ of $\Lambda\subset\Pi'$ is defined as usual by

$$\Lambda_\perp:=\{p\in\Pi:\langle\lambda,p\rangle=0,\ \forall\lambda\in\Lambda\}. \tag{3.27}$$

Because of its importance for us, and in preparation for the proof of Theorem 4.11, we verify directly the following

(3.28) **Corollary.** *If the subspace* Λ *of* Π' *is D-invariant, then* $p(D)\Lambda = 0$ *for all* $p \in \Lambda_\perp$.

Proof. For $\lambda \in \Lambda$, $p \in \Lambda_\perp$ and $\alpha \in \mathbb{Z}_+^s$,

$$\alpha(p(D)\lambda) = \langle p(D)\lambda, (\,)^\alpha \rangle = \langle D^\alpha \lambda, p \rangle = 0,$$

by (2.2), (2.3), and the D-invariance of Λ, respectively. □

Property H: Tensor product

The tensor product of two power series spaces commutes with the least map:

(3.29) **Proposition.** *Let* M, N *be subspaces of* $\Pi'(\mathbb{R}^m)$, $\Pi'(\mathbb{R}^n)$ *respectively. Then,* $\mathrm{M} \otimes \mathrm{N}$, *regarded as a subspace of* $\Pi'(\mathbb{R}^{m+n})$, *satisfies*

$$(\mathrm{M} \otimes \mathrm{N})_\downarrow = \mathrm{M}_\downarrow \otimes \mathrm{N}_\downarrow. \tag{3.30}$$

Proof. For $\mu \in \mathrm{M}$ and $v \in \mathrm{N}$, $(\mu \otimes v)_\downarrow = \mu_\downarrow \otimes v_\downarrow$, hence $(\mathrm{M} \otimes \mathrm{N})_\downarrow \supset \mathrm{M}_\downarrow \otimes \mathrm{N}_\downarrow$. This completes the proof for finite-dimensional M and N, since in this case both sides of (3.30) are of dimension dim M dim N (by Proposition 2.10 applied to M, N and $\mathrm{M} \otimes \mathrm{N}$). The general case now follows by expressing $\mathrm{M} \otimes \mathrm{N}$ as the union of an increasing sequence $(\mathrm{M}^{(j)} \otimes \mathrm{N}^{(j)})_{j=1}^\infty$ of subspaces, where each $\mathrm{M}^{(j)}$ and $\mathrm{N}^{(j)}$ is a finite-dimensional subspace of M and N, respectively. □

This proposition applies to a "rectangular array" of interpolation conditions: assume that we are given finite-dimensional $\mathrm{M}_1, \ldots, \mathrm{M}_s \subset \Pi'(\mathbb{R})$ and define

$$\mathrm{M} := \mathrm{M}_1 \otimes \mathrm{M}_2 \otimes \ldots \otimes \mathrm{M}_s.$$

Then, with $(\mu_{j,k})_{k=0}^{\kappa_j}$ in $\Pi(\mathbb{R})$ a basis for $(\mathrm{M}_j)_\downarrow$, $j = 1, \ldots, s$,

$$\mathrm{M}_\downarrow = \mathrm{span}\{\mu_{1,\alpha_1} \otimes \mu_{2,\alpha_2} \otimes \ldots \otimes \mu_{s,\alpha_s} : \alpha \in \Gamma\}, \tag{3.31}$$

where

$$\Gamma := J(\kappa) := \{\alpha \in \mathbb{Z}_+^s : \alpha \leqq \kappa\}. \tag{3.32}$$

In particular, we get the following result:

(3.33) **Corollary.** *Let* $\{\mathrm{M}_j\}_{j=1}^s$ *and* M *be as above, and assume that, for each* j, $(\mathrm{M}_j)_\downarrow = \Pi_{\kappa_j}(\mathbb{R})$. *Then*

$$\mathrm{M}_\downarrow = \Pi_\Gamma := \mathrm{span}\{(\,)^\alpha : \alpha \in \Gamma\}. \tag{3.34}$$

In case $\Theta \subset \mathbb{R}^s$ consists of the vertices of a rectangular grid, this corollary shows that Π_Θ coincides with the "natural" solution, i.e., the polynomial space of *coordinate* degree κ.

Corollary 3.33 can be extended from rectangular arrays to **order-closed arrays** (or, **lower sets** in the terminology of [LL]), i.e., to subsets Γ' of Γ which satisfy

$$\mathbb{Z}_+^s \ni \alpha \leqq \beta \in \Gamma' \Rightarrow \alpha \in \Gamma'.$$

For this, we equip each M_j in the corollary with a basis $\{\mu_{j,0}, \ldots, \mu_{j,\kappa_j}\}$ for which

$$(M_{j,k})_\downarrow = \Pi_k, \quad \forall 0 \leqq k \leqq \kappa_j, \quad 1 \leqq j \leqq s, \tag{3.35}$$

with $M_{j,k} := \mathrm{span}\{\mu_{j,0}, \ldots, \mu_{j,k}\}$. For each $\alpha \in \Gamma$ (with Γ as in (3.32)), define

$$\Lambda_\alpha := M_{1,\alpha_1} \otimes M_{2,\alpha_2} \otimes \ldots \otimes M_{s,\alpha_s}.$$

Finally, for a given $\Gamma' \subset \Gamma$, we set

$$\Lambda_{\Gamma'} := \sum_{\alpha \in \Gamma'} \Lambda_\alpha$$

and conclude the following.

(3.36) **Corollary.** *For every order-closed* $\Gamma' \subset \Gamma$,

$$(\Lambda_{\Gamma'})_\downarrow = \Pi_{\Gamma'} := \mathrm{span}\{(\,)^\alpha : \alpha \in \Gamma'\}. \tag{3.37}$$

Proof. The map

$$(\,)^\alpha \mapsto \mu_{1,\alpha_1} \otimes \mu_{2,\alpha_2} \otimes \ldots \otimes \mu_{s,\alpha_s}, \qquad \alpha \in \Gamma'$$

induces a linear isomorphism between $\Pi_{\Gamma'}$ and $\Lambda_{\Gamma'}$, hence their dimensions agree. On the other hand, by Corollary 3.33 and the monotonicity property (Property B),

$$\Pi_{J(\alpha)} = (\Lambda_\alpha)_\downarrow \subset (\Lambda_{\Gamma'})_\downarrow \qquad \forall \alpha \in \Gamma',$$

therefore $\Pi_{\Gamma'} \subset (\Lambda_{\Gamma'})_\downarrow$, and the desired result then follows, since by the above and Proposition 2.10, $\dim \Pi_{\Gamma'} = \dim \Lambda_{\Gamma'} = (\dim \Lambda_{\Gamma'})_\downarrow$. □

A particular example is obtained by choosing each $\mu_{j,k}$ to be the point-evaluation $\delta_{\theta_{j,k}}$ (with $\theta_{j,k} \in \mathbb{R}$ and $\theta_{j,k} \neq \theta_{j,k'}$ for $k \neq k'$). In this case $\mathrm{IP}(\Lambda_{\Gamma'})$ is a Lagrange interpolation problem with respect to an order-closed Θ and the least solution turns out to coincide again with the "natural" monomial space $\Pi_{\Gamma'}$. It is not the (known) fact that $\Pi_{\Gamma'}$ does solve $\mathrm{IP}(\Lambda_{\Gamma'})$ that should be emphasized, but the fact that the least solution coincides with this preferred solution. We note that actually the only facts used to derive this result (aside from the correctness of total degree spaces for Lagrange interpolation at arbitrary subsets of $\mathbb{R}$) are the monotonicity, the tensor product property, and the minimal degree property, of the least map. Any other map satisfying these three properties would provide here $\Pi_{\Gamma'}$ as the solution space.

4 Homogenization

The least map

$$\Lambda \mapsto \Lambda_\downarrow := \operatorname{span}\{\lambda_\downarrow : \lambda \in \Lambda\}, \tag{4.1}$$

defined on subspaces of Π', is a typical example of an **internal homogenization** map (cf. [NV]). Such maps make use of the *graded* structure of Π'. The least map is complemented by the homogenization map

$$P \mapsto P_\uparrow := \operatorname{span}\{p_\uparrow : p \in P\} \tag{4.2}$$

defined on subspaces $P \subset \Pi$, where $p_\uparrow$ denotes the *leading term* (cf. 3.22)) of the polynomial p. We discuss in this section some properties concerning these maps and their interrelation.

The spaces $P_\uparrow$ and $\Lambda_\downarrow$ are both homogeneous (or graded), i.e., are spanned by homogeneous polynomials. The map $p \mapsto p_\uparrow$ (resp. $\lambda \mapsto \lambda_\downarrow$) is non-linear, and is neither injective nor surjective when considered as a map from P to $P_\uparrow$ (resp. Λ to $\Lambda_\downarrow$). We already noted the monotonicity of the least map; the leading map $P \mapsto P_\uparrow$ is just as obviously monotone.

For any $P \subset \Pi$, the action of $\Lambda \subset \Pi'$ on

$$P_k := P \cap \Pi_k \tag{4.3}$$

is entirely determined by $T_k \Lambda$, with T_k the **Taylor map**, i.e., the map on Π' which associates with each $\lambda \in \Pi' = \mathbb{R}[\![X]\!]$ its Taylor polynomial $T_k \lambda$ of degree k. In terms of the power series coefficients,

$$\alpha(T_k \lambda) = \begin{cases} \alpha(\lambda), & |\alpha| \leqq k; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for any subspace $\Lambda \subseteq \Pi'$,

$$(\Lambda_\perp)_k = ((T_k \Lambda)_\perp)_k.$$

Here and below, we use the subscript $_k$ to indicate the collection of all polynomials of degree $\leqq k$ in a set (cf. (4.3)), and continue to use the subscript $_\perp$ to indicate the kernel of a set of linear functionals on Π (cf. (3.27)).

The next result shows that the two homogenization processes preserve dimensions in the following strong sense.

(4.4) **Proposition.** (a) *For any subspace* $\Lambda \subset \Pi'$ *and any* $k \in \mathbb{Z}_+$, $\dim(\Lambda_\downarrow)_k = \dim T_k \Lambda$. *In particular*, $\dim \Lambda_\downarrow = \dim \Lambda$.
(b) *For any subspace* $P \subset \Pi$ *and any* $k \in \mathbb{Z}_+$, $\dim(P_\uparrow)_k = \dim P_k$. *In particular*, $\dim P_\uparrow = \dim P$.

Proof. (b) Set $S_j := (\mathrm{id} - T_j)_{|P}$ (id being the identity map). Note that $\deg p = j$ iff $S_j p = 0$ and $S_{j-1} p \neq 0$, and so

$$\dim(P_\uparrow \cap \Pi_j^0) = \dim S_{j-1}(\ker S_j) = \dim \ker S_j - \dim \ker S_{j-1},$$

using the fact that $\ker S_{j-1} \subset \ker S_j$. Summing this equality over $j=0, 1, \ldots, k$, we obtain

$$\dim P_{\uparrow k} = \dim \ker S_k - \dim \ker S_{-1}.$$

Yet, $S_{-1} = \mathrm{id}$, and therefore $\dim \ker S_{-1} = 0$, while $\ker S_k = P_k$, hence $\dim P_k = \dim (P_\uparrow)_k$. Letting $k \to \infty$, we obtain also that $\dim P = \dim P_\uparrow$.

The proof of (a) is very similar to that of (b) (see [BR1] for details). □

We will also need the following observations regarding homogeneous bases for $\Lambda_\downarrow$. While $\Lambda_\downarrow$ is a homogeneous polynomial space, hence has homogeneous algebraic bases, an algebraic basis for Λ is of little interest when Λ is not finite-dimensional. But any subspace Λ of Π' has a **weak basis**, i.e., there are sequences $(\lambda_i)_i$ in Λ so that, for every $\lambda \in \Lambda$, there is a unique a so that $\lambda = \sum_i a(i)\, \lambda_i$, with the sum taken pointwise, i.e., $\langle \lambda, p \rangle = \sum_{\operatorname{ord} \lambda_i \leqq \deg p} a(i) \langle \lambda_i, p \rangle$ for all $p \in \Pi$.

(4.5) **Lemma.** *Let Λ be a subspace of Π'. Any homogeneous (algebraic) basis for $\Lambda_\downarrow$ is of the form $(\lambda_{i\downarrow})_i$ for some (weak) basis $(\lambda_i)_i$ for Λ. In particular, $\Lambda_\perp = \bigcap_i \ker \lambda_i$, for each homogeneous basis $(\lambda_{i\downarrow})_i$ for $\Lambda_\downarrow$.*

Proof. Since any homogeneous element of $\Lambda_\downarrow$ is necessarily of the form $\lambda_\downarrow$ for some $\lambda \in \Lambda$, we may assume that our homogeneous algebraic basis for $\Lambda_\downarrow$ is of the form $(\lambda_{i\downarrow})_i$ for some sequence $(\lambda_i)_i$ in Λ.

We now prove that such a sequence $(\lambda_i)_i$ is necessarily a (weak) basis for Λ. The proof is by induction: Let $\lambda \in \Lambda$. Assume that we have already determined $a(i)$ for $\operatorname{ord} \lambda_i < k$ so that $\lambda = \sum_{\operatorname{ord} \lambda_i < k} a(i)\, \lambda_i$ on $\Pi_{<k}$, with the sum being finite, since $(\lambda_{i\downarrow})_{\operatorname{ord} \lambda_i < k}$ are linearly independent. Then

$$\mu := \lambda - \sum_{\operatorname{ord} \lambda_i < k} a(i)\, \lambda_i$$

is in Λ and has order at least k (since it vanishes on $\Pi_{<k}$). If $\operatorname{ord} \mu = k$, then $\mu_\downarrow = \sum_{\operatorname{ord} \lambda_i = k} a(i)\, \lambda_{i\downarrow}$ for some numbers $a(i)$. Else, choose $a(i) = 0$ for $\operatorname{ord} \lambda_i = k$. In either case, $\lambda = \sum_{\operatorname{ord} \lambda_i \leqq k} a(i)\, \lambda_i$ on Π_k, with the new coefficients uniquely determined since $\{\lambda_{i\downarrow} : \operatorname{ord} \lambda_i = k\}$ are linearly independent, by assumption. This advances the induction hypothesis.

If now $p \in \bigcap_i \ker \lambda_i$, then $\langle \lambda, p \rangle = \sum_i a(i) \langle \lambda_i, p \rangle = 0$ for any $\lambda \in \Lambda$, hence $p \in \Lambda_\perp$. This proves that $\bigcap_i \ker \lambda_i \subset \Lambda_\perp$, while the converse inclusion is trivial. □

Here is a simple, yet useful, observation.

(4.6) **Lemma.** *Let $\lambda \in \Pi'$ and $p \in \Pi$. If $\langle \lambda, p \rangle = 0$, then $\langle \lambda_\downarrow, p_\uparrow \rangle = 0$ as well.*

Proof. If $\operatorname{ord} \lambda \neq \deg p$, then $p_\uparrow$ and $\lambda_\downarrow$ are two homogeneous polynomials of different degrees and hence $\langle \lambda_\downarrow, p_\uparrow \rangle = 0$ trivially. Otherwise, $\deg p = \operatorname{ord} \lambda$, a case in which $\langle \lambda, p \rangle = \langle \lambda_\downarrow, p_\uparrow \rangle$. □

In analogy to $\Lambda_\perp$, we define

$$P^\perp := \{\lambda \in \Pi' : P \subset \ker \lambda\},$$

the **annihilator in** Π' **of** $P \subseteq \Pi$. We note that, with the identification $\Pi' = \mathbb{R}[\![X]\!]$, any subspace P of Π is also a subspace of Π' and that, for a homogeneous subspace P of Π, the essential difference between $P_\perp$ and $P^\perp$ lies in the fact that the latter contains also *infinite* linear combinations. Further,

$$P = P^\perp{}_\perp$$

for any subspace P of Π, and also

$$P = P_{\perp\perp} \tag{4.7}$$

for any *homogeneous* subspace P of Π.

We now come to the main result of this section. It concerns the interaction among the maps $_\downarrow$, $_\uparrow$, $_\perp$ and $^\perp$.

(4.8) **Theorem.** *Let P and Λ be subspaces of Π and Π' respectively. Then*

(a) $(\Lambda_\downarrow)_\perp = (\Lambda_\perp)_\uparrow$;
(b) $(P^\perp)_\downarrow = (P_\uparrow)_\perp$.

Proof. (a) We first show that $(\Lambda_\downarrow)_\perp \supset (\Lambda_\perp)_\uparrow$. Let $q \in (\Lambda_\perp)_\uparrow$. To prove that $q \in (\Lambda_\downarrow)_\perp$, we need to show that $\langle \mu, q \rangle = 0$ for $\mu \in \Lambda_\downarrow$. Since both $\Lambda_\downarrow$ and $(\Lambda_\perp)_\uparrow$ are homogeneous, we may assume without loss that μ and q are homogeneous. This in turn implies the existence of $p \in \Lambda_\perp$ and $\lambda \in \Lambda$ such that $p_\uparrow = q$ and $\lambda_\downarrow = \mu$, so that we have to prove that $\langle \lambda_\downarrow, p_\uparrow \rangle = 0$. But this follows from Lemma 4.6, since, by the choice of λ and p, one has $\langle \lambda, p \rangle = 0$.

For the converse inclusion, it is now sufficient to show that, for every $k \in \mathbb{Z}_+$,

$$\dim (\Lambda_\downarrow)_{\perp k} = \dim (\Lambda_\perp)_{\uparrow k} \tag{4.9}$$

(with $Q_k := Q \cap \Pi_k$ for any $Q \subset \Pi$, as before). We have $\mathrm{M}_{\perp k} = (T_k \mathrm{M})_{\perp k}$ for any $\mathrm{M} \subset \Pi'$, since $\langle \lambda, p \rangle = \langle T_k \lambda, p \rangle$ for every $\lambda \in \Pi'$ and every $p \in \Pi_k$. Further, by Proposition 4.4 (b) (with $P = \Lambda_\perp$), we have $\dim (\Lambda_\perp)_{\uparrow k} = \dim \Lambda_{\perp k}$. Therefore, (4.9) is equivalent to

$$\dim (T_k \Lambda_\downarrow)_{\perp k} = \dim (T_k \Lambda)_{\perp k}. \tag{4.10}$$

For any $\mathrm{M} \subset \Pi_k$, $\mathrm{M}_{\perp k}$ is the orthogonal complement of M in Π_k with respect to the *inner product* $\langle \cdot, \cdot \rangle$. Since both $T_k \Lambda_\downarrow = \Lambda_{\downarrow k}$ and $T_k \Lambda$ are subspaces of Π_k, (4.10) is therefore equivalent to

$$\dim \Lambda_{\downarrow k} = \dim T_k \Lambda,$$

and this is Proposition 4.4 (a).

As for (b), it is obtained by choosing $\Lambda = P^\perp$ in (a), hence $(P^\perp{}_\downarrow)_\perp = \Lambda_{\perp\uparrow} = P^\perp{}_{\perp\uparrow} = P_\uparrow$, which implies $(P^\perp{}_\downarrow)_{\perp\perp} = P_{\uparrow\perp}$, and this gives (b), by (4.7). □

For the D-invariant case, the last theorem implies the following.

(4.11) **Theorem.** *Let Λ be a D-invariant subspace of Π', and let p be a polynomial.*

(a) *If $p(D)\Lambda_\downarrow = 0$, then $q(D)\Lambda = 0$, for some $q \in \Pi$ with $q_\uparrow = p_\uparrow$.*

(b) *If* $p(D)\Lambda=0$, *then* $p_\uparrow(D)\Lambda_\downarrow=0$.

Proof. (a) Since $\Lambda_\downarrow$ is homogeneous, $p(D)\Lambda_\downarrow=0$ implies that $p_\uparrow(D)\Lambda_\downarrow=0$, and therefore $p_\uparrow\in\Lambda_{\downarrow\perp}$, hence also $p_\uparrow\in\Lambda_{\perp\uparrow}$, by Theorem 4.8. This implies the existence of some $q\in\Lambda_\perp$ with $q_\uparrow=p_\uparrow$. Since Λ is D-invariant, it follows from Corollary 3.28 that $q(D)\Lambda=0$.

(b) If $p(D)\Lambda=0$, then $p\in\Lambda_\perp$, hence $p_\uparrow\in\Lambda_{\perp\uparrow}=\Lambda_{\downarrow\perp}$, by Theorem 4.8. Since Λ is D-invariant, so is $\Lambda_\downarrow$ (by Proposition 3.17), therefore $p_\uparrow(D)\Lambda_\downarrow=0$ by Corollary 3.28. □

5 The least solution and its minimal degree property

Since any $F\in\Lambda^*$ is necessarily of the form $\langle\cdot, q\rangle_{|\Lambda}$ for some $q\in\Pi$, our interpolation problem (of finding $p\in P$ with $F=\langle\cdot, p\rangle_{|\Lambda}$) is essentially finite-dimensional, even if Λ is not. For, if such q has degree k, then it is sufficient to find

$$p\in P_k=P\cap\Pi_k$$

such that

$$(5.1)\qquad \langle T_k\lambda, p\rangle=\langle T_k\lambda, q\rangle \qquad \forall\lambda\in\Lambda,$$

since

$$(5.2)\qquad \langle T_k\lambda, r\rangle=\langle\lambda, T_k r\rangle=\langle\lambda, r\rangle \qquad \forall r\in\Pi_k,$$

hence (5.1) implies that $\langle\lambda, p\rangle=\langle\lambda, q\rangle=F(\lambda)$ for all $\lambda\in\Lambda$. Further, the solution p is unique (in P) if and only if $\Lambda_\perp\cap P=0$, while (with (5.2))

$$(5.3)\qquad \Lambda_\perp\cap P=0 \Leftrightarrow (T_k\Lambda)_\perp\cap P_k=0\;\forall k.$$

Finally, the correctness of the (finite-dimensional) pair $\langle T_k\Lambda, P_k\rangle$ is well-known to be equivalent to the conditions

$$(5.4)\qquad \dim T_k\Lambda\leqq\dim P_k, \qquad (T_k\Lambda)_\perp\cap P_k=0.$$

Thus, having (5.4) hold for every k is a sufficient condition for the correctness of $\langle\Lambda, P\rangle$, and we have proved the following lemma.

(5.5) **Lemma.** *Let P and Λ be subspaces of Π and Π', respectively, which satisfy*

$$(5.6)\qquad \dim T_k\Lambda\leqq\dim P_k, \qquad \forall k,$$

with $P_k := P\cap\Pi_k$. Then the following are equivalent:
(a) *$\langle\Lambda, P\rangle$ is correct;*
(b) $\Lambda_\perp\cap P=0$;
(c) *For all k, $(T_k\Lambda)_\perp\cap P_k=0$;*
(d) *For all k, $\langle T_k\Lambda, P_k\rangle$ is correct.*

(5.7) **Corollary.** *If P and Λ are homogeneous subspaces of Π and Π', respectively, then the following conditions are equivalent (even without the explicit assumption (5.6)).*

(a) *$\langle\Lambda, P\rangle$ is correct;*

(b) *For all k*, $\langle \Lambda_k, P_k \rangle$ *is correct*;
(c) *For all k*, $\langle \Lambda \cap \Pi_k^0, P \cap \Pi_k^0 \rangle$ *is correct*;

Proof. Note that $T_k \Lambda = \Lambda_k$ for a homogeneous Λ, hence (b) here is (d) of Lemma 5.5. We already observed that (d) of Lemma 5.5 implies (a) for arbitrary Λ and P. For the converse, it is sufficient to prove that (a) implies (5.6). So assume that $\dim \Lambda_k > \dim P_k$ for some k. Then it follows that Λ_k contains some nontrivial λ which vanishes on P_k. By the homogeneity of P, it therefore vanishes on all of P, yet belongs to Λ by the homogeneity of Λ. Thus $\langle \Lambda, P \rangle$ is not correct.

For the equivalence of (b) and (c), note that the correctness of $\langle \Lambda_k, P_k \rangle$ is equivalent to the invertibility of the Gramian matrix $(\langle \lambda_i, p_j \rangle)_{i,j}$ for some (hence, any) bases $(\lambda_i)_i$ and $(p_j)_j$ for Λ_k and P_k, respectively. By taking, in particular, homogeneous bases, ordered by degree, such a Gramian becomes block-diagonal, hence invertible if and only if these diagonal blocks are invertible. □

(5.8) **Theorem.** *For any subspace* Λ *of* Π', $\Lambda_\downarrow \in \mathrm{IP}(\Lambda)$.

Proof. By Proposition 4.4, $P := \Lambda_\downarrow$ satisfies (5.6). Hence, by Lemma 5.5, it suffices to prove that $\Lambda_\perp \cap \Lambda_\downarrow = 0$. Let $p \in \Lambda_\perp \cap \Lambda_\downarrow$. Since $\Lambda_\downarrow$ is homogeneous, $p_\uparrow \in \Lambda_\downarrow$, hence there exists $\lambda \in \Lambda$ such that $\lambda_\downarrow = p_\uparrow$. By assumption $\langle \lambda, p \rangle = 0$, hence, by Lemma 4.6, $\langle p_\uparrow, p_\uparrow \rangle = \langle \lambda_\downarrow, p_\uparrow \rangle = 0$, which implies that $p = 0$. □

If

$$\dim T_k \Lambda < \dim P_k$$

for some k, then P_k contains some nontrivial $p \in (T_k \Lambda)_\perp$, and, since $p \in \Pi_k$, it follows (from (5.2)) that $p \in \Lambda_\perp$, therefore $p \in (\Lambda_\perp \cap P) \backslash 0$, showing that $\langle \Lambda, P \rangle$ is not correct in this case. Consequently, having

$$\dim T_k \Lambda \geqq \dim P_k \tag{5.9}$$

hold for every k is a necessary condition for the correctness of $\langle \Lambda, P \rangle$. Since $P = \Lambda_\downarrow$ is a solution (by Theorem 5.8) for which equality holds in (5.9) for all k (by Proposition 4.4), we conclude that $\Lambda_\downarrow$ is minimally correct for Λ in the sense of Definition 3.6.

(5.10) **Theorem.** *For every subspace* Λ *of* Π', $\Lambda_\downarrow$ *is a minimal-degree solution.*

Further, $P \in \mathrm{IP}(\Lambda)$ is in $\mathrm{MIP}(\Lambda)$ if and only if

$$\dim P_k = \dim (\Lambda_\downarrow)_k, \qquad \forall k. \tag{5.11}$$

We now show that all minimal-degree solutions can be characterized entirely in terms of $\Lambda_\downarrow$.

(5.12) **Theorem.** *Let* Λ *be a subspace of* Π', *and* P *be a subspace of* Π *that satisfies the minimal-degree conditions* (5.11). *Then the following conditions are equivalent*:

(a) $P \in \mathrm{IP}(\Lambda)$;
(b) $\Lambda_\perp \cap P = 0$;
(c) $\Lambda_{\downarrow\perp} \cap P = 0$.

Proof. The equivalence (a)⇔(b) was already established in Lemma 5.5.

(b)⇒(c): If $\Lambda_{\downarrow\perp} \cap P \neq 0$, then it would contain some p of degree $k \geqq 0$. Choose $(\lambda_i)_i \subset \Lambda$ such that $(\lambda_{i\downarrow})_i$ is a basis for $\Lambda_\downarrow$. By Lemma 5.5, (b) implies that $\langle T_{k-1}\Lambda, P_{k-1}\rangle$ is correct, while, by Lemma 4.5 (with Λ and $\Lambda_\downarrow$ replaced by $T_{k-1}\Lambda$ and $(\Lambda_\downarrow)_{k-1}$ respectively), $(T_{k-1}\lambda_i)_{\text{ord}\,\lambda_i<k}$ are linearly independent. Hence, we can find $q \in P_{k-1}$ such that

$$\langle \lambda_i, q\rangle = \langle T_{k-1}\lambda_i, q\rangle = \langle \lambda_i, p\rangle \qquad \forall \text{ ord}\,\lambda_i < k,$$

the first equality since $\deg q < k$. Further, if $\text{ord}\,\lambda_i \geqq k$, then $\langle \lambda_i, q\rangle = 0$ (since $\deg q < k$), while $\langle \lambda_i, p\rangle = (\lambda_{i\downarrow}, p\rangle = 0$ (since $\deg p = k$ and by choice of p, respectively), thus $\langle \lambda_i, q\rangle = \langle \lambda_i, p\rangle$ also in this case. We thus conclude that $p - q \in \bigcap_i \ker \lambda_i$, which implies, by Lemma 4.5, that $p - q \in \Lambda_\perp$. This contradicts assumption (b), since $\deg p > \deg q$, and therefore $p - q \in P \backslash 0$.

(c)⇒(b): This is proved analogously, but with λ_i and $\lambda_{i\downarrow}$ interchanged. In particular, Lemma 4.5 is not needed for this implication. □

We note for completeness that any of the possible four conditions of the form $M \cap Q = 0$, with M one of Λ or $\Lambda_\downarrow$, and Q one of P or $P_\uparrow$, is equivalent to the correctness of $\langle \Lambda, P\rangle$ under the minimal-degree conditions (5.11).

(5.13) **Corollary.** *Let Λ and P be subspaces of Π' and Π respectively, satisfying the minimal-degree conditions* (5.11). *Then* $\Lambda_\perp \cap P = 0 \Leftrightarrow \Lambda_{\downarrow\perp} \cap P = 0 \Leftrightarrow \Lambda_\perp \cap P_\uparrow = 0 \Leftrightarrow \Lambda_{\downarrow\perp} \cap P_\uparrow = 0$.

Proof. The first and last equivalence are special cases of Theorem 5.12. Further, $\Lambda_{\downarrow\perp} \cap P_\uparrow = 0$ implies $\Lambda_\perp \cap P = 0$ by Lemma 4.6. It is therefore sufficient to prove that $\Lambda_{\downarrow\perp} \cap P = 0$ implies that $\Lambda_{\downarrow\perp} \cap P_\uparrow = 0$, and this we do by an argument similar to that for the equivalence (b)⇔(c) of Theorem 5.12. For this, let $(\lambda_i)_i$ be a homogeneous basis for $\Lambda_\downarrow$ and let $p \in \Lambda_{\downarrow\perp} \cap P_\uparrow$. If $p \neq 0$, then, since $\Lambda_{\downarrow\perp} \cap P_\uparrow$ is homogeneous, we may assume without loss that p is homogeneous, hence that $p = r_\uparrow$ for some $r \in P$ with $\deg r =: k \geqq 0$. By Lemma 5.5, our assumption would then provide some $q \in P_{k-1}$ so that $\langle \lambda_i, q\rangle = \langle \lambda_i, r\rangle$ for all $\text{ord}\,\lambda_i < k$, while $\langle \lambda_i, q\rangle = 0 = \langle \lambda_i, r_\uparrow\rangle = \langle \lambda_i, r\rangle$ for all $\text{ord}\,\lambda_i \geqq k$. Consequently, $r - q$ would be a nontrivial element of $\Lambda_{\downarrow\perp} \cap P$. □

In the remainder of this section, we examine certain relations among the various elements of IP(Λ). In particular, we take advantage of the fact that a polynomial space Q is also a subspace of Π' to consider conditions under which $P \in \text{IP}(Q)$ for $P, Q \in \text{IP}(\Lambda)$. This also gives us an opportunity to examine the related question of whether the **algebraic** dual Q' of a polynomial space Q is representable by a polynomial space. Since Q' is much richer than its w^*-dual in case $\dim Q \nless \infty$, we actually cannot hope to represent such Q' by some $P \subset \Pi$. But, since the algebraic dual of a polynomial space is not as rich as the algebraic dual of an arbitrary subspace Λ if Π', we can hope that some subspace P of Π is w^*-densely imbedded into Q' by the map

(5.14) $$P \to Q' : p \mapsto p_{|Q}$$

which carries $p \in P$ to the linear functional $p_{|Q}$ on Q given by

(5.15) $$p_{|Q} : Q \to \mathbb{R} : q \mapsto \langle q, p\rangle.$$

If this is the case, then we say that P **is dual to** Q. Our results concerning polynomial interpolation readily yield conditions on P to be dual to a given Q. In addition, such considerations throw further light on the special role played by the least solution in the set of all minimal solutions and in the set of all homogeneous solutions.

(5.16) **Lemma.** *Let P and Q be subspaces of Π. If $P \in \mathrm{IP}(Q)$, then P is dual to Q.*

Proof. Since $P \in \mathrm{IP}(Q)$, we have $Q_\perp \cap P = 0$, hence the map $p \mapsto p_{|Q}$ is 1-1 on P. Further, to show that $P_{|Q}$ is w^*-dense in Q', observe that, since Q is polynomial, there exists, for any $\lambda \in Q'$, some $r_k \in \Pi_k$ so that $r_{k|Q} = \lambda$ on Q_k, $k = 1, 2, \ldots$, hence λ is the w^*-limit of $r_{k|Q}$ as $k \to \infty$. Since $P \in \mathrm{IP}(Q)$, there exists a corresponding sequence $(p_k)_k$ in P with $p_{k|Q} = r_{k|Q}$ for all k. □

The converse does not hold in general since the w^*-closure of $P_{|Q}$ may well contain polynomials not in P. For example, with P the linear span of the univariate polynomials $p_k := 1 + (\,)^k$, $k = 1, 2, \ldots$, and $Q = \Pi$, the linear functional δ_0 (represented by $p = 1$) is in the w^*-limit of $P \subset \Pi'$, hence so is all of Π, the latter being obviously dense in Π', and therefore P is dual to Π in the above sense. On the other hand, $\langle \Pi, P \rangle$ fails to be correct, since there is no $p \in P$ for which $\langle p, \cdot \rangle = \delta_0$ even though $\delta_0 \in \Pi'$.

In the next two results, we study in greater detail the above duality notion, as well as the interpolation problem $\mathrm{IP}(Q)$ for a polynomial Q.

(5.17) **Proposition.** *Let P and Q be polynomial spaces satisfying the conditions*

$$\dim T_k Q \leq \dim P_k, \qquad \forall k \in \mathbb{Z}_+. \tag{5.18}$$

Then the following conditions are equivalent:

(a) *$\langle Q, P \rangle$ is correct (i.e., $P \in \mathrm{IP}(Q)$);*
(b) *P is dual to Q;*
(c) *$\langle T_k Q, P_k \rangle$ is correct for every $k \in \mathbb{Z}_+$.*

Proof. The equivalence of (a) and (c) is obtained by substituting $\Lambda = Q$ in Lemma 5.5, and using the equivalence of (a) and (d) there. Also, assuming (b), we get $Q_\perp \cap P = 0$, and this implies (a) here because of the implication (b)$\Rightarrow$(a) in Lemma 5.5. Finally, the implication (a)$\Rightarrow$(b) holds even without the aid of (5.18), as is proved in Lemma 5.16. □

More can be said in case P and Q are homogeneous:

(5.19) **Corollary.** *Let P and Q be homogeneous subspaces of Π. Then conditions* (a), (b), *and* (c) *of Proposition 5.17 are equivalent. Furthermore, P is dual to Q if and only if Q is dual to P. Also, $P \in \mathrm{IP}(Q)$ if and only if $Q \in \mathrm{IP}(P)$.*

Proof. The equivalence of (a) and (c) was already established in Corollary 5.7. Further, (c) implies (5.18), hence implies (b), by Proposition 5.17. Thus, by the same proposition, it suffices to prove that (b) implies (5.18). For this, assume by way of contradiction that $\dim T_k Q > \dim P_k$ for some k. Then it follows that $T_k Q$ contains some nontrivial q perpendicular to P_k, hence to all of P, by the homogeneity of P. Further, this q is in Q by the homogeneity of Q. Since q is not zero, there exists $F \in Q'$ with $F q = 1$, and no such F can be in the w^*-closure of $P_{|Q}$, hence P cannot be dual to Q.

Finally, since Q is homogeneous, $T_k Q = Q_k$, and hence condition (c) of Proposition 5.17 is symmetric in P and Q, and we may change the roles of P and Q in this condition. Thus, from the equivalence of the three conditions in Proposition 5.17, we get the rest of the claim. □

We showed in Theorem 5.12 that $\Lambda_\downarrow$ can be used to single out MIP(Λ) in the collection of all polynomial spaces satisfying the minimal degree conditions (5.11). The next corollary shows that $\Lambda_\downarrow$ also singles out all *homogeneous* elements of MIP(Λ) among *all* polynomial spaces.

(5.20) **Corollary.** *Assume that P is a homogeneous subspace of Π and Λ is a subspace of Π'. Then the following conditions are equivalent*:

(a) $P \in \mathrm{MIP}(\Lambda)$;
(b) $P \in \mathrm{IP}(\Lambda_\downarrow)(\Leftrightarrow \Lambda_\downarrow \in \mathrm{IP}(P))$;
(c) *P is dual to $\Lambda_\downarrow$ ($\Leftrightarrow \Lambda_\downarrow$ is dual to P).*

Proof. The equivalence of (b) and (c) is obtained by substituting $Q = \Lambda_\downarrow$ in Corollary 5.19.

Assume (b). First, the implication (a)⇒(c) of Corollary 5.19 (with $Q := \Lambda_\downarrow$ and with $T_k Q = Q_k$ by the homogeneity of Q) shows that $\langle \Lambda_{\downarrow k}, P_k \rangle$ is correct for every k, in particular $\dim \Lambda_{\downarrow k} = \dim P_k$ for every k. Second, the assumption here guarantees that $\Lambda_{\downarrow\perp} \cap P = 0$. Employing the implication (c)⇒(a) in Theorem 5.12, we obtain that $P \in \mathrm{MIP}(\Lambda)$, which is (a) here.

Finally, assume (a). The implication (a)⇒(c) in Theorem 5.12 shows that $\Lambda_{\downarrow\perp} \cap P = 0$, but then the implication (b)⇒(a) there (with Λ replaced by $\Lambda_\downarrow$) shows that $P \in \mathrm{IP}(\Lambda_\downarrow)$, which is (b) here. □

The above corollary states that MIP(Λ) and IP($\Lambda_\downarrow$) contain the same homogeneous spaces. It should be clear that, for any homogeneous Q other than $\Lambda_\downarrow$, it is never true that MIP(Λ) and IP(Q) contain the same homogeneous spaces, since this would mean that IP($\Lambda_\downarrow$) and IP(Q) contain the same homogeneous spaces, and this is false, by Corollary 5.7: Indeed, Corollary 5.7 implies that, for a homogeneous Q, for any k and any algebraic complement C (in Π_k^0) of the orthogonal complement of $Q \cap \Pi_k^0$ (in Π_k^0), we obtain a homogeneous $P \in \mathrm{IP}(Q)$ by taking any homogeneous space in IP(Q) but replacing its kth homogeneous part by C. Thus, any algebraic complement of the orthogonal complement of $Q \cap \Pi_k^0$ occurs as $P \cap \Pi_k^0$ for some homogeneous $P \in \mathrm{IP}(Q)$. This shows that the homogeneous spaces in IP(Q) determine the orthogonal complement of $Q \cap \Pi_k^0$ (in Π_k^0), hence determine $Q \cap \Pi_k^0$ for every k, therefore determine Q.

6 The D-invariance case

In the case of the Lagrange interpolation problem IP(Θ), the linear functional space is the exponential space Exp_Θ, hence is always D-invariant. The D-invariance of the linear functional space is equivalent to $\Lambda_\perp$ being an ideal, and thus allows us to employ some elements of ideal theory for the analysis of $\Lambda_\perp$. This point is pursued in the present section.

We begin with some general remarks about D-invariant subspaces of Π'.

(6.1) **Proposition.** *Let Λ be a subspace of Π'. Consider the following:*

(a) *Λ is D-invariant;*
(b) *$\Lambda_\perp$ is an ideal (in Π).*

Then (a)$\Rightarrow$(b), *and, if Λ is closed, then* (b)$\Rightarrow$(a) *as well.*

Proof. For $\alpha\in\mathbb{Z}_+^s$, we consider the map

$$\chi^\alpha:\ \Pi\to\Pi:\ p\mapsto(\ )^\alpha p. \tag{6.2}$$

Since $\langle\lambda,(\ )^\alpha p\rangle=\langle D^\alpha\lambda,p\rangle$ for every $p\in\Pi$ and $\lambda\in\Pi'$, by (2.3), the map χ^α is the transpose of the map D^α: $\Pi'\to\Pi'$. This implies that Λ is an invariant subspace of D^α if and only if $\Lambda_\perp$ is an invariant subspace of χ^α (with the "if" implication making use of the fact that $\Lambda=\Lambda_\perp{}^\perp$, namely that Λ is closed). In particular, Λ is D-invariant, (i.e., invariant under all possible D^α) if and only if $\Lambda_\perp$ is invariant under all possible χ^α, i.e., is an ideal. □

In general the annihilator $\Lambda_\perp$ of a given linear functional space Λ is infinite-dimensional, hence a characterization of Λ in terms of its annihilator requires infinitely many conditions. The D-invariance assumption changes the situation: since $\Lambda_\perp$ is a polynomial ideal, it is finitely generated, say by $G\subset\Pi$. The finitely many polynomials in G characterize the (closure of the) original space Λ, if we regard them as *differential operators* rather than linear functionals. Precisely, for $G\subset\Pi$, defining

$$\ker G:=\{\lambda\in\Pi':g(D)\lambda=0,\ \forall g\in G\},$$

we have

(6.3) **Proposition.** *For a subset G of Π, let I_G be the ideal (in Π) generated by G. Then*

$$\ker G=I_G{}^\perp. \tag{6.4}$$

In addition, $p\in I_G$ if and only if the differential operator $p(D)$ vanishes on $\ker G$.

Proof. For $\lambda\in\Pi'$ and with $I_p:=p\Pi$,

$$\begin{aligned}
&p(D)\lambda=0\\
&\Leftrightarrow\langle p(D)\lambda,(\ )^\alpha\rangle=0,\qquad\forall\alpha\in\mathbb{Z}_+^s\\
&\Leftrightarrow\langle\lambda,(\ )^\alpha p\rangle=0,\qquad\forall\alpha\in\mathbb{Z}_+^s\\
&\Leftrightarrow\lambda\in I_p{}^\perp,
\end{aligned}\tag{6.5}$$

where the equivalence of the second and third statements is a consequence of (2.3). Thus (6.4) follows from the fact that $\lambda\in I_G{}^\perp$ if and only if $\lambda\in I_p{}^\perp$ for all $p\in G$.

The other statement follows from Corollary 3.28, since $\ker G$ is D-invariant. □

The linkage between kernels of differential operators and annihilators of linear functionals that was obtained in Proposition 6.3 allows us to convert some of the results of Sect. 4 to the present context.

The following is a rewrite of Theorem 4.11 in the language of this section.

(6.6) **Corollary.** *Let G be a polynomial set, and p a polynomial.*

(a) *If $p(D)((\ker G)_\downarrow)=0$, then $q(D)(\ker G)=0$, for some $q\in\Pi$ with $q_\uparrow=p_\uparrow$;*

(b) *If* $p(D)(\ker G)=0$, *then* $p_\uparrow(D)((\ker G)_\downarrow)=0$.

Next, substituting $P=I_G$ into Theorem 4.8 (and using (6.4)), the following corollary is obtained from Proposition 6.3.

(6.7) **Corollary.** *Let* I_G *be the ideal generated by the subset* G *of* Π. *Then*

$$(\ker G)_\downarrow = (I_G)_{\uparrow\perp}. \tag{6.8}$$

The above corollaries (which were first established in [BR2]) are useful tools in the analysis of certain interpolation problems, and, moreover, admit important applications in other areas of Approximation Theory (e.g., box splines). We first comment on the connection of Corollary 6.6 to polynomial interpolation.

Suppose that our original polynomial interpolation problem is reversed. Rather than having the linear functional space Λ as given, we hold a (D-invariant and, say, finite-dimensional) polynomial space P, and seek (say Lagrange) interpolation problems IP(Θ) whose least solution Π_Θ coincides with the given P. Since Π_Θ is always homogeneous, we must assume that so is P. Assume that, further, a collection F of polynomials for which $\ker F = P$ has been identified (the case might be that P is not known explicitly and is *a priori* defined as $\ker F$ for some $F\subset\Pi$). Since $P_\perp$ is homogeneous, we may assume without loss that all the polynomials in F are homogeneous (otherwise, each one of them can be replaced by its homogeneous components). Now, we perturb F in the following way: with each $h\in F$ we associate $g\in\Pi$ that satisfies $g_\uparrow=h$, thus obtaining a new set G of (possibly) non-homogeneous polynomials. By construction, $F\subset(I_G)_\uparrow$, hence also $I_F\subset(I_G)_\uparrow$, and hence

$$(\ker F=) \quad I_{F\perp}\supset(I_G)_{\uparrow\perp}.$$

Combining this with Proposition 6.3 and Theorem 4.8, we arrive at the following.

(6.9) **Corollary.** *Let* F *be a set of homogeneous polynomials, and let* $G\subset\Pi$ *be such that* $F\subset\{g_\uparrow : g\in G\}$. *Then*

$$\ker F\supset(\ker G)_\downarrow. \tag{6.10}$$

Since we are assuming that $P=\ker F$ is finite-dimensional, so is $(\ker G)_\downarrow$. Moreover, in order to get *equality* in (6.10), it suffices, in view of Proposition 4.4 (for the choice $\Lambda:=\ker G$), to show that $\dim\ker F\leqq\dim\ker G$.

If only F and G are known (i.e., if the original polynomial space P is known only implicitly, i.e., is *defined* as $\ker F$), it may be hard to estimate either dim $\ker F$ or dim $\ker G$. On the other hand, it might be easier to find (at least some of) the *exponentials* e_θ in $\ker G$. This is so, since $e_\theta\in\ker G$ if and only if θ is a common zero for the polynomials in G, (equivalently, the point θ lies in the (affine) algebraic variety of the ideal I_G). If G vanishes on some $\Theta\subset\mathbb{R}^s$, then each of the exponentials e_θ, $\theta\in\Theta$, lies in $\ker G$, and we get the simple estimate $\dim\ker G\geqq\#\Theta$. These observations lead to

(6.11) **Corollary.** *Let* F *be a homogeneous polynomial set, and* G *a polynomial set satisfying* $F\subset\{g_\uparrow : g\in G\}$. *Let* Θ *be a finite set of common zeros of* G. *Then*

(a) $\ker F\supset\Pi_\Theta$; *in particular* $\dim\ker F\geqq\#\Theta\,(=\dim\Pi_\Theta)$.

(b) *If* $\dim\ker F=\#\Theta$, *then*

(b1) $\ker G = \mathrm{Exp}_\Theta := \mathrm{span}\{e_\theta\}_{\theta\in\Theta}$;

(b2) $\ker F$ *is the least solution for the Lagrange interpolation problem* $\mathrm{IP}(\Theta)$, *i.e.*, $\ker F = \Pi_\Theta$.

The last corollary admits various applications. As a first setting, assume that a (finite-dimensional) polynomial space is defined as the joint kernel $\ker F$ of some homogeneous differential operators. The first part of Corollary 6.11 provides a way to obtain a lower bound for the dimension of $\ker F$ in terms of the cardinality of the variety of the ideal I_G. This results [BR2] in a painless derivation of the lower bound for the dimension of the space $\Pi(M)$ of all polynomials in the span of the integer translates of a box spline M. If G is chosen in such a way that also (b) is valid, one obtains a way to construct a basis for $\ker F$: if Θ is known and $\ker F = \Pi_\Theta$, then we only have to apply one of the algorithms [BR1, BR3] that compute Π_Θ from Θ. This leads [BR2] to an algorithmic way to construct a basis for the above-mentioned $\Pi(M)$, by an application of these "least map algorithms" to the (explicitly known) exponential space in the span of the integer translates of a suitably chosen exponential box spline.

We mention in passing that, in [BDR], Corollary 6.11 is exploited in a different way. The main result of [BDR] shows that a certain explicitly known polynomial space P (of significance in box spline theory) is $\ker F$ for very simple polynomials F (each of which is a power of a directional derivative). Perturbing the polynomials in F in a suitable way, we obtain there a polynomial set G whose common zero set Θ constitutes the integer points in the support of a box spline. It then follows from Corollary 6.11 that $P = \Pi_\Theta$. The various known properties of P (e.g., its homogeneous dimensions) provide in this way a better understanding of the interpolation problem $\mathrm{IP}(\Theta)$ (which was previously considered in [DM]), leading thereby to some optimality results for box splines.

When we want to adopt such an approach in general, we encounter at least two essential difficulties. In the first place, for the given D-invariant homogeneous space P, we need to find a set F of reasonably simple polynomials such that $\ker F = P$. Then, we need to find a way to obtain a perturbed set G with (at least) $\dim\ker F$ common zeros. Even then, there is no guarantee for the resulting interpolation problem to be of any interest.

7 Reduction to the Lagrange interpolation problem

Finding the space Π_Θ that solves the Lagrange interpolation problem associated with the finite Θ may appear to be very hard in general. Nevertheless, the results of the previous section exhibit the fact that certain tools and observations can be applied to facilitate the study of D-invariant interpolation problems, and this is particularly true for the Lagrange interpolation problem because of its explicit structure. It is therefore useful, especially for an interpolation problem $\mathrm{IP}(\Lambda)$ which is not D-invariant, to identify the space $\Lambda_\downarrow$ with a certain Π_Θ space, or one of its subspaces. We describe in this section a certain effort in this direction, and discuss some specific examples corresponding to this setting.

We start with the following simple fact:

(7.1) **Proposition.** *Assume that* $\mathrm{M}_\downarrow \cap \mathrm{N}_\downarrow = 0$ *for some subspaces* $\mathrm{M}, \mathrm{N} \subset \Pi'$. *Then* $\mathrm{M}+\mathrm{N}$ *is direct, and*

$$(M+N)_\downarrow = M_\downarrow \oplus N_\downarrow. \tag{7.2}$$

Proof. This is a consequence of Lemma 4.5, but here is a direct proof. If $\lambda \in \mathrm{M} \cap \mathrm{N}$, then $\lambda_\downarrow \in \mathrm{M}_\downarrow \cap \mathrm{N}_\downarrow$, hence $\lambda_\downarrow = 0$, hence also $\lambda = 0$, and the sum $\mathrm{M} + \mathrm{N}$ is indeed direct. Further, the sum $\mathrm{M}_\downarrow + \mathrm{N}_\downarrow$ is direct by assumption, and is included in $(\mathrm{M} + \mathrm{N})_\downarrow$, by the monotonicity of the least map (cf. (3.5)).

To prove the opposite inclusion, note that since $\mathrm{M}_\downarrow \cap \mathrm{N}_\downarrow = 0$, we must have

$$\operatorname{ord}(\mu + \nu) = \min\{\operatorname{ord} \mu, \operatorname{ord} \nu\}$$

for $\mu \in \mathrm{M}$ and $\nu \in \mathrm{N}$, since otherwise $\mu_\downarrow + \nu_\downarrow = 0$ and hence $\mu_\downarrow \in \mathrm{M}_\downarrow \cap \mathrm{N}_\downarrow$. It follows then that $(\mu + \nu)_\downarrow \in \{\mu_\downarrow, \nu_\downarrow, \mu_\downarrow + \nu_\downarrow\} \subset \mathrm{M}_\downarrow + \mathrm{N}_\downarrow$. □

Next, we discuss the following instructive example.

(7.3) *Example.* Let $s = 2$ and assume that Θ is a finite set in the right half plane. We use here (u, v) for the generic point in $\mathbb{R}^2$. We associate with each $\theta \in \Theta$ the line integral

$$\ell_\theta \colon p \mapsto \int_{-\theta_1}^{\theta_1} p(t, \theta_2)\, dt,$$

i.e., each integration segment is horizontal and symmetric across the v-axis. The corresponding generating function is then (up to a multiplicative constant) $\ell_\theta{}^\vee(u, v) = e^{\theta_2 v} \dfrac{\sinh(\theta_1 u)}{u}$. In view of Proposition 3.9, we may obtain $\Lambda_\downarrow$ in the form $\mathrm{M}_\downarrow / u$, with $\mathrm{M}^\vee$ the exponential space

$$\mathrm{M}^\vee := \operatorname{span}\{(u, v) \mapsto e^{\theta_2 v} \sinh(\theta_1 u)\}_{\theta \in \Theta}. \tag{7.4}$$

This space has dimension $\#\Theta$ and is a subspace of Exp_T, with $T := \Theta \cup \Theta'$, and Θ' being the image of Θ under reflection across the v-axis. Furthermore, the monomials appearing in the power expansion of each of the basis functions of $\mathrm{M}^\vee$ in (7.4) contain exclusively odd powers of u. On the other hand, defining N by

$$\mathrm{N}^\vee := \operatorname{span}\{(u, v) \mapsto e^{\theta_2 v} \cosh(\theta_1 u)\}_{\theta \in \Theta},$$

we get another subspace of Λ, and all the monomials appearing in the power expansion of any $\nu \in \mathrm{N}$ have only even powers of u. Hence $\mathrm{M}_\downarrow \cap \mathrm{N}_\downarrow = 0$. Since $\mathrm{M} + \mathrm{N} = \mathrm{Exp}_T$, we obtain from Proposition 7.1 that $\Pi_T = (\mathrm{Exp}_T)_\downarrow = \mathrm{M}_\downarrow \oplus \mathrm{N}_\downarrow$, which implies that $\mathrm{M}_\downarrow$ consists of *all* polynomials in Π_T which are odd in u. Application of Proposition 3.9 then yields the following:

$\Lambda_\downarrow$ is the subspace of Π_T/u consisting of all polynomials which are even functions in u.

Assume further that Θ here lies on the right unit semicircle. Then T lies on the unit circle, and Theorem 3.26 implies that Π_T consists of harmonic polynomials. Further, since $\#\mathrm{T}$ is even $(= 2\#\Theta =: 2n)$, Π_T contains all harmonic polynomials in Π_{n-1} and one homogeneous harmonic polynomial of degree n. The description of $\Lambda_\downarrow$ given in the previous paragraph thus implies that $\Lambda_\downarrow \cap \Pi_{n-2}$ is spanned by the polynomials

$$\frac{\operatorname{Im}(iu - v)^k}{u}, \qquad k = 1, 2, \ldots, n-1.$$

Since $\dim \Lambda_\downarrow = \# \Theta = n$, we must have an additional polynomial in the space, necessarily of degree $n-1$, namely the polynomial $\frac{\operatorname{Im}(iu-v)^n}{u}$. Since the space of all homogeneous polynomials of degree n in Π_T has dimension 1, it is necessarily spanned by $\operatorname{Im}(iu-v)^n$, regardless of the distribution of Θ. Note that we have obtained a complete description of Π_T for $T=\Theta \cup \Theta'$, and that Π_T depends on $\# \Theta$, but not on the distribution of the original Θ. □

In the rest of the section, we consider spaces Λ which are the composition of a single *univariate* power series with a collection of s-variate homogeneous polynomials. To avoid possible confusion between the aforementioned univariate power series and elements of $\Pi'(\mathbb{R}^s)$, we use the letter φ exclusively for the former. The setting is of interest, primarily since it includes every Lagrange interpolation problem IP(Θ); there the univariate power series φ is the exponential function

$$e\colon t \mapsto e^t,$$

and the homogeneous polynomials are the linear polynomials

$$x \mapsto \theta x, \qquad \theta \in \Theta.$$

For a power series λ, we use K_λ to denote its **support**, i.e.,

$$K_\lambda := \{\alpha \in \mathbb{Z}_+^s : \alpha(\lambda) \neq 0\}, \tag{7.5}$$

with $\alpha(\lambda)$ the αth coefficient of λ; cf. (2.1). Thus, $K_\varphi \subset \mathbb{Z}_+$, for any univariate φ. We assume that the linear functional space $\Lambda \subset \Pi'$ is of the form

$$\Lambda = \operatorname{span}\{\varphi \circ g : g \in G\}, \tag{7.6}$$

where φ is some univariate power series and $G \subset \Pi_k^0$ for some k.

The basic observation concerning the setting (7.6) is recorded in the following proposition.

(7.7) **Proposition.** *Assume that $\Lambda \subset \Pi'$ is of the form* (7.6). *Then the space $\Lambda_\downarrow$ depends only on G and K_φ, hence is independent of the specific (non-zero) values $\{\alpha(\varphi)\colon \alpha \in K_\varphi\}$.*

Proof. Each homogeneous polynomial in $\Lambda_\downarrow$ has the form $\lambda_\downarrow$ for some $\lambda := \sum_{g \in G} c_g \varphi \circ g$. Since the g's are all homogeneous and of the same degree, say k, each $\varphi \circ g$ is graded in the form

$$\varphi \circ g = \sum_{j \in K_\varphi} j(\varphi) g^j, \tag{7.8}$$

where g^j is homogeneous and of degree jk. This implies that the decomposition of λ into its homogeneous terms takes the form

$$\lambda = \sum_{j \in K_\varphi} j(\varphi) r_j,$$

with r_j being the homogeneous polynomial $\sum_{g\in G} c_g g^j$, hence is independent of φ. Since, up to the non-zero multiplicative constant $j(\varphi)$, $\lambda_\downarrow$ is the nonzero r_j of smallest $j\in K_\varphi$, our claim follows. □

In view of this proposition, we make the following definition:

(7.9) **Definition.** Let G be a finite set of homogeneous polynomials, all of the same degree, and let K be an arbitrary subset of $\mathbb{Z}_+$. We define

$$\Pi_{K,G}:=(\operatorname{span}\{\varphi\circ g: g\in G\})_\downarrow,$$

with $\varphi=\varphi_K$ some (any) univariate power series satisfying $K_\varphi=K$. In case $G=\{\theta\cdot\}_{\theta\in\Theta}$ for some $\Theta\subset\mathbb{R}^s$, we use

$$\Pi_{K,\Theta}$$

rather than $\Pi_{K,G}$.

The space $\Pi_{K,G}$ is well-defined by Proposition 7.7, and $\Pi_{\mathbb{Z}_+,\Theta}=\Pi_\Theta$. We record this in the following corollary:

(7.10) **Corollary.** *Let $\Theta\subset\mathbb{R}^s$ be a finite set, and φ a univariate power series that satisfies $K_\varphi=\mathbb{Z}_+$. Then, for $\Lambda:=\operatorname{span}\{\varphi(\theta\cdot)\}_{\theta\in\Theta}$, we have*

$$\Lambda_\downarrow=\Pi_\Theta. \tag{7.11}$$

The above corollary follows indeed from Proposition 7.7, since $K_e=\mathbb{Z}_+$ for the univariate exponential function e, and the functions $\{\theta\cdot\}_\theta$ are all homogeneous and linear.

The next result provides information about the case when K forms an arithmetic progression, i.e., the case when $K=k+n\mathbb{Z}_+$ for some non-negative integers k, n. In this theorem we make use of the polynomial space Π_Θ for a finite *complex* $\Theta\subset\mathbb{C}^s$, which is defined in the same way as in the real case (the only difference being that $\bar{\Pi}_\Theta$, rather than Π_Θ itself, solves IP(Θ)). Also, for a fixed positive integer n, we define on $\mathbb{C}^s$ the following equivalence relation

$$\theta\sim\vartheta\Leftrightarrow\theta=\xi\vartheta,\quad \text{for some } \xi\in\mathbb{C} \quad\text{with}\quad \xi^n=1.$$

We denote by $[\theta]$ the equivalence class containing θ, and by Θ' any subset of $\Theta\subset\mathbb{C}^s$ which contains exactly one representative from each equivalence class $[\theta]$, $\theta\in\Theta$.

(7.12) **Theorem.** *Let Θ be a finite subset of $\mathbb{C}^s$, n be a positive integer and $0\leq k<n$. Let ξ be a primitive nth root of unity (say $\xi=e^{2\pi i/n}$). Set $\mathrm{T}:=\bigcup_{j=1}^n \xi^j\Theta$ and $K:=K_k:=k+n\mathbb{Z}_+$. Then* (a) *$\Pi_{K,\Theta}=(G_k)_\downarrow$, where $G_k:=\operatorname{span}\{g_\theta\colon \theta\in\Theta\}$, with*

$$g_\theta:=g_{\theta,k}:=\sum_{j=1}^n \overline{\xi^{-kj}}\, e_{\xi^j\theta}. \tag{7.13}$$

(b) *If $0\notin\Theta$, then*

$$\dim\Pi_{K,\Theta}=\#\Theta'=\#\mathrm{T}/n. \tag{7.14}$$

In particular,

(b1) $\dim \Pi_{K,\Theta} = \#\Theta$ *if and only if the sets* $\{\xi^k \Theta\}_{k=1}^n$ *are pairwise disjoint*;

(b2) *for real* Θ, $\dim \Pi_{K,\Theta} = \#\Theta$ *if and only if either* n *is odd, or else* n *is even and* $\Theta \cap (-\Theta) = \emptyset$.

(c) $\Pi_{K,\Theta}$ *is spanned by all homogeneous polynomials in* Π_{T} *where degree is in* K.

Proof. Since ξ is primitive, $\{\xi^m\}_{m=1}^n$ are the n different characters of the group $\mathbb{Z}_n$, and hence, for every non-negative m,

$$\sum_{j=1}^{n} \overline{\xi^{-kj}}\, \xi^{mj} \neq 0 \Leftrightarrow k = m \bmod n. \tag{7.15}$$

Since each of the homogeneous terms in the power expansion of g_θ has the form

$$\frac{(\theta\cdot)^l}{l!} \sum_{j=1}^{n} \overline{\xi^{-kj}}\, \xi^{lj},$$

we conclude that

$$g_\theta = \sum_{m\in\mathbb{Z}_+} c(m)(\theta\cdot)^{k+nm},$$

for some θ-independent non-zero coefficients $c(m)$, and (a) follows from the definition of $\Pi_{K,\Theta}$.

(b) The fact that $\#\mathrm{T} = n\,\#\Theta'$ readily follows from the observation that $\theta\in\mathrm{T}$ iff $[\theta]\cap\Theta' \neq \emptyset$, which implies that $\mathrm{T} = \bigcup_{\theta\in\Theta'} [\theta]$. Since, with g_θ and G_k as above, $g_\theta \in \mathrm{Exp}_{[\theta]}$, we conclude that $\{g_\theta\}_{\theta\in\Theta'}$ are linearly independent. On the other hand, one checks that, for $\theta \sim \vartheta$, the functions g_θ and g_ϑ are dependent (regardless of the underlying k). Therefore, $\dim G_k = \#\Theta'$, and hence, by Proposition 4.4, also $\dim \Pi_{K,\Theta} = \#\Theta'$. This proves (7.14), which implies the rest of (b).

To prove (c), it suffices to show $\bigoplus_{k=0}^{n-1} \Pi_{K_k,\Theta} = \Pi_{\mathrm{T}}$. By (a), $G_{k\downarrow} = \Pi_{K_k,\Theta}$. Also, it is clear that $g_\theta \in \mathrm{Exp}_{\mathrm{T}}$ for every $\theta\in\Theta$, hence, $G_k \subset \mathrm{Exp}_{\mathrm{T}}$, and, by the monotonicity of the least map, $\Pi_{K,\Theta} \subset \Pi_{\mathrm{T}}$. On the other hand, the sum

$$\sum_{k=0}^{n-1} \Pi_{K_k,\Theta}$$

of subspaces of Π_{T} is direct, since each $\Pi_{K_k,\Theta}$ is spanned by homogeneous polynomials of degrees $\in K_k$, and the sets $K_0, \ldots, K_{n-1}$ are pairwise disjoint, and, consequently,

$$\bigoplus_{k=0}^{n-1} \Pi_{K_k,\Theta} \subset \Pi_{\mathrm{T}}. \tag{7.16}$$

If $0\notin\mathrm{T}$, then equality must hold in (7.16), since, by (b),

$$\dim \Pi_{\mathrm{T}} = \#\mathrm{T} = n\,\#\Theta' = \sum_{k=0}^{n-1} \dim \Pi_{K_k,\Theta}.$$

But this readily extends to the case when $0 \in \mathrm{T}$, since adding 0 to T adds constants to $\mathrm{Exp}_{\mathrm{T}}$, hence does not affect $\{G_k\}_{k=1}^{n-1}$, and increases dim G_0 by 1, hence also increases dim $\Pi_{K_0,\Theta}$ by 1. □

Remark. The observation, just made at the end of the proof of (c) of Theorem 7.12, implies that also the exclusion of 0 from Θ in part (b) of Theorem 7.12 was for convenience. Addition of 0 to the set Θ will increase $\dim \Pi_{K_0,\Theta}$ by 1, and will leave all other $\Pi_{K_k \Theta}$ unchanged.

The following example provides some illustration for the last result.

(7.17) *Example.* Let $\Theta := \{\pm\theta\}$ for some $\theta \in \mathbb{C}^s \backslash 0$ and let $n=2$, $k=1$. Then, by Theorem 7.12, $\dim \Pi_{2\mathbb{Z}+1,\Theta}=1$. Indeed, we find that the linear polynomial $(\theta\cdot)$ is in $\Pi_{2\mathbb{Z}+1,\Theta}$, yet no higher-degree polynomial is in this space, since a dependence relation $c_1(\theta\cdot)+c_2(-\theta\cdot)=0$ implies that $c_1(\theta\cdot)^j+c_2(-\theta\cdot)^j=0$ for every $j \in 2\mathbb{Z}+1$.

We also note the following interaction of the spaces $\Pi_{n\mathbb{Z}+k,\Theta}$ with differentiation:

(7.18) **Proposition.** *Let p be a homogeneous polynomial of degree m. Then, in the notations of Theorem* 7.12,

$$p(D)\,\Pi_{K_k,\Theta} \subset \Pi_{K_{(k-m)_n},\Theta},$$

where $j_n \in \{0, \ldots, n-1\}$ *is the residue of j* mod *n.*

Proof. It suffices to prove the result for $p=(\,)^\alpha$, $|\alpha|=m$. Let $g_{\theta,k}$ be as in (7.13). Then

$$D^\alpha g_{\theta,k} = \theta^\alpha \sum_{j=1}^{n} \overline{\xi^{-(k-m)j}}\, e_{\xi^j \theta} = \theta^\alpha\, g_{\theta,(k-m)_n} \in G_{(k-m)_n}.$$

Therefore, $D^\alpha G_k \subset G_{(k-m)_n}$, and thus combining (a) of Theorem 7.12 with (3.16) and (3.5), we obtain

$$D^\alpha \Pi_{K_k,\Theta} \subset (D^\alpha G_k)_\downarrow \subset (G_{(k-m)_n})_\downarrow = \Pi_{K_{(k-m)^n},\Theta}. \quad \square$$

We end this section with the following application of the above results.

(7.19) *Example.* Let $\{\ell_\theta\}_{\theta\in\Theta}$ be a finite set of line integrals of the form

$$\ell_\theta \colon p \mapsto \int_a^b p(\eta + t\theta)\, dt,$$

where $\theta \in \Theta \subset \mathbb{R}^s \backslash 0$, $\eta \in \mathbb{R}^s$, $a, b \in \mathbb{R}$, and η, a, b are θ-independent. In this case, the generating function associated with ℓ_θ has the form

$$\ell_\theta{}^\vee = e_\eta \frac{e_{b\theta} - e_{a\theta}}{(\theta\cdot)}.$$

Set

$$\Lambda := \operatorname{span} \{\ell_\theta\}_{\theta\in\Theta}.$$

With φ the univariate function

$$\varphi: t \mapsto \frac{e^{bt}-e^{at}}{t},$$

we observe that $\Lambda^{\vee} = e_\eta \operatorname{span}\{\varphi(\theta\cdot)\colon \theta\in\Theta\}$. From Proposition 3.9, we conclude that

$$\Lambda_{\downarrow} = (\operatorname{span}\{\varphi(\theta\cdot)\colon \theta\in\Theta\})_{\downarrow},$$

and thus $\Lambda_{\downarrow}$ is of the form $\Pi_{K_\varphi,\Theta}$. Since $K_\varphi = \mathbb{Z}_+$ unless $a=-b$ (we exclude the trivial case $a=b$), in which case $K_\varphi = 2\mathbb{Z}_+$, we thus conclude from Theorem 7.12 the following

(7.20) **Corollary.** *In the terms just introduced,*

$$\Lambda_{\downarrow} = \begin{cases} \Pi_\Theta, & \text{if } a \neq -b; \\ (\operatorname{span}\{\cosh(\theta\cdot)\colon \theta\in\Theta\})_{\downarrow}, & \text{if } a=-b. \end{cases}$$

The least space associated with the latter case consists of all even functions in $\Pi_{(-\Theta)\cup\Theta}$.

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