Math. Z. 215, 519–531 (1994)

Mathematische Zeitschrift

A priori estimates for higher order hyperbolic equations*

Mitsuru Sugimoto
College of General Education, Osaka University, Toyonaka, Osaka 560, Japan

Received 12 June 1992; in final form 26 February 1993

0 Introduction

We express the solution to the Cauchy problem of the wave equation

$$\begin{cases} \Box u = 0 \\ u_{|t=0} = g_0, \ u_{|t=0} = g_1 \end{cases}$$

as

$$u(t) = E_0(t)g_0 + E_1(t)g_1 \ .$$

Then the L^p-estimate (Peral [10]) and the L^p–L^q-estimate (Strichartz [14]) of the operators $E_j(t)$ $(j = 0, 1)$ are well known. These estimates are used to show the regularity properties of the solution (L^p-estimate) and to prove the existence of the global solution in case that the wave equation has such perturbations as semi-linear term or potential term (L^p–L^q-estimate).

The subject of this paper is to extend them to more general hyperbolic equations

(CP)
$$\begin{cases} Pu = 0 \\ D_t^j u_{|t=0} = g_j \quad (j = 0, 1, \ldots, m-1) \ . \end{cases}$$

Here the operator $P = P(D_t, D_x)$ is associated with a homogeneous polynomial $p(\tau, \xi) = (\tau - \varphi_1(\xi)) \ldots (\tau - \varphi_m(\xi))$ of order m $((\tau, \xi) \in \mathbf{R} \times \mathbf{R}^n)$, and the characteristic roots $\{\varphi_l\}_{l=1}^m$ are ordered as $\varphi_1(\xi) > \ldots > \varphi_m(\xi)$ $(\xi \neq 0)$. Then the solution

$$u(t) = \sum_{j=0}^{m-1} E_j(t) g_j$$

to the problem (CP) is of the form

(1)
$$E_j(t) = \sum_{l=1}^{m} F^{-1} e^{it\varphi_l(\xi)} a_{l,j}(\xi) F$$

* Dedicated to Professor Tosinobu Muramatu on his sixtieth birthday

($j = 0, 1, \ldots, m-1$), where the functions $a_{l,j}$ is homogeneous of order $-j$ and F (resp. F^{-1}) denotes the Fourier (resp. inverse Fourier) transform. Since we can see that the symbol $\sum_{l=1}^{m} e^{it\varphi_l(\xi)} a_{l,j}(\xi)$ has no singularity at the origin, our problem is reduced to

Question. When is the operator

$$M_k(D) = F^{-1} m_k(\xi) F; \quad m_k(\xi) = e^{i\varphi(\xi)} |\xi|^{-k} \chi(\xi)$$

L^p-bounded ($1 < p < \infty$) or L^p–$L^{p'}$-bounded ($1 < p \leqq 2$, $1/p + 1/p' = 1$)? Here the function $\varphi(\xi) \in C^\omega(\mathbf{R}^n \setminus 0)$ is homogeneous of order 1 and positive, and the function $\chi(\xi) \in C^\infty(\mathbf{R}^n)$ equals to 1 for large $|\xi|$ and vanishes near the origin.

We remark that the general L^p–L^q-boundedness ($1 < p \leqq q < \infty$) is obtained from the analytic interpolation of the L^p and L^p–$L^{p'}$-boundedness, and the L^p–L^q-boundedness ($1 < q < p < \infty$) cannot be obtained unless $M_k(D) = 0$ (Hörmander [6, Theorem 1.1]).

In this question, geometrical properties of the hypersurface

$$\Sigma = \{\xi \in \mathbf{R}^n; \varphi(\xi) = 1\}$$

are important since the singularity of the kernel of the operator $M_k(D)$ is related to them. There is an answer to the question above in a favorable case, due to Littman [8], Brenner [4] and Miyachi [9].

Theorem A. *We assume that the Gaussian curvature of the hypersurface Σ never vanishes. Then*

(a) *The operator $M_k(D)$ is L^p-bounded provided* $k \geqq (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$.

(b) *The operator $M_k(D)$ is L^p–$L^{p'}$-bounded provided* $k \geqq (n+1)\left(\frac{1}{p} - \frac{1}{2}\right)$.

These results are optimal in the sense that the operator $M_k(D)$ with $\varphi(\xi) = |\xi|$ is not L^p-bounded (resp. L^p–$L^{p'}$-bounded) provided $k < (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$ $\left(\text{resp. } k < (n+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right)$.

When we estimate the solution to (CP), the case $m = 2$ (e.g. wave equation) corresponds to the case that the assumption of Theorem A is satisfied (Remark 3). But, to higher order equations, we must remove the assumption on the hypersurface Σ. Recently, there are some papers dealing with this subject (Beals [2], Seeger et al. [12], Sugimoto [16]). Especially (a) in Theorem A has been proved without the assumption [12].

Then our next aim is to prove (b) similarly without the assumption, but we claim in this paper that it is impossible. This is the essential difference between the L^p-boundedness and the L^p–$L^{p'}$-boundedness.

Before going into details, we shall begin with a trivial result. If we combine the L^p–L^2 and L^2–$L^{p'}$-boundedness of the Riesz potentials (Hardy–Littlewood–Sobolev's theorem, see Stein [13]) with the L^2-boundedness of the operator $M_0(D)$ (Plancherel's theorem), we have (b′) The operator $M_k(D)$ is L^p–$L^{p'}$-bounded provided $k \geqq 2n\left(\frac{1}{p} - \frac{1}{2}\right)$.

There is a great gap between (b) and (b′), but the following result, due to Brenner [5], fill it to a certain degree.

Theorem B. *Let* $\rho = \min_{\xi \neq 0} \operatorname{rank} \varphi''(\xi)$. *Then the operator* $M_k(D)$ *is* L^p–$L^{p'}$-*bounded provided* $k \geqq (2n - \rho)\left(\frac{1}{p} - \frac{1}{2}\right)$.

We remark that we obtain the inequality $0 \leqq \rho \leqq n - 1$ from the homogeneity of the function $\varphi(\xi)$. The most favorable case $\rho = n - 1$, which is equivalent to the condition that the Gaussian curvature of the hypersurface Σ never vanishes, corresponds to (b) and the most unfavorable case $\rho = 0$ to (b′).

Theorem B, however, makes no contribution towards the boundedness of the operator $M_k(D)$ with

(2) $$\varphi(\xi) = (\xi_1^{2N} + \xi_2^{2N} + \cdots + \xi_n^{2N})^{1/(2N)}, \quad (N = 1, 2, \ldots)$$

since $\rho = 0$ (resp. $\rho = n - 1$) provided $N \geqq 2$ (resp. $N = 1$). Is this result optimal for this special case?

In Sect. 1, our main theorems (Theorems 1 and 2) say that a geometrical property of the hypersurface Σ has an essential effect on the L^p–$L^{p'}$-boundedness. As a special case, we have

Theorem C. *The operator* $M_k(D)$ *with* $\varphi(\xi)$ *as equality* (2) *is* L^p–$L^{p'}$-*bounded if and only if* $k \geqq \left(2n - \frac{n-1}{N}\right)\left(\frac{1}{p} - \frac{1}{2}\right)$.

This result suggests that Theorem B is not necessarily a good scale which interpolates the results (b) and (b′).

In Sect. 2, we shall show that the geometrical property stated in main theorems is derived only from the order of the operator P in (CP) under a convexity condition for characteristics (Theorem 3). Because of this fact, we can easily apply our theorems to the problem of higher order equations, and show a priori estimates for them (Theorem 4). They are extension of the results of Strichartz [15] which treats the wave equation.

1 L^p–$L^{p'}$-estimates

Let Σ be a hypersurface in $\mathbf{R}^n$, and let T be a tangent hyperplane at the point $p \in \Sigma$. Then for any plane H that contains the normal line of Σ at p, the line $T \cap H$ tangent to the curve $\Sigma \cap H$. We denote the order of this contact by $\gamma(\Sigma; p, H)$, and set

$$\gamma(\Sigma) = \sup_{p,H} \gamma(\Sigma; p, H).$$

For example, we have $\gamma(\Sigma) = 2N$ for the hypersurface $\Sigma = \{\xi \in \mathbf{R}^n; \varphi(\xi) = 1\}$ with $\varphi(\xi) = (\xi_1^{2N} + \xi_2^{2N} + \cdots + \xi_n^{2N})^{1/(2N)}$ $(N = 1, 2, \ldots)$.

Now, we shall state the boundedness of the operator $M_k(D)$, which is an answer to question in introduction.

Theorem 1 *We assume that the hypersurface $\Sigma = \{\xi \in \mathbf{R}^n; \varphi(\xi) = 1\}$ is convex. Then the operator $M_k(D)$ is L^p–$L^{p'}$-bounded $(1 < p \leqq 2,\ 1/p + 1/p' = 1)$ provided* $k \geqq \left(2n - \dfrac{2(n-1)}{\gamma(\Sigma)}\right)\left(\dfrac{1}{p} - \dfrac{1}{2}\right)$.

Remark 1 If the Gaussian curvature of the hypersurface Σ never vanishes, then Σ is convex (Cf. Kobayashi and Nomizu [7, Chap. 7]) and $\gamma(\Sigma) = 2$. Accordingly, Theorem 1 contains (b) in Theorem A as a special case.

Remark 2 The convexity, real analyticity, and the compactness imply that the hypersurface Σ is strictly convex, that is, every tangent plane of Σ never lies on Σ except for the tangent point. Then the order $\gamma(\Sigma)$ is finite and even.

This result is optimal in the following sense.

Theorem 2 *The operator $M_k(D)$ with $\varphi(\xi) = (\xi_1^{2N} + \xi_2^{2N} + \cdots + \xi_n^{2N})^{1/(2N)}$ $(N = 1, 2, \ldots)$ is not L^p–$L^{p'}$-bounded provided* $k < \left(2n - \dfrac{n-1}{N}\right)\left(\dfrac{1}{p} - \dfrac{1}{2}\right)$.

We shall prove Theorem 1. Since the case $n = 1$ is trivial, we may assume $n \geqq 2$. In the following, the capital "C" (with some suffices) in estimates always denotes a positive constant (depending on the suffices) which may be different in each occasion.

First of all, we introduce the Besov spaces $B^s_{p,q}$ defined by the norms

$$\| v \|_{B^s_{p,q}} = \left(\sum_{j=0}^{\infty} (2^{js} \| F^{-1} \Phi_j(\xi) F v \|_{L^p})^q \right)^{1/q}.$$

Here $\{\Phi_j\}_{j=1}^{\infty}$ is a partition of unity of Littlewood–Paley. For more information about these spaces, see, for example, Bergh and Löfström [3]. Then the following lemma is important, which is a special case of [3, Theorem 6.4.4].

Lemma 1 *We have the continuous inclusions $L^p \subset B^0_{p,2}$ $(1 < p \leqq 2)$ and $B^0_{p',2} \subset L^{p'}$ $(2 \leqq p' < \infty)$.*

By virtue of this lemma, our problem is reduced to the proof of the estimate

$$\| M_k(D) \Phi_j(D) u \|_{L^{p'}} \leqq C \| u \|_{L^p} \tag{1.1}$$

with $k = \left(2n - \dfrac{2(n-1)}{\gamma(\Sigma)}\right)\left(\dfrac{1}{p} - \dfrac{1}{2}\right)$. Here the constant C is independent of the numbers $j = 1, 2, \ldots$. If we write

$$\Phi_j(\xi) = \Phi_j(\xi) \Psi\left(\frac{\varphi(\xi)}{2^j}\right)$$

with a function $\Psi(t) \in C_0^\infty$ $(t > 0)$, we may prove estimate (1.1) with the operator $\Phi_j(D)$ replaced by $\Psi(\varphi(D)/2^j)$. Since estimate (1.1) with $p = 2$ is trivial by the Plancherel theorem, the estimate with $p = 1$ yields the general case by the analytic interpolation. Hence, all we have to show is the L^∞-estimate for the kernel of the operator $M_k(D)\Psi(\varphi(D)/2^j)$, that is, the estimate

$$\left\| F^{-1}\left[m_k(\xi)\Psi\left(\frac{\varphi(\xi)}{2^j}\right)\right](x)\right\|_{L^\infty} \leqq C \tag{1.2}$$

with $k = n - \dfrac{n-1}{\gamma(\Sigma)}$.

On the other hand, by the compactness of the sphere S^{n-1} and the rotation invariance of the geometrical properties, we may assume $m_k(\xi) = e^{i\varphi(\xi)}a_k(\xi)$ with a homogeneous function $a_k(\xi)$ of order $-k$ supported in a sufficiently small open conic neighbourhood of the point $e_n = (0, \ldots, 0, 1) \in S^{n-1}$. Then we have only to pay attention to x near the point $-\nabla\varphi(e_n) \in \mathbf{R}^n$, because the equality

$$F^{-1}\left[m_k(\xi)\Psi\left(\frac{\varphi(\xi)}{2^j}\right)\right](x)$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i\{x\cdot\xi+\varphi(\xi)\}}(L^*)^l\left(a_k(\xi)\Psi\left(\frac{\varphi(\xi)}{2^j}\right)\right)d\xi$$

holds for all positive integer l. Here

$$L = \frac{(x+\nabla\varphi)\cdot\nabla\xi}{i|x+\nabla\varphi|^2}$$

and L^* is the transpose of L.

Now, we may express the hypersurface Σ locally as

$$\Sigma = \{(y, h(y));\ y \in U\}$$

by the implicit function theorem since Euler's identity $\varphi(e_n) = e_n\cdot\nabla\varphi(e_n) > 0$ yields $\varphi'_{x_n}(e_n) = \varphi(e_n) > 0$. Here $U \subset \mathbf{R}^{n-1}$ is an open neighbourhood of the origin and $h\colon U \to \mathbf{R}$ is a real analytic function. The strictly convexity of the hypersurface Σ (Remark 2) implies that the function h is concave and the map $h'\colon U \to h'(U) \subset \mathbf{R}^{n-1}$ is homeomorphism.

In this situation, we shall rewrite estimate (1.2) in terms of the function h. For x near the point $-\nabla\varphi(e_n)$, we define $z \in U$ by $\Sigma \ni (z, h(z)) = v^{-1}(-x/|x|)$. Here the map

$$v\colon \Sigma \ni p \mapsto \frac{\nabla\varphi(p)}{|\nabla\varphi(p)|} \in S^{n-1}$$

is the Gauss map of the hypersurface Σ. If we write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1})$, it is equivalent to the equality

$$h'(z) = -\frac{x'}{x_n}$$

because of the trivial equality $-x/|x| = \nabla\varphi/|\nabla\varphi|(z, h(z))$ and of the fact that the vector $(-h'(z), 1)$ is normal to the hypersurface Σ at the point $(z, h(z))$. We remark that the variable x_n is away from 0 by Euler's identity again. Then, by the change of variables $\xi \mapsto (ty, th(y))$ and $t \mapsto x_n^{-1} t$ $(t > 0,\ y \in U)$, we have

$$F^{-1}\left[m_k(\xi)\Psi\left(\frac{\varphi(\xi)}{2^j}\right)\right](x)$$
$$= \frac{(x_n)^{k-n}}{(2\pi)^n}\int_0^\infty \int_U e^{it(x_n^{-1}+h(y)-h'(z)\cdot y)} t^{n-1-k}\Psi\left(\frac{t}{2^j x_n}\right) g(y)\,dt\,dy\,.$$

Here $g \in C_0^\infty(U)$ is a function which is supported in a sufficiently small neighbourhood of the origin. Hence, we have for x near the point $-\nabla\varphi(e_n)$

$$\left|F^{-1}\left[m_k(\xi)\Psi\left(\frac{\varphi(\xi)}{2^j}\right)\right](x)\right|$$
$$\leqq C\int_0^\infty \left|\left\{\int_U e^{it(x_n^{-1}+h(y)-h'(z)\cdot y)} g(y)\,dy\right\}\Psi\left(\frac{t}{2^j x_n}\right) t^{n-1-k}\right| dt$$
$$= C\int_0^\infty \left|I(t;z)\Psi\left(\frac{t}{2^j x_n}\right)t^{n-1-k}\right| dt$$
$$= C 2^{j(n-k)}\int_0^\infty \left|I(2^j t;z)\Psi\left(\frac{t}{x_n}\right)t^{n-1-k}\right| dt\,,$$

where

$$(1.3)\qquad I(t;z) = \int_{\mathbf{R}^{n-1}} e^{itE(y;z)} g(y)\,dy;\quad E(y;z) = h(y) - h(z) - h'(z)\cdot(y-z)\,.$$

If we combine this inequality with the estimate

$$(1.4)\qquad |I(t,z)| \leqq C t^{-\frac{n-1}{\gamma(\Sigma)}}\quad (t > 0,\ z \in U)\,,$$

we have estimate (1.2).

Now, we shall prove estimate (1.4). We rewrite equality (1.3) with the polar coordinates as

$$I(t;z) = \int_{S^{n-2}} G(t;z,\omega)\,d\omega;\quad G(t;z,\omega) = \int_0^\infty e^{itF(\rho;z,w)}\beta(\rho;z,\omega)\,d\rho\,,$$

where

$$F(\rho;z,\omega) = h(\rho\omega + z) - h(z) - \rho h'(z)\cdot\omega,\quad \beta(\rho;z,\omega) = g(\rho\omega + z)\rho^{n-2}\,.$$

For the sake of simplicity, we shall often abbreviate parameters z and ω. We split the function $G(t)$ into the following two parts:

$$G_1(t) = \int_0^\infty e^{itF(\rho)}\beta_1(\rho,t)\,d\rho;\quad \beta_1(\rho,t) = \beta(\rho)\psi(t^{\frac{1}{\gamma(\Sigma)}}\rho)\,,$$
$$G_2(t) = \int_0^\infty e^{itF(\rho)}\beta_2(\rho,t)\,d\rho;\quad \beta_2(\rho,t) = \beta(\rho)(1-\psi)(t^{\frac{1}{\gamma(\Sigma)}}\rho)\,,$$

where the function $\psi(\rho) \in C^\infty(\mathbf{R})$ equals to 1 for large ρ and vanishes near the origin. The estimate for the part $G_2(t)$ is easy. In fact, we have

$$\begin{aligned} |G_2(t)| &\leqq \int_0^\infty |\beta_2(\rho, t)|\, d\rho \\ &\leqq C \int_0^\infty |\rho^{n-2}(1-\psi)(t^{\frac{1}{\gamma(\Sigma)}}\rho)|\, d\rho \\ &\leqq C t^{-\frac{n-1}{\gamma(\Sigma)}} . \end{aligned}$$

On the other hand, integration by parts yields

$$G_1(t) = \int_0^\infty e^{itF(\rho)}(L^*)^l \beta_1(\rho, t)\, d\rho$$

for $l = 0, 1, 2, \ldots$. Here

$$L = \frac{1}{itF'(\rho)} \frac{\partial}{\partial \rho}$$

and L^* is the transpose of L. By induction, we can easily have

$$(L^*)^l = \left(\frac{i}{t}\right)^l \sum C_{r,p,s_1,\ldots,s_p} \frac{F^{(s_1)} \ldots F^{(s_p)}}{(F')^{l+p}}(\rho) \frac{\partial^r}{\partial \rho^r} ,$$

where the summation $\sum$ is a finite sum of r, p, s_1, $\ldots$, $s_p \geqq 0$ which satisfy $r + s_1 + \cdots + s_p = l + p$. Then we shall use

Lemma 2 *Let $\delta > 0$ be sufficiently small. Then there exist constants C, $C_m > 0$ such that the estimates*

$$|F'(\rho)| \geqq C\rho^{\gamma(\Sigma)-1},$$

$$|F^{(m)}(\rho)| \leqq C_m \rho^{1-m} |F'(\rho)|$$

hold for $0 \leqq \rho \leqq \delta$, $|z| \leqq \delta$, $\omega \in S^{n-2}$, and $m = 0, 1, 2 \ldots$.

Proof. The following proof is essentially due to Randol [11, Lemmas 4, 5]. First we note that the function $F(\rho)$ is real analytic for fixed z and ω. For the expansion $F(\rho) = \sum_{j=2}^\infty a_j(z, \omega)\rho^j$, we set

$$\pi(\rho) = \sum_{j=2}^{\gamma(\Sigma)} |\, j a_j(z, \omega)| \rho^{j-1} .$$

Since the definition of the order $\gamma(\Sigma)$ yields $\sum_{j=2}^{\gamma(\Sigma)} |a_j(z, \omega)| \neq 0$, we have the estimate

$$\pi(\rho) \geqq C\rho^{\gamma(\Sigma)-1} \tag{1.5}$$

for $0 \leqq \rho \leqq \delta$, $|z| \leqq \delta$ and $\omega \in S^{n-2}$. Here $\delta > 0$ is sufficiently small and the constant C is independent of ρ, z and ω. Accordingly all we have to show is the estimates

$$|F'(\rho)| \geqq C\pi(\rho) , \tag{1.6}$$

$$|F^{(m)}(\rho)| \leqq C_m \rho^{1-m} \pi(\rho) . \tag{1.7}$$

Now, we write $F^{(m)}(\rho) = \sum_{j=m}^{\infty} \frac{j!}{(j-m)!} a_j(z, \omega)\rho^{j-m}$. Then we can easily have

$$\left|\sum_{j=m}^{\gamma(\Sigma)} \frac{j!}{(j-m)!} a_j(z, \omega)\rho^{j-m}\right| \leqq C_m \rho^{1-m}\pi(\rho) .$$

As for the remainder term, we use Cauchy's estimate, that is

$$\begin{aligned}\left|\frac{j!}{(j-m)!} a_j(z, \omega)\right| &\leqq (2\delta)^{-(j-m)} \max_{|\zeta|=2\delta} |F^{(m)}(\zeta)| \\ &\leqq C_m(2\delta)^{-(j-m)} .\end{aligned}$$

Here the constant C_m is independent of z, ω and j. Then we have

$$\begin{aligned}\left|\sum_{j=\gamma(\Sigma)+1}^{\infty} \frac{j!}{(j-m)!} a_j(z, \omega)\rho^{j-m}\right| &\leqq C_m \sum_{j=\gamma(\Sigma)+1}^{\infty} \left(\frac{\rho}{2\delta}\right)^{j-m} \\ &\leqq C_m \rho^{\gamma(\Sigma)+1-m} \\ &\leqq C_m \rho^{2-m}\pi(\rho)\end{aligned} \tag{1.8}$$

for $0 \leqq \rho \leqq \delta$. Here we have used estimate (1.5). Combining these estimates, we have estimate (1.7). On the other hand, by the concavity of the function $h(y)$ and the equality $F'(0) = 0$, we can see that the function $|F'(\rho)|$ is non decreasing. Hence we have

$$\begin{aligned}|F'(\rho)| &= \max_{0 \leqq t \leqq \rho} |F'(t)| \\ &\geqq \max_{0 \leqq t \leqq \rho} \left|\sum_{j=2}^{\gamma(\Sigma)} j a_j(z, \omega) t^{j-1}\right| - \max_{0 \leqq t \leqq \rho} \left|\sum_{j=\gamma(\Sigma)+1}^{\infty} j a_j(z, \omega) t^{j-1}\right| \\ &\geqq \max_{0 \leqq t \leqq 1} \left|\sum_{j=2}^{\gamma(\Sigma)} j a_j(z, \omega)\rho^{j-1} t^{j-1}\right| - C_1 \max_{0 \leqq t \leqq \rho} |t\pi(t)| \\ &\geq (C - C_1\rho)\pi(\rho) .\end{aligned}$$

Here we have used the compatibility of norms $\max_{0 \leqq t \leqq 1} |\sum_{j=1}^{\gamma(\Sigma)} k_j t^{j-1}|$ and $\sum_{j=1}^{\gamma(\Sigma)} |k_j|$ on $\mathbf{C}^{\gamma(\Sigma)}$, and used estimate (1.8) with $m = 1$. Thus we have estimate (1.6) for sufficiently small ρ and finished the proof of Lemma 2.

If we use Lemma 2 and the estimate

$$\left|\frac{\partial^r \beta_1}{\partial \rho^r}(\rho, t)\right| \leqq C\rho^{h-2-r} ,$$

we have for large number l and a constant $c > 0$

$$\begin{aligned}|G_1(t)| &\leqq \frac{C}{t^l} \sum \int_0^{\infty} \left|\frac{F^{(s_1)} \ldots F^{(s_p)}}{(F')^{l+p}}(\rho) \frac{\partial^r \beta_1}{\partial \rho^r}(\rho, t)\right| d\rho \\ &\leqq \frac{C}{t^l} \int_{ct^{-\frac{1}{\gamma(\Sigma)}}}^{\infty} \rho^{n-2-l\gamma(\Sigma)} d\rho \\ &\leqq Ct^{-\frac{n-1}{\gamma(\Sigma)}} .\end{aligned}$$

Hence we have estimate (1.4) and finished the proof of Theorem 1.

In the next place, we shall prove Theorem 2. For positive functions $f \in C_0^\infty(\mathbf{R}^{n-1})$ and $\Phi \in C_0^\infty(\mathbf{R})$ such that $f(0), \Phi(1) \neq 0$, we set

$$u_j(x) = (2\pi)^n 2^{j\left(\frac{n-1}{2N} - n\right)\left(1 - \frac{1}{p}\right)} F^{-1}[f(2^{j\left(\frac{1}{2N} - 1\right)}\xi')\Phi(2^{-j}\xi_n)](x),$$

where $\xi = (\xi', \xi_n)$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$. Then we can easily see that the set $\{u_j\}_{j=0}^\infty$ is bounded in the space L^p. On the other hand, if supp f and supp Φ are sufficiently small and $k < \left(2n - \frac{n-1}{N}\right)\left(\frac{1}{p} - \frac{1}{2}\right)$, the set $\{M_k(D)u_j\}_{j=0}^\infty$ is not bounded in the space $L^{p'}$, that is, the estimate

$$\| M_k(D)u_j(x) \|_{L^{p'}} \geqq C2^{j\left\{\left(2n - \frac{n-1}{N}\right)\left(\frac{1}{p} - \frac{1}{2}\right) - k\right\}} \tag{1.9}$$

holds for a constant $C > 0$. In fact, we have for large numbers j

$$\begin{aligned} M_k(D)u_j(x) &= 2^{j\left(\frac{n-1}{2N} - n\right)\left(1 - \frac{1}{p}\right)} \int e^{i(x\cdot\xi + \varphi(\xi))} |\xi|^{-k} f(2^{j\left(\frac{1}{2N} - 1\right)}\xi')\Phi(2^{-j}\xi_n)\, d\xi \\ &= 2^{j\left\{\left(n - \frac{n-1}{2N}\right)\frac{1}{p} - k\right\}} \int e^{i\left(2^{j\left(1 - \frac{1}{2N}\right)}x'\cdot\xi' + 2^j x_n\cdot\xi_n + 2^j\varphi\left(2^{-\frac{j}{2N}}\xi', \xi_n\right)\right)} \\ &\quad \times \frac{f(\xi')\Phi(\xi_n)}{|(2^{-\frac{j}{2N}}\xi', \xi_n)|^k}\, d\xi \end{aligned}$$

$(x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}))$. Hence, we have

$$\begin{aligned} \| M_k(D)u_j(x) \|_{L^{p'}} &= 2^{j\left(\frac{n-1}{2N} - n\right)\frac{1}{p'}} \left\| M_k(D)u_j(2^{j\left(\frac{1}{2N} - 1\right)}x', 2^{-j}x_n) \right\|_{L^{p'}} \\ &= 2^{j\left\{\left(2n - \frac{n-1}{2N}\right)\left(\frac{1}{p} - \frac{1}{2}\right) - k\right\}} A_j, \end{aligned}$$

where

$$\begin{aligned} A_j &= \left\| \int e^{i(x\cdot\xi + 2^j\varphi(2^{-\frac{j}{2N}}\xi', \xi_n))} \frac{f(\xi')\Phi(\xi_n)}{|(2^{-\frac{j}{2N}}\xi', \xi_n)|^k}\, d\xi \right\|_{L^{p'}(\mathbf{R}_x^n)} \\ &\geqq \left\| \int \cos(x\cdot\xi + 2^j\varphi(2^{-\frac{j}{2N}}\xi', \xi_n)) \frac{f(\xi')\Phi(\xi_n)}{|(2^{-\frac{j}{2N}}\xi', \xi_n)|^k}\, d\xi \right\|_{L^{p'}(\mathbf{R}_x^n)} \\ &= \left\| \int \cos(x\cdot\xi + 2^j(\varphi(2^{-\frac{j}{2N}}\xi', \xi_n) - \xi_n)) \frac{f(\xi')\Phi(\xi_n)}{|(2^{-\frac{j}{2N}}\xi', \xi_n)|^k}\, d\xi \right\|_{L^{p'}(\mathbf{R}_x^n)}. \end{aligned}$$

If we notice the equality

$$\begin{aligned} \varphi(\xi) &= (\xi_1^{2N} + \cdots + \xi_{n-1}^{2N} + \xi_n^{2N})^{1/(2N)} \\ &= \xi_n + \frac{1}{2N}\xi_n^{1-2N}(\xi_1^{2N} + \cdots + \xi_{n-1}^{2N}) \\ &\quad + \frac{1}{2N}\left(\frac{1}{2N} - 1\right)\int_0^1 (1-\theta)(\theta(\xi_1^{2N} + \cdots + \xi_{n-1}^{2N}) + \xi_n^{2N})^{1/(2N)-2}\, d\theta \\ &\quad \times (\xi_1^{2N} + \cdots + \xi_{n-1}^{2N})^2, \end{aligned}$$

we have for x sufficiently close to the origin

$$\int \cos(x \cdot \xi + 2^j(\varphi(2^{-\frac{j}{2N}}\xi', \xi_n) - \xi_n)) \frac{f(\xi')\Phi(\xi_n)}{|(2^{-\frac{j}{2N}}\xi', \xi_n)|^k} d\xi$$

$$\geqq \frac{1}{2}\int \frac{f(\xi')\Phi(\xi_n)}{|\xi|^k} d\xi ,$$

which implies $\inf_j A_j \neq 0$. Then we have estimate (1.9) and finished the proof of Theorem 2.

2 Hyperbolic operators with convex characteristics

We shall apply the results given in the last section to the problem of higher order hyperbolic equations. In the rest of this paper, $P = P(D_t, D_x)$ denotes a homogeneous constant coefficient partial differential operator of degree m in $D_t, D_{x_1}, \ldots, D_{x_n}$ which is strictly hyperbolic, that is, the symbol $p(\tau, \xi)$ is factorized as

$$p(\tau, \xi) = (\tau - \varphi_1(\xi)) \ldots (\tau - \varphi_m(\xi)); \; \varphi_1(\xi) > \cdots > \varphi_m(\xi) \; (\xi \neq 0) .$$

We shall say that the operator P satisfies the convexity condition provided all the Hessians $\varphi_l''(\xi)$ $(l = 1, 2, \ldots, m)$ are semi-definite for $\xi \neq 0$. Then the following theorem is fundamental.

Theorem 3 *We assume that the operator P satisfies the convexity condition. Then there exists a polynomial $\alpha(\xi)$* of order 1 such that $\varphi_{m/2}(\xi) > \alpha(\xi) > \varphi_{m/2+1}(\xi)$ (if m is even) or $\alpha(\xi) = \varphi_{(m+1)/2}(\xi)$ *(if m is odd). Moreover, the hypersurfaces $\Sigma_l = \{\xi \in \mathbf{R}^n; \tilde{\varphi}_l(\xi) = \pm 1\}$ with $\tilde{\varphi}_l(\xi) = \varphi_l(\xi) - \alpha(\xi)$ $(l \neq (m+1)/2)$ are convex and* $\gamma(\Sigma_l) \leqq 2[m/2]$.

Remark 3 The hyperbolicity of the polynomial $p(\tau, \xi)$ implies that the Hessians $\varphi_1''(\xi)$ and $\varphi_m''(\xi)$ are always semi-definite $(\xi \neq 0)$. (See, Atiyah et al. [1, Corollary 3.23].) Especially in the case $m = 2$, we obtain $\gamma(\Sigma_1) = \gamma(\Sigma_2) = 2$ which implies that the Gaussian curvature of them never vanishes.

We shall prove Theorem 3. In order to prove the first half, we shall use the following lemmata.

Lemma 3 *Let Γ be an open convex cone, and let E be the edge of Γ, that is, $E = \{\eta; \Gamma + t\eta \subset \Gamma$ for all $t \in \mathbf{R}\}$. If $E = \{0\}$, then the set $\bar{\Gamma}\backslash 0$ is contained in an open halfspace.*

Proof. Let K be the dual cone of Γ, that is, $K = \{x; x \cdot \xi \geqq 0$ for all $\xi \in \Gamma\}$. Then $E = \{0\}$ implies that K has a non-empty interior. (See, Atiyah et al. [1, p. 124].) Hence, we have $\bar{\Gamma}\backslash 0 \subset \{\xi; a \cdot \xi > 0\}$ for some $a \in \mathbf{R}^n$.

Lemma 4 *We have $\varphi_l(\xi) = -\varphi_{m-l+1}(-\xi)$ for $l = 1, 2, \ldots, [(m+1)/2]$.*

Proof. We note $p(\tau, \xi) = (-1)^m p(-\tau, -\xi)$. In other words, for any l there exists some j such that $\varphi_l(\xi) = -\varphi_j(-\xi)$. We claim $\varphi_1(\xi) = -\varphi_m(-\xi)$. In fact, if assume $\varphi_1(\xi) = -\varphi_k(-\xi)$ for some $k \neq m$, we have $\varphi_m(\xi) = -\varphi_j(-\xi)$ for some $j \neq 1$. On the other hand, for $\xi \neq 0$ we have $\varphi_m(\xi) < \varphi_k(\xi) = -\varphi_1(-\xi)$ so that $\varphi_1(\xi) < -\varphi_m(-\xi) = \varphi_j(\xi)$, which contradicts the choice of φ_1. Then, by the same argument, we have successively $\varphi_l(\xi) = -\varphi_{m-l+1}(-\xi)$ for $l = 2, 3, \ldots, [(m+1)/2]$.

First we assume that m is even. Since $\varphi''_{m/2}(\xi)$ is semi-definite for $\xi \neq 0$, say positive semi-definite, the cone $\Gamma_{m/2} = \{(\tau, \xi); \tau > \varphi_{m/2}(\xi)\}$ is convex. On the other hand, by Lemma 4, the edge E of the cone $\Gamma_{m/2}$ is also the edge of the cone $\Gamma_{m/2+1} = \{(\tau, \xi); \tau < \varphi_{m/2+1}(\xi)\}$ so that $E = \{0\}$ by the inequality $\varphi_{m/2}(\xi) > \varphi_{m/2+1}(\xi)$ for $\xi \neq 0$. Then, by Lemma 3, there exists a polynomial $\alpha(\xi)$ of order 1 such that $\bar{\Gamma}_{m/2} \backslash 0 \subset \{(\tau, \xi); \tau > \alpha(\xi)\}$, in other words, $\varphi_{m/2}(\xi) > \alpha(\xi)$ for $\xi \neq 0$. By Lemma 4 again, we have $\varphi_{m/2}(\xi) > \alpha(\xi) > \varphi_{m/2+1}(\xi)$ for $\xi \neq 0$. In the case that $\varphi''_{m/2}(\xi)$ is negative semi-definite, we have similarly $\varphi_{m/2}(\xi) < \alpha(\xi) < \varphi_{m/2+1}(\xi)$, which contradicts the choice of $\varphi_{m/2}$ and $\varphi_{m/2+1}$.

Secondly we assume that m is odd. From Lemma 4, we obtain the equality $\varphi_{(m+1)/2}(\xi) = -\varphi_{(m+1)/2}(-\xi)$. Then we can see that the function $\varphi_{(m+1)/2}$ is convex and concave, therefore it is a polynomial of order 1.

In the next place, we shall prove the latter half of Theorem 3. We assume $\gamma(\Sigma_l; p, H) > m' = 2[m/2]$ for some l, p and H. After an appropriate rotation, we may express the hypersurface Σ_l locally as

$$\Sigma_l = \{(y, h(y)); y \in U \subset \mathbf{R}^{n-1}\},$$

where $p = (0, h(0))$. We set

$$F(\rho) = h(\rho\omega) - h(0) - \rho h'(0) \cdot \omega$$

for $\rho > 0$ and $\omega \in S^{n-2}$. Then we have $F(\rho) = o(\rho^{m'})$ for some $\omega \in S^{n-2}$. We remark that $F(\rho)$ does not identically equal to 0 by the compactness and real analyticity of Σ_l. If we set $\tilde{p}(\tau, \xi) = p(\tau + \alpha(\xi), \xi)$, we have $\tilde{p}(\pm 1, \xi) = 0$ with $\xi_n = h(\xi') = h(0) + \rho h'(0) \cdot \omega + F(\rho)$ $(\xi = (\xi', \xi_n), \xi' = (\xi_1, \ldots, \xi_{n-1}) = \rho\omega)$. Then we have the identity

$$F(\rho)^{m'} + b_1(\rho) F(\rho)^{m'-1} + b_2(\rho) F(\rho)^{m'-2} + \cdots + b_{m'}(\rho) \equiv 0.$$

Here $b_j(\rho)$ is a polynomial of order j at most. If $b_{m'} \equiv b_{m'-1} \equiv \cdots \equiv b_{m''+1} \equiv 0$ and $b_{m''} \not\equiv 0$, it is reduced to

$$F(\rho)(F(\rho)^{m''-1} + b_1(\rho) F(\rho)^{m''-2} + \cdots + b_{m''-1}(\rho)) \equiv -b_{m''}(\rho).$$

The left hand side of this equality is $o(\rho^{m'})$ while the right hand side is a polynomial of order $m''(\leqq m')$ at most. This is a contradiction, and we have Theorem 3.

Now, we shall consider the Cauchy problem

$$\text{(CP}'\text{)} \qquad \begin{cases} Pu = f \\ D_t^j u_{|t=0} = 0 \quad (j = 0, 1, \ldots, m-1). \end{cases}$$

The solution to it is expressed as

$$u(t) = \int_0^t E_{m-1}(t - \tau) f(\tau)\, d\tau, \tag{2.1}$$

where the operator $E_{m-1}(t)$ is given by equality (1) in introduction. If we combine Theorem 1 with Theorem 3 and notice the equality

$$[E_{m-1}(t) g](x) = t^{m-1} [E_{m-1}(1)(g(t \cdot))](t^{-1} x),$$

L^p–$L^{p'}$-estimates for the operator $E_{m-1}(t)$ are reduced to those for the operator $M_k(D)$ with $k = m - 1$ and $\gamma(\Sigma) = 2[m/2]$. Here we have used the fact that the operator $F^{-1}e^{i\alpha(\xi)}F$ is nothing but a translation. Accordingly, we have easily the estimate

$$\| E_{m-1}(t)g \|_{H^s_{p'}} \leqq Ct^{m-1-2n(\frac{1}{p}-\frac{1}{2})} \| g \|_{H^s_p} \tag{2.2}$$

if $m - 1 = \left(2n - \dfrac{n-1}{[m/2]}\right)\left(\dfrac{1}{p} - \dfrac{1}{2}\right)$.

From this estimate, we can obtain a priori estimates for the problem (CP′). For example, equality (2.1) and estimate (2.2) yields the estimate

$$\| u(t) \|_{H^s_{p'}(\mathbf{R}^n_x)} \leqq C\int_0^t (t-\tau)^{m-1-2n(\frac{1}{p}-\frac{1}{2})} \| f(\tau) \|_{H^s_p} d\tau$$

$$\leqq C|t|^{m-1-2n(\frac{1}{p}-\frac{1}{2})} * \| f(t) \|_{H^s_p} .$$

On the other hand, the boundedness of the Riesz potential of dimension 1 says that convolutions with $|t|^{\alpha-1}$ are $L^q(\mathbf{R})$–$L^{q'}(\mathbf{R})$-bounded ($1 < q < 2, 1/q + 1/q' = 1$) provided $\alpha = 2\left(\dfrac{1}{q} - \dfrac{1}{2}\right)$. (See Stein [13, Chap. 5, Theorem 1].) If we use it with $\alpha = m - 2n\left(\dfrac{1}{p} - \dfrac{1}{2}\right)$, we have

Theorem 4 *Let indices* $1 < p, q < 2$ *be as* $m - 1 = \left(2n - \dfrac{n-1}{[m/2]}\right)\left(\dfrac{1}{p} - \dfrac{1}{2}\right)$, $2\left(\dfrac{1}{q} - \dfrac{1}{2}\right) = m - 2n\left(\dfrac{1}{p} - \dfrac{1}{2}\right)$, *and let* $1/p + 1/p' = 1/q + 1/q' = 1$, $s \in \mathbf{R}$. *We assume that the operator P satisfies the convexity condition. Then there exists a constant C such that the solution u to problem* (CP′) *satisfies the estimate*

$$\| u \|_{L^{q'}(\mathbf{R}_t;\, H^s_{p'}(\mathbf{R}^n_x))} \leqq C \| f \|_{L^q(\mathbf{R}_t;\, H^s_p(\mathbf{R}^n_x))}$$

for any given data f.

Remark 4 Theorem 4 with the case of the wave equation is given by Strichartz [15, Theorem 1].

Remark 5 Assumptions for indices in Theorem 4 implies $m < n + 1$.

It is a routine work to prove existence and uniqueness for semi-linear equations by the method of iteration using a priori estimates. For example, we shall consider the problem

$$\text{(CP}''\text{)} \qquad \begin{cases} Pu = H(u) \\ D_t^j u_{|t=0} = g_j \quad (j = 0, 1, \ldots, m-1), \end{cases}$$

where $H(\cdot) = H(\cdot, t, x)$ is a scalar function.

Corollary 1 *Let indices p, q, p′, q′, s be the same as in Theorem* 4, *and let the following assumptions be satisfied*:

[I] (*convexity*) *The operator P satisfies the convexity condition.*

[II] (*non-linearity*) *For any* $u \in L^{q'}(\mathbf{R}_t; H^s_{p'}(\mathbf{R}^n_x))$, *the non-linear term satisfies* $H(u) \in L^q(\mathbf{R}_t; H^s_p(\mathbf{R}^n_x))$. *Moreover, for any* $\varepsilon > 0$, *there exists a decomposition* $-\infty = t_0 < t_1 < \cdots < t_k = \infty$ *such that the estimates*

$$\|H(u) - H(v)\|_{L^q(I_j;\, H^s_p(\mathbf{R}^n_x))} \leqq \varepsilon \|u - v\|_{L^{q'}(I_j;\, H^s_{p'}(\mathbf{R}^n_x))}$$

holds for the intervals $I_j = (t_j\, t_{j+1})$ $(j = 0, 1, \ldots, k-1)$.

[III] (*regularity*) *The solution* $\sum_{j=0}^{m-1} E_j(t) g_j$ *of the associated linear problem* (CP) *in introduction is in the space* $L^{q'}(\mathbf{R}_t; H^s_{p'}(\mathbf{R}^n_x))$.

Then the problem (CP″) *has a unique solution in* $L^{q'}(\mathbf{R}_t; H^s_{p'}(\mathbf{R}^n_x))$.

The proof of this result is carried out in the same way as that of Theorem 2 in Strichartz [14] which treats the wave equation. Hence, we shall omit it.

References

1. Atiyah, M.F., Bott, R., Gårding, L.: Lacunas for hyperbolic differential operators with constant coefficients I. Acta Math. **124**, 109–189 (1970)
2. Beals, M.: L^p boundedness of Fourier integral operators. Mem. Am. Math. Soc. **264** (1982)
3. Bergh, J., Löfström, J.: Interpolation Spaces. Berlin Heidelberg New York: Springer 1976
4. Brenner, P.: On L_p–$L_{p'}$ estimates for the wave-equation. Math. Z. **145**, 251–254 (1975)
5. Brenner, P.: L_p–$L_{p'}$-estimates for Fourier integral operators related to hyperbolic equations. Math. Z. **152**, 273–286 (1977)
6. Hörmander, L.: Translation invariant operators in L^p spaces. Acta Math. **104**, 93–140 (1960)
7. Kobayashi, S., Nomizu, K.: Foundation of Differential Geometry, vol. II. New York: Interscience 1969
8. Littman, W.: L^p–L^q-estimates for singular integral operators arising from hyperbolic equations. In: Spencer, D.C. (ed.) Partial Differential Equations. (Proc. Symp. Pure Math., vol. 23, pp. 479–481) Providence, R.I.: Am. Math. Soc. 1973
9. Miyachi, A.: On some estimates for the wave equation in L^p and H^p. J. Fac. Sci., Univ. Tokyo, Sect. IA **27**, 331–354 (1980).
10. Peral, J.: L^p estimates for the wave equation. J. Funct. Anal. **36**, 114–145 (1980)
11. Randol, B.: On the asymptotic behavior of the Fourier transform of the indicator function of a convex set. Trans. Am. Math. Soc. **139**, 279–285 (1969)
12. Seeger, A., Sogge, C.D., Stein, E.M.: Regularity properties of Fourier integral operators. Ann. Math., II. Ser. **134**, 231–251 (1991)
13. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton: Princeton University Press 1970
14. Strichartz, R.: Convolutions with kernels having singularities on a sphere. Trans. Am. Math. Soc. **148**, 461–471 (1970)
15. Strichartz, R.: A priori estimates for the wave equation and some applications. J. Funct. Anal. **5**, 218–235 (1970)
16. Sugimoto, M.: On some L^p-estimates for hyperbolic equations. Ark. Mat. **30**, 149–163 (1992)