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# Symplectic homology I

## Open Sets in $\mathbb{C}^n$

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* Andreas Floer died on May 15, 1991. This is the one of several joint papers which were in preparation and are concerned with a symplectic homology theory

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## 1 Introduction and outline of the main results

### *1.1 Some general remarks*

In recent years, much progress has been achieved in symplectic geometry and the variational theory of Hamiltonian dynamics. The variational existence theory for periodic solutions of Hamiltonian systems initiated by P. Rabinowitz, [35, 36], and Gromov's theory of pseudoholomorphic curves in symplectic geometry, [26, 27], are corner stones for this development.

A few years after Rabinowitz's seminal work, Conley and Zehnder, [7], observed that the variational methods can be successfully used in studying symplectic fixed point problems. This lead them to the solution of one of the Arnold conjectures, [3, 4].

Motivated by an influential paper by Witten, [45], Floer was able to merge the variational and Gromov's elliptic theory, which lead to the so-called *Floer homology* for the Lagrangian intersection problem, [15, 16, 17, 19, 20], see also [18, 28], for the corresponding Ljusternik–Schnirelmann theory, and [34, 38] for a survey.

In their study of periodic solutions of Hamiltonian systems with prescribed energy, [9], Ekeland and Hofer introduced an interesting, very rigid invariant for a convex energy surface. In [10, 11], motivated by [42], and [32], it was observed that Hamiltonian dynamics can be effectively used in studying symplectic rigidity phenomena. In [11], for example, infinitely many new symplectic invariants, so-called symplectic capacities have been constructed using the variational theory, see also [14, 29, 31, 44].

The aim of our series of papers on symplectic homology is concerned with combining Floer homology and Capacity theory. This will lead to a variety of new symplectic invariants, and interesting applications.

In the present paper SH I we construct a theory for open bounded subsets of $\mathbb{C}^n$. This construction already exhibits the key points of any more general theory. The present theory could presumably also be constructed using generating function type techniques as in [44]. However, such techniques can in general not be carried through on more general manifolds (at least not at the moment). The second paper SH II, [22], extends the theory to more general symplectic manifolds, which turns out not to be very difficult. Jointly with K. Wysocki we give applications of SH I, II in [24, 25]. In [24] we compute the symplectic homology of simple shapes and prove Gromov's conjecture concerning the classification of symplectic open polydisks. Moreover, we make some statements about the space of symplectic embeddings from one polydisk into another. In [25], we apply the more general results to show the invariance of the action spectrum of suitable symplectic manifolds with a contact type boundary. This particular application belongs to a circle of ideas concerned with the question what does the interior of a symplectic manifold know about its boundary. A particular striking phenomenon is the Benci–Sikorav rigidity for sets of the form $T^n \times U$ in $T^*(T^n)$ , [5], [41]. For this type of problem see for negative as well as positive results, [12, 13, 14].

*1.2 Periodic hamiltonian trajectories in a symplectic rigidity theory*

Symplectic homology is a device to detect and measure symplectic rigidity. An important ingredient is the study of periodic orbits of Hamiltonian systems and closed characteristics on Hamiltonian energy surfaces. In this paper we shall deal only with the $\mathbb{C}^n$-case and refer the reader to SH II for the general case.

Let us view $\mathbb{C}^n$ as a $2n$-dimensional $\mathbb{R}$-vectorspace. We denote the usual complex Hermitian inner product by $(*, *)$ . Associated to $(*, *)$ we have the real inner product $\langle *, * \rangle = \mathrm{Re}(*, *)$ and the symplectic form $\omega = \mathrm{Im}(*, *)$ .

A smooth real hypersurface $S$ in $\mathbb{C}^n$ carries an important structure induced by the symplectic form $\omega$. Namely define a one-dimensional distribution $\mathscr{L}_S \to S$ by

$$\mathscr{L}_S = \{(x, \xi) \in TS \mid \omega(\xi, \eta) = 0 \quad \text{for all } \eta \in T_x S\} . \tag{1}$$

The integral curve through $x \in S$ will be denoted by $L_S(x)$ . If $S$ is compact, the closed integral curves $L_S(x) \simeq S^1$ are of particular interest. This collection of closed integral curves, also called periodic Hamiltonian trajectories, will be denoted by $\mathscr{P}(S)$. An element in $\mathscr{P}(S)$, say $P$, carries a numerical value $A(P) \in \mathbb{R}$ defined as follows. First of all, we may assume that $S$ is connected. Then $\mathbb{C}^n \backslash S$ (assuming $S$ to be compact) has in view of Alexander duality a unique bounded component $B_S$. We take a smooth map $H$: $\mathbb{C}^n \to \mathbb{R}$ such that

$$S = H^{-1}(0), \quad dH(x) \neq 0 \quad \text{for } x \in S$$

$$\inf_{x \in B_S} H(x) < 0 .$$

Then the Hamiltonian vectorfield $X_H$ defined by

$$i_{X_H}\omega = dH$$

defines a nowhere vanishing section of $\mathscr{L}_S \to S$ and hence an orientation. Having this orientation in mind we have a canonical orientation of $P \in \mathscr{P}(S)$ in view of $TP = \mathscr{L}_S|_P$. We define

$$A(P) = \int \lambda|_P$$

for $P \in \mathscr{P}(S)$, where $\lambda$ is any 1-form on $\mathbb{C}^n$ satisfying $d\lambda = \omega$.

Given a symplectic diffeomorphism $\Psi$: $\mathbb{C}^n \xrightarrow{\sim} \mathbb{C}^n$ we have the following rules

$$A(P) = A(\Psi(P)) \tag{2}$$

$$(T\Psi)\mathscr{L}_S = \mathscr{L}_{\Psi(S)}$$

$$\mathscr{P}(\Psi(S)) = \Psi(\mathscr{P}(S)) .$$

The crucial fact, which we will explore in this and the following papers, is that periodic trajectories occur naturally as obstructions in a symplectic rigidity theory. The following heuristic considerations clarify this statement.

Let us start with Gromov's celebrated *Squeezing theorem*, [26, 27]. Relying strongly on his theory of pseudoholomorphic curves, Gromov showed that the $r$-ball $B^{2n}(r)$ can be symplectically embedded into the $R$-cylinder $Z^{2n}(R) := B^2(R) \times \mathbb{C}^{n-1}$ if and only if $R \geqq r$. In [10] this result is proved using the variational study of periodic orbits of Hamiltonian systems. The occurrence of periodic solution seems to be absolutely unexpected in particular in view of the original proof of the squeezing theorem. In order to shed some light on this fact, let us assume we have an optimal embedding of some open bounded set $U$ into $Z^{2n}(R)$. Optimal here means that there is no such symplectic embedding for $R' < R$. Without loss of generality we may assume $U \subset Z^{2n}(R)$. In addition let us also assume that $U$ is bounded with a smooth boundary $\partial U$.

Next we try the impossible: *Squeezing U into some smaller cylinder*. Symplectic isotopies are generated by (time-dependent) Hamiltonians. Locally, the optimal way seems to be the following: One tries to push points in $\partial U \cap \partial Z^{2n}(R)$ into $Z^{2n}(R)$ by taking a Hamiltonian vectorfield which points inside $Z^{2n}(R)$ at points in $\partial U \cap \partial Z^{2n}(R)$ and which is allowed to point outside of $Z^{2n}(R)$ at points of $\partial Z^{2n}(R)$ which stay away from $\bar{U}$. (However the vectorfield should be sufficiently small there.) A Hamiltonian $H$ achieving all that will increase along the integral curves of $\mathscr{L}_{\partial U}$ on the *leaves* in $\partial U \cap \partial Z^{2n}(R)$ and perhaps will be decreasing on the parts in $\partial Z^{2n}(R)$ staying away from $\bar{U}$. Obviously the only local obstruction is a common close characteristic contained in $\partial U \cap \partial Z^{2n}(R)$, i.e. $\mathscr{P}(\partial U) \cap \mathscr{P}(\partial Z^{2n}(R))$ contains a closed characteristic $P_0$ satisfying $P_0 \subset \partial U \cap \partial Z^{2n}(R)$. In fact in the case that such a $P_0$ exists the Hamiltonian $H$ has to be strictly increasing along $P_0$ which is absurd. Assuming $\mathscr{P}(\partial U) \cap \mathscr{P}(\partial Z^{2n}(R)) = \emptyset$ we can construct a Hamiltonian close to every section $\mathbb{C} \times \{a\}$, $a \in \mathbb{C}^{n-1}$. Using a partition of unity in the section parameter $a \in \mathbb{C}^{n-1}$ only, one can globalize this construction (at least for nice sets). Hence if $\mathscr{P}(\partial U) \cap \mathscr{P}(\partial Z^{2n}(R)) = \emptyset$ we can construct a Hamiltonian $H$ such that the associated time-1-map $\Psi$ satisfies $\Psi(\bar{U}) \subset Z^{2n}(R)$ implying that $U$ was not *optimally squeezed* contrary to our assumption. So our element in $\mathscr{P}(\partial U)$ turns out (in our simple minded heuristics) to be an obstruction for the squeezing problem (under favourable assumptions). Surprisingly it will turn out that there is real mathematics behind this consideration!

To make this more precise we recall the well-known fact that closed characteristics on hypersurfaces and periodic solutions of Hamiltonian systems with prescribed energy are closely related (the dual character of time and energy in Hamiltonian mechanics).

The crucial phenomenon for our construction is the following fact exhibited in [32]: Given some open bounded set $A$ of $\mathbb{C}^n$ and a Hamiltonian $H: \mathbb{C}^n \to [0, +\infty)$, which vanishes on $A$, but grows sufficiently fast outside of $A$, the associated Hamiltonian system $\dot{x} = X_H(x)$, will have many 1-periodic solutions (nontrivial ones) geometrically close to $\partial A$. Here "close" depends on the growth rate of $H$ outside of $A$. These 1-periodic solutions can be found as critical points of a functional on the loop space of $\mathbb{C}^n$. This is the so-called principle of the least action. It is a difficult variational principle, due to the fact that it is indefinite in the sense that the Morse indices of the critical points are always infinite.

This difficulty is handled by Floer's elliptic Morse theory, see [26, 27] for more information. The elliptic Morse theory combines Gromov's theory of pseudo-holomorphic curves [26, 27] with Conley's idea of connection matrices for flows. Our main construction merges the theory of symplectic capacities, [10, 11, 32, 43], with elliptic Morse theory.

Before we go into more details we give an outline of symplectic homology in the next sections.

### *1.3 The Conley–Zehnder index*

We denote by Sp the group of linear symplectic maps in $(\mathbb{C}^n, \omega)$. $\mathrm{Sp}^*$ is the subset of Sp consisting of all symplectic maps which do not have 1 in their spectrum. Let $\pi_1 = \pi_1(\mathrm{Sp}, \{\mathrm{Id}\})$ be the fundamental group. It is well known that $\pi_1 \simeq \mathbb{Z}$. As a generator we take the class – called Maslov class – given by the loop

$$\sigma(t) = \begin{bmatrix} e^{2\pi i t} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}, \quad t \in [0, 1]$$

with $S^1 = \mathbb{R}/\mathbb{Z}$. We define a homomorphism $\mu$: $\pi_1 \to \mathbb{Z}$ by $\mu([\sigma]) = 1$.

Next consider smooth arcs $\Psi$: $[0, 1] \to \mathrm{Sp}$ satisfying

$$\Psi(0) = \mathrm{Id}, \quad \Psi(1) \in \mathrm{Sp}^* . \tag{3}$$

We call two such arcs $\Psi$ and $\Phi$ equivalent provided there is a smooth map $F$: $[0, 1] \times [0, 1] \to \mathrm{Sp}$ such that

$$\begin{gathered} F(0, t) = \Psi(t), \quad F(1, t) = \Phi(t) \\ F([0, 1] \times \{1\}) \subset \mathrm{Sp}^* \\ F([0, 1] \times \{0\}) = \{\mathrm{Id}\} . \end{gathered} \tag{4}$$

We denote by $\mathscr{A}$ the set of equivalence classes. Let us denote by $\alpha$, $\beta$ the following two arcs

$$\alpha(t) = \begin{bmatrix} e^{\pi i t} & & & \\ & e^{\pi i t} & & 0 \\ & & \ddots & \\ & 0 & & e^{\pi i t} \end{bmatrix}$$

and

$$\beta(t) = \begin{bmatrix} e^{-\tau}\mathrm{Re} + ie^{\tau}\mathrm{Im} & & 0 \\ & e^{\pi i t} & \\ & \ddots & \\ 0 & & e^{\pi i t} \end{bmatrix} .$$

We observe that $\pi_1$ acts on $\mathscr{A}$ by

$$\begin{gathered} \pi_1 \times \mathscr{A} \to A\colon [\sigma][\Phi] = [\sigma\Phi] \\ (\sigma\Phi)(t) = \sigma(t)\,\Phi(t) . \end{gathered}$$

We have [8, 21, 39, 40]

**Theorem 1** *There exists a unique map $\mu_{CZ}$: $\mathscr{A} \to \mathbb{Z}$ satisfying*

$$\begin{gathered} \mu_{CZ}([\sigma][\Phi]) = 2 + \mu_{CZ}([\Phi]) \\ \mu_{CZ}([\alpha]) = n \\ \mu_{CZ}([\beta]) = n - 1 . \end{gathered}$$

Next we introduce the class $\mathscr{H}$ of smooth Hamiltonians $H$: $S^1 \times \mathbb{C}^n \to \mathbb{R}$ such that

$$H \,|\, (S^1 \times \bar{U}) < 0 \tag{5}$$

for some open bounded set $U \subset \mathbb{C}^n$ (possibly $U = \emptyset$). Moreover there exists a $\langle *, * \rangle$-positive definite matrix $A$ such that

(6) $$|H'(t, u) - Au|\,|u|^{-1} \to 0 \quad \text{as} \quad |u| \to +\infty$$

uniformly for $t \in S^1$, where $H'$ is the $\langle *, * \rangle$-gradient with respect to the $u$-variable. Moreover the linear Hamiltonian system

(7) $$-i\dot{x} = Ax, \quad x(0) = x(1)$$

has only the trivial solution. Further there exists a constant $c > 0$ such that

(8) $$|H''(t, u)\,h| \leqq c|h| \qquad \text{for all } t \in S^1,\ u \in \mathbb{C}^n,\ h \in \mathbb{C}^n$$

$$\left|\frac{\partial H'}{\partial t}(t, u)\right| \leqq c(1 + |u|) \quad \text{for all } (t, u) \in S^1 \times \mathbb{C}^n .$$

We call a Hamiltonian $H \in \mathscr{H}$ regular if all 1-periodic solutions are non-degenerate, i.e. the linearization of the time-1-map $\Psi_H$ at $x_0 = x(0)$, where $x \in \mathscr{P}_H = \{x\colon S^1 \to \mathbb{C}^n \mid \dot{x} = X_{H_t}(x)\}$, belongs to Sp*. Denote the collections of all regular Hamiltonians by $\mathscr{H}_{\text{reg}}$. If $H \in \mathscr{H}_{\text{reg}}$ and $x \in \mathscr{P}_H$ then the linear Hamiltonian system

$$\dot{h}(t) = X'_{H_t}(x(t))\,h(t), \quad t \in [0, 1]$$

defines an arc $t \to \Psi_t^H$, $\Psi_0^H = \text{Id}$, with $\Psi_1^H \in \text{Sp}^*$. Hence we can define an index $\text{Ind}(x, H)$ by letting

(9) $$\text{Ind}(x, H) = \mu_{CZ}([\Psi^H]) .$$

**Definition 2** For $H \in \mathscr{H}_{\text{reg}}$ we call $\text{Ind}(x, H) \in \mathbb{Z}$ as given in (9) the Conley-Zehnder index of the 1-periodic solution $x$ of $\dot{x} = X_{H_t}(x)$.

So, in some sense $\text{Ind}(x, H)$ gives a local information concerning a periodic orbit $x$. In order to give relations between this local information we need Instanton homology or the so-called *Floer homology*. It has been previously exploited by Amann-Zehnder, [2], and Conley-Zehnder, [8], that asymptotically quadratic Hamiltonians admit a good existence theory.

### *1.4 Instanton homology*

We denote by $\mathscr{J}$ the collection of all smooth $t$-depending $\omega$-calibrated almost complex structures $J$ such that

$$J(t, u) = i \quad \text{for} \quad |u| \text{ large and } t \in S^1 .$$

$\omega$-calibrated means that

$$g_J(t, u)\,(h, k) := \omega(h, J(t, u)\,k)$$

defines a $(t \in S^1)$-depending Riemannian metric on $\mathbb{C}^n$ which is standard outside of a compact set.

For $H \in \mathscr{H}_{\text{reg}}$ and $J \in \mathscr{J}$ consider the partial differential equation with asymptotic boundary conditions ($\nabla_J H$ is the gradient for the second variable for the inner product $g_J(t, u) = \omega \circ (\text{Id} \times J(t, u))$)

(10) $$u_s - J(t, u)\,u_t - (\nabla_J H)(t, u) = 0$$
$$u\colon \mathbb{R} \times S^1 \to \mathbb{C}^n$$
$$u(s, *) \to x^{\pm} \quad \text{as} \quad s \to \pm\infty ,$$

where $x^{\pm} \in \mathscr{P}_H$ and the limits are in the $C^1$-sense ($\Rightarrow C^\infty$ sense via elliptic regularity theory, see [33]). The solutions of (10) can be considered as zeroes of a smooth nonlinear Fredholm map $\partial_{H,J}$: $\mathscr{B}^{1,p}(x^-, x^+) \to L^p(Z, \mathbb{C}^n)$ for some $p > 2$, see the later chapters. The Fredholm index satisfies, [15, 34, 38, 39, 40]

**Theorem 3** *For every zero $u$ of $\partial_{H,J}$ the linearization $\partial'_{H,J}(u)$: $H^{1,p}(Z, \mathbb{C}^n) \to L^p(Z, \mathbb{C}^n)$ has the Fredholm index*

$$\mathrm{Ind}(\partial'_{H,J}(u)) = \mathrm{Ind}(x^-, H) - \mathrm{Ind}(x^+, H) .$$

Given $H \in \mathscr{H}_{\mathrm{reg}}$ and $J \in \mathscr{J}$ it is shown in Sect. 3 that for every multi index $\alpha$ there exists a constant $c_\alpha$ such that for every pair $(x^-, x^+) \in \mathscr{P}_H \times \mathscr{P}_H$ a solution of (10) satisfies

$$|(D^\alpha u)(s, t)| \leqq c_\alpha \quad \text{for all} \quad (s, t) \in Z . \tag{11}$$

Moreover, in Sect. 5 it is proved that given $J \in \mathscr{J}$ there exists a $\tilde{J}$ arbitrarily close to $J$ in $C^\infty$, and identically equal "$i$" outside some compact set such that the operators $\partial_{H,\tilde{J}}$ have 0 as a regular value. Consequently for such a regular $\tilde{J} \in \mathscr{J}_{\mathrm{reg}}(H)$, $H \in \mathscr{H}_{\mathrm{reg}}$ the solution set of (10) has a structure of a finite-dimensional manifold. We denote this manifold for $H \in \mathscr{H}_{\mathrm{reg}}$ and $J \in \mathscr{J}_{\mathrm{reg}}(H)$ and $x^-, x^+ \in \mathscr{P}_H$ by $\mathscr{M}(x^-, x^+; H, J)$. In view of Theorem 3 we have

$$\dim \mathscr{M}(x^-, x^+; H, J) = \mathrm{Ind}(x^-, H) - \mathrm{Ind}(x^+, H) . \tag{12}$$

Moreover all those manifolds are orientable. Using the so-called glueing construction, see [17, 19, 34], there is a natural way to produce an orientation of $\mathscr{M}(x, z; H, J)$ , provided one is given for $\mathscr{M}(x, y; H, J)$ and $\mathscr{M}(y, z; H, J)$ . This is related to the fact that given a trajectory in $\mathscr{M}(x, y) := \mathscr{M}(x, y; H, J)$ and one in $\mathscr{M}(y, z)$ there exists one geometrically close to its union which lies in $\mathscr{M}(x, z)$ and can be found by an implicit function theorem. A choice of orientation compatible with the above natural procedure is called a coherent orientation, see [21], and Sect. 5 of this paper.

Now following [16, 19], we define for $a \in \mathbb{R} \cup \{+\infty\}$ the graded free Abelian group

$$C^a(H, J) = \bigoplus_{k \in \mathbb{Z}} C^a_k \tag{13}$$
$$C^a_k = \bigoplus \mathbb{Z}x ,$$

where the sum in the second line is taken over all $x \in \mathscr{P}_H$ satisfying $\Phi_H(x) < a$ and $\mathrm{Ind}(x, H) = k$. Here

$$\Phi_H(x) = \frac{1}{2}\int_0^2 \langle -i\dot{x}, x\rangle\, dx - \int_0^1 H(t, x(t))dt .$$

Next we define a group homomorphism $\partial$: $C^a_k \to C^a_{k-1}$ by

$$\partial x = \sum \tau(x, y) y , \tag{14}$$

where the sum is taken over all those $y \in \mathscr{P}_H$ satisfying $\mathrm{Ind}(y, H) = \mathrm{Ind}(x, H) - 1$. Here $\tau(x, y) \in \mathbb{Z}$ is obtained as follows. We consider for $y \in \mathscr{P}_H$ as above the 1-dimensional (perhaps empty) manifold $\mathscr{M}(x, y; H, J)$ which carries an orientation (coming from the choice of a coherent orientation). All orbits in $\mathscr{M}(x, y; H, J)$ are isolated and $\mathscr{M}(x, y; H, J)$ decomposes into several components of the type $\{\rho * u \mid \rho \in \mathbb{R}\}$, where

$$(\rho * u)(s, t) = u(s + \rho, t)$$

is the natural $\mathbb{R}$-action. Each component of the form $\{\rho * u \mid \rho \in \mathbb{R}\}$ has two orientations, namely $o(u)$ from the coherent orientation, see Sect. 4, and the orientation given by $[u_s]$, with $u_s = \frac{\partial u}{\partial s}$. We define $\tau(u) \in \{1, -1\}$ by

$$o(u) = \tau(u)[u_s] \tag{15}$$

and put

$$\tau(x, y) = \sum \tau(u) , \tag{16}$$

where the sum is taken over all points in the reduced connecting orbit space

$$\hat{\mathscr{M}}(x, y; H, J) := \mathscr{M}(x, y; H, J)/\mathbb{R} ,$$

where we divide by the $\mathbb{R}$-action. The crucial result proved in [16, 17, 19] is

**Theorem 4** $\partial^2 = 0$.

Let us denote by $\mathscr{N}$ the product $\mathscr{H} \times \mathscr{J}$ and by $\mathscr{N}_{\text{reg}}$ the subset consisting of all pairs $(H, J) \in \mathscr{N}$ such that the associated first order elliptic partial differential equation of type (10) can be formulated as the problem of finding zeroes for a regular Fredholm section. In particular $H \in \mathscr{H}_{\text{reg}}$ and $J \in \mathscr{J}_{\text{reg}}(H)$. For $(H, J) \in \mathscr{N}_{\text{reg}}$ and $-\infty < a \leqq b \leqq +\infty$ we define

$$C_*^{[a,b)}(H, J) := C_*^b(H, J)/C_*^a(H, J) \tag{17}$$

with the induced boundary operator $\partial$ and put

$$S^{[a,b)}(H, J) = \operatorname{kern}(\partial_k)/\operatorname{Im}(\partial_{k+1}) \tag{18}$$

with $\partial_k \colon C_k^{[a,b)} \to \mathbb{C}_{k-1}^{[a,b)}$. Moreover we put

$$S_*^{[a,b)}(H, J) = \bigoplus_{k \in Z} S_k^{[a,b)}(H, J) . \tag{19}$$

### *1.5 Monotonicity*

Consider $(K, \tilde{J})$, $(H, J) \in \mathscr{N}_{\text{reg}}$. We define a partial ordering by

$$(H, J) \leqq (K, \tilde{J}) :\Leftrightarrow H(t, x) \leqq K(t, x) \tag{20}$$

for all $t \in S^1$ and $x \in \mathbb{C}^n$. A monotone homotopy between pairs $(H, J)$ and $(K, \tilde{J})$ in $\mathscr{N}_{\text{reg}}$ is a pair $(L, \hat{J})$ consisting of a smooth map $L \colon \mathbb{R} \times S^1 \times \mathbb{C}^n \to \mathbb{R}$ and a smooth map $(s, t, u) \to \hat{J}(s, t, u)$, where $\hat{J}(s, t, u)$ is an $\omega$-calibrated complex multiplication on $\mathbb{C}^n$ such that for suitable $s_0 > 0$, $R > 0$

$$\begin{aligned} \hat{J}(s, t, u) &= i && \text{for } |u| \geqq R \\ \hat{J}(s, t, u) &= J(s, u) && \text{for } s \leqq -s_0 \\ \hat{J}(s, t, u) &= \tilde{J}(s, u) && \text{for } s \geqq s_0 . \end{aligned} \tag{21}$$

Moreover

(22)
$$\frac{\partial L}{\partial s}(s,t,u) \geqq 0$$
$$L(s,t,u) = H(t,u) \quad \text{for } s \leqq -s_0$$
$$L(s,t,u) = K(t,u) \quad \text{for } s \geqq s_0 \,.$$

Further there exists a smooth map $\mathbb{R} \to \mathscr{L}_{\mathbb{R}}(\mathbb{C}^n): s \to A(s)$ associating to $s \in \mathbb{R}$ a $\langle *, * \rangle$-positive definite matrix such that

(23)
$$A(s) = A(-s_0) \quad \text{for } s \leqq -s_0$$
$$A(s) = A(s_0) \quad \text{for } s \geqq s_0$$
$$\frac{d}{ds} A(s) \geqq 0$$

and the following is satisfied

(24) If the linear Hamiltonian system $-i\dot{x} = A(\hat{s})x$ on $(0, 1)$ has a nontrivial solution satisfying $x(0) = x(1)$ for some $\hat{s} \in \mathbb{R}$ then $\left.\frac{d}{ds} A(s)\right|_{s=\hat{s}}$ is positive definite .

Finally all these data are required to satisfy the estimates

(25)
$$|L'(s,t,u) - A(s)u| \, |u|^{-1} \to 0$$
$$\left|\frac{\partial}{\partial s} L'(s,t,u) - \frac{d}{ds} A(s)u\right| \, |u|^{-1} \to 0$$

uniformly in $(s, t)$ as $|u| \to +\infty$ and there exists a constant $c > 0$ such that

(26)
$$\left|\frac{\partial}{\partial t} L'(s,t,u)\right| \leqq c(1 + |u|)$$
$$|L''(s,t,u)h| \leqq c|h|$$

for all $(s, t, u) \in \mathbb{R} \times S^1 \times \mathbb{C}^n$ and $h \in \mathbb{C}^n$.

Given such a monotone homotopy $(L, \hat{J})$ between $(H, J) \in \mathcal{N}_{\text{reg}}$ and $(K, \tilde{J}) \in \mathcal{N}_{\text{reg}}$ with $H \leqq K$ we consider the partial differential equation

(27)
$$u_s - \hat{J}(s,t,u)u_t - (\nabla_{\hat{J}} L)(s,t,u) = 0$$
$$u(s, *) \to x \in \mathscr{P}_H \quad \text{as } s \to -\infty$$
$$u(s, *) \to x \in \mathscr{P}_K \quad \text{as } s \to +\infty \,.$$

Again it is shown that there is a $C^\infty$- apriori estimate for solutions of (27) independent of the choices of $x \in \mathscr{P}_H$ and $y \in \mathscr{P}_K$. Further it is shown that there are always generic monotone homotopies so that the solutions of (27) can be considered as regular zeroes of some nonlinear Fredholm operator. The combinatorics of the solutions of (27) can be used to construct natural homomorphisms

(28)
$$S_*^{[a,b)}(H, J) \to S_*^{[a,b)}(K, \tilde{J})$$

which are independent (!) of the chosen generic monotone homotopy (see the later sections).

*1.6 Symplectic homology*

By the preceding results we have for numbers $-\infty < a \leqq b \leqq +\infty$ constructed a functor $S^{[a,b)}$ from the partially ordered set $(\mathcal{N}_{\text{reg}}, \leqq)$, viewed as a category, into the category of graded Abelian groups.

Next assume $U \subset \mathbb{C}^n$ is a given bounded open set. Denote by $\mathcal{N}_{\text{reg}}(U)$ the collection of all pairs $(H, J) \in \mathcal{N}_{\text{reg}}$ satisfying

(29) $$H \mid (S^1 \times \bar{U}) < 0 .$$

It will be shown later that for $(H, J), (K, \tilde{J}) \in \mathcal{N}_{\text{reg}}(U)$ one can construct an $(A, \hat{J}) \in \mathcal{N}_{\text{reg}}(U)$ satisfying

(30) $$(H, J) \leqq (A, \hat{J})$$
$$(K, \tilde{J}) \leqq (A, \hat{J}) .$$

In other words $\mathcal{N}_{\text{reg}}(U)$ is a directed set. Associated to this directed set we have the functor obtained by restriction

(31) $$(\mathcal{N}_{\text{reg}}(U), \leqq) \xrightarrow{S^{[a,b)}} \mathcal{G}Ab .$$

Here $\mathcal{G}Ab$ is the category of graded Abelian groups. We define the symplectic homology (with coefficient in $\mathbb{Z}$) of an open bounded set $U \subset \mathbb{C}^n$ by

(32) $$S_*^{[a,b)}(U) = \varinjlim S_*^{[a,b)}(H, J) ,$$

where the direct limit is taken over $\mathcal{N}_{\text{reg}}(U)$.

We should note that we could tensorize the chain complex in (13) by any Abelian group $G$ first, before we carry out the previous construction. This would lead to symplectic homology with coefficients in $G$.

$S_*^{[a,b)}(U)$ measures symplectic properties of $U$ as a subset of $\mathbb{C}^n$. We denote by $\mathcal{D}$ the compactly supported symplectic diffeomorphism group in $\mathbb{C}^n$. Given $(H, J) \in \mathcal{N}_{\text{reg}}(U)$ and $\Psi \in \mathcal{D}$ we define a pair $(H_\Psi, J_\Psi) \in \mathcal{N}_{\text{reg}}(\Psi(U))$

(33) $$H_\Psi(t, u) = H(t, \Psi^{-1}(u))$$
$$J_\Psi(t, u) = T\Psi(\Psi^{-1}(u))\, J(t, \Psi^{-1}(u))\, T\Psi^{-1}(u) .$$

If $u$ solves the partial differential equation associated to $(H, J)$ then $\Psi(u)$ solves the partial differential equation associated to $(H_\Psi, J_\Psi)$. We note that $\Psi$ induces via (37) a bijection

$$\mathcal{N}_{\text{reg}}(U) \to \mathcal{N}_{\text{reg}}(\Psi(U)) .$$

Obviously it follows immediately that $\Psi$ induces an isomorphism

(34) $$\Psi_\#: S_*^{[a,b)}(U) \xrightarrow{\sim} S_*^{[a,b)}(\Psi(U)) .$$

If $U \subset V$ are both bounded open subsets then

$$\mathcal{N}_{\text{reg}}(V) \subset \mathcal{N}_{\text{reg}}(U) .$$

This gives a natural map called monotonicity morphism

$$S_*^{[a,b)}(V) \xrightarrow{mm} S_*^{[a,b)}(U)\,. \tag{35}$$

Next assume $\Psi \in \mathscr{D}$ and $U$, $V$ are bounded open subsets of $\mathbb{C}^n$ such that $\Psi(U) \subset V$. We define a group homomorphism $\Psi^*$: $S_*^{[a,b)}(V) \to S_*^{[a,b)}(U)$ by the factorization

$$\begin{array}{ccc} S_*^{[a,b)}(V) & \xrightarrow{\Psi^*} & S_*^{[a,b)}(U) \\ & {\scriptstyle mm}\searrow \quad \nearrow {\scriptstyle (\Psi_\#)^{-1}} & \\ & S^{[a,b)}(\Psi(U)) & \end{array} \tag{36}$$

The previous construction of $S_*^{[a,b)}$ and $\Psi^*$ turns out to have many useful properties. For example

**Theorem 5** *Assume $\Psi_s(U) \subset V$ for $s \in [0,1]$ where $U$ and $V$ are bounded open sets in $\mathbb{C}^n$ and $s \to \Psi_s$ is a smooth arc in $\mathscr{D}$. Then the map*

$$s \to \Psi_s^* \in \operatorname{mor}(S_*^{[a,b)}(V), S_*^{[a,b)}(U))$$

*is constant.*

This property will be referred to as isotopy invariance.

Next assume a triplet of numbers is given, such that $-\infty < a \leqq b \leqq c \leqq +\infty$. We have an obvious exact triangle $\Delta_{a,b,c}(H,J)$ for every $(H,J) \in \mathscr{N}_{\mathrm{reg}}$, induced by the short exact sequence

$$0 \to C^{[a,b)}(H,J) \to C^{[a,c)}(H,J) \to C^{[b,c)}(H,J) \to 0\,.$$

This gives an exact triangle $\Delta_{a,b,c}$ of symplectic homology groups.

**Theorem 6** *Given an open bounded subset $U$ of $\mathbb{C}^n$ we have an associated triangle $\Delta_{a,b,c}(U)$ for numbers $-\infty < a \leqq b \leqq c \leqq +\infty$, namely*

$$\begin{array}{ccc} S_*^{[a,b)}(U) & \longrightarrow & S_*^{[a,c)}(U) \\ & {\scriptstyle \partial_*}\nwarrow \quad \swarrow & \\ & S_*^{[b,c)} & \end{array} \tag{37}$$

Moreover a $\Psi \in \mathscr{D}$ with $\Psi(U) \subset V$ induces a map $\Psi^*$: $\Delta_{a,b,c}(V) \to \Delta_{a,b,c}(U)$.

Next we observe that for numbers $-\infty < a \leqq b \leqq +\infty$, and $-\infty < a' \leqq b' \leqq +\infty$ with $a \leqq a'$, $b \leqq b'$, and $(H,J) \in \mathscr{N}_{\mathrm{reg}}$ we have a natural morphism

$$C_*^{[a,b)}(H,J) \to C_*^{[a',b')}(H,J) \tag{38}$$

inducing natural maps. In fact

**Theorem 7** *Let $U \subset \mathbb{C}^n$ be an open bounded set and let $-\infty < a \leqq b \leqq c \leqq +\infty$, $-\infty < a' \leqq b' \leqq c' \leqq +\infty$ be numbers satisfying*

$$-\,a \leqq a',\ b \leqq b',\ c \leqq c'\,.$$

*Then there exists a natural homomorphism*

$$\Delta_{a,b,c}(U) \to \Delta_{a',b',c'}(U) . \tag{39}$$

It turns out that for $\Psi(U) \subset V$, $\Psi \in \mathscr{D}$ the following diagram commutes

$$\begin{array}{ccc} \Delta_{a,b,c}(V) & \xrightarrow{\Psi^*} & \Delta_{a,b,c}(U) \\ \downarrow & & \downarrow \\ \Delta_{a',b',c'}(V) & \xrightarrow[\Psi^*]{} & \Delta_{a',b',c'}(U) \end{array}$$

All the previous properties will be proved in the following. In this paper we shall not give any application, but refer the reader to the forthcoming papers [24, 25]. In [24] we compute the symplectic homology of some simple shapes and prove the conjecture of M. Gromov concerning the symplectic classification of open polydisks. Moreover we study the space of symplectic embeddings of one polydisc into another. In [22] we generalize the present results to more general symplectic manifolds and apply these in [25] to prove some results concerning the symplectic stability of the action spectrum, a problem which has to be seen in relation to the "theme" what does the interior of a symplectic manifold know about its boundary, see also [12, 13, 14].

## 2 Apriori estimates

### *2.1 A version of the maximum principle*

Let $Z = \mathbb{R} \times S^1$ and assume $\alpha: Z \to \mathbb{R}$ is a smooth map. Given $\delta > 0$ we denote by $\Gamma_\delta$ the set of all bi-infinite sequences $(s_k)_{k\in\mathbb{Z}}$ satisfying

$$0 < s_{k+1} - s_k \leqq \delta \tag{40}$$

$$s_k \to \pm\infty \quad \text{as} \quad k \to \pm\infty .$$

Given $s = (s_k) \in \Gamma_\delta$ we define

$$[\alpha]^s := \sup\{\alpha(s_k, t) \mid k \in \mathbb{Z},\ t \in S^1\} \tag{41}$$

and put

$$[\alpha]_\delta := \inf\{[\alpha]^s \mid s \in \Gamma_\delta\} . \tag{42}$$

Clearly $[\alpha]_\delta \in (-\infty, +\infty]$.

**Proposition 8** *Assume constants $a, b, \lambda \geqq 0$ and $\delta > 0$ are given such that*

$$\delta^2 \lambda < \pi^2 .$$

*Then there exists a constant $C = C(a, b, \lambda, \delta) > 0$ such that every smooth map $\alpha: Z \to \mathbb{R}$ with*

$$-\Delta\alpha - \lambda\alpha \leqq a \quad \text{on} \quad Z$$

$$[\alpha]_\delta < b$$

*satisfies*

$$\sup\{\alpha(s, t) \mid (s, t) \in Z\} \leqq C .$$

*Proof.* Let $\Omega := (x, y) \times S^1$ with $-\infty < x < y < +\infty$ and $y - x \leqq \delta$. Then

$$\gamma_1 := \pi^2 (y-x)^{-2}$$

is the smallest eigenvalue of the eigenvalue problem

$$-\Delta u = \gamma u \quad \text{on} \quad \Omega\,, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

with associated eigenfunction $u(s, t) = \sin((s-x)(y-x)^{-1}\pi)$. By assumption

$$\lambda < \pi^2 \delta^{-2} \leqq \pi^2 (y-x)^{-2}\,.$$

Let $s = (s_k) \subset \Gamma_\delta$ be a sequence satisfying

$$\sup\{\alpha(s_k, t) \mid k \in \mathbb{Z}, t \in S^1\} < b$$

and put $x = s_k$ and $y = s_{k+1}$. Then with $\Omega = (x, y) \times S^1$ we have

$$(43) \qquad \begin{aligned} -\Delta\alpha - \lambda\alpha &\leqq a \quad \text{on } \Omega \\ \alpha &< b \quad \text{on } \partial\Omega\,. \end{aligned}$$

As a consequence of the classical Krein–Rutmann result, see [1], the maximum principle is valid for (43) provided $\lambda < \gamma_1 = \gamma_1(\Omega)$. If

$$\begin{aligned} -\Delta\beta - \lambda\beta &= a \quad \text{on } \Omega \\ \beta &= b \quad \text{on } \partial\Omega \end{aligned}$$

then $\alpha \leqq \beta$ on $\bar{\Omega}$ and $\beta$ is clearly independent of the $t$-variable in $S^1$. Consequently it satisfies

$$\begin{aligned} -\beta'' - \lambda\beta &= a \quad \text{on } (x, y) \\ \beta(x) = \beta(y) &= b\,. \end{aligned}$$

With $\sigma := (y-x)\delta^{-1} \in (0, 1]$, assuming without loss of generality that $x = 0$ we define

$$\hat{\beta}(s) = \beta(\sigma s)\,.$$

Then

$$\begin{aligned} -\hat{\beta}'' - \sigma^2\lambda\hat{\beta} &= \sigma^2 a \quad \text{on } (0, \delta) \\ \hat{\beta}(0) = \hat{\beta}(\delta) &= b\,. \end{aligned}$$

If $\tilde{\beta}$ solves

$$\begin{aligned} -\tilde{\beta}'' - \lambda\tilde{\beta} &= a \quad \text{on } (0, \delta) \\ \tilde{\beta}(0) = \tilde{\beta}(\delta) &= 0 \end{aligned}$$

we deduce $\tilde{\beta} \geqq 0$ since $\lambda < \dfrac{\pi^2}{\delta^2}$. Moreover $\hat{\beta} \geqq 0$. Hence on $(0, \delta)$

$$-\hat{\beta}'' - \delta\hat{\beta} \leqq -\hat{\beta}'' - \lambda\sigma^2\hat{\beta} = \sigma^2 a \leqq a = -\tilde{\beta}'' - \lambda\tilde{\beta}$$

and $\hat{\beta} = \tilde{\beta}$ on $\{0, \delta\}$. This implies $\hat{\beta} \leqq \tilde{\beta}$. Consequently

$$\sup_{(s,t)\in\bar{\Omega}} \alpha(s, t) \leqq \sup_{s\in[x,y]} \hat{\beta}(s) \leqq \sup_{s\in[x,y]} \tilde{\beta}(s) =: C(a, b, \delta, \lambda)\,.$$

Since $k$ was arbitrarily chosen we infer

$$\sup \alpha(Z) \leqq C(a, b, \delta, \lambda)\,.$$

□

*2.2 Variational estimates*

Assume $L$: $\mathbb{R} \times S^1 \times \mathbb{C}^n \to \mathbb{R}$ is smooth and satisfies

$$\frac{\partial L}{\partial s}(s, t, u) \geqq 0 \quad \text{on} \quad \mathbb{R} \times S^1 \times \mathbb{C}^n \tag{44}$$

and

$$|L'(s, t, u)| \leqq c(1 + |u|) \quad \text{on} \quad \mathbb{R} \times S^1 \times \mathbb{C}^n . \tag{45}$$

Moreover we assume that there exists a suitable $s_0 > 0$ such that

$$\begin{aligned} H(t, u) &= L(s, t, u) \quad \text{for } s \leqq -s_0 \\ K(t, u) &= L(s, t, u) \quad \text{for } s \leqq s_0 , \end{aligned} \tag{46}$$

where $H, K \in \mathscr{H}$. Further $\hat{J}$ *is a* $(s, t)$-depending $\omega$-calibrated almost complex structure satisfying

$$\begin{aligned} \hat{J}(s, t, u) &= i \quad && \text{for } |u| \text{ large} \\ \hat{J}(s, t, u) &= J(t, u) && \text{for } s \leqq -s_0 \\ \hat{J}(s, t, u) &= \tilde{J}(t, u) && \text{for } s \geqq s_0 . \end{aligned} \tag{47}$$

Now consider a smooth solution $u$: $\mathbb{R} \times S^1 \to \mathbb{C}^n$ of

$$u_s - \hat{J}(s, t, u)\, u_t - (\nabla_{\hat{J}} L)(s, t, u) = 0 . \tag{48}$$

We have

**Lemma 9** *Let* $\hat{J}$ *and* $L$ *be as described above. Then there exist real constants* $-\infty < c_1 \leqq c_2 < +\infty$ *such that every solution* $u$: $Z \to \mathbb{C}^n$ *of* (48) *satisfying*

$$\inf_{s \in \mathbb{R}} \Phi_{L(s)}(u(s)) > -\infty , \quad \sup_{s \in \mathbb{R}} \Phi_{L(s)}(u(s)) < +\infty \tag{49}$$

*fulfills*

$$\Phi_{L(s)}(u(s)) \in [c_1, c_2] \quad \text{for all } s \in \mathbb{R} . \tag{50}$$

Here $u(s)(t) := u(s, t)$ *and for a smooth loop* $x$: $S^1 \to \mathbb{C}^n$

$$\Phi_{L(s)}(x) = \frac{1}{2} \int_0^1 \langle -i\dot{x}, x \rangle dt - \int_0^1 L(s, t, x(t))\, dt . \tag{51}$$

*Proof*. We compute

$$\frac{d}{ds} \Phi_{L(s)}(u(s)) = -\int_0^1 \frac{\partial L}{\partial s}(s, t, u(s, t)) dt - \| \Phi'_{L(s)}(u(s)) \|^2_{s, u(s)} .$$

Here for a smooth loop $x$ and $s \in \mathbb{R}$

$$\|h\|^2_{s, x} = \int_0^1 g_{\hat{J}}(s, t, x(t))(h(t), h(t)) dt .$$

By our assumption on $\hat{J}$ this is equivalent to the usual $L^2$-norm, where the equivalence is uniform in $x$ and $s$. By assumption the map $s \to \Phi_{L(s)}(u(s))$ is decreasing. Hence

$$\lim_{s\to-\infty} \Phi_H(u(s)) = \sup_{s\in\mathbb{R}} \Phi_{L(s)}(u(s))$$

$$\lim_{s\to+\infty} \Phi_K(u(s)) = \inf_{s\in\mathbb{R}} \Phi_{L(s)}(u(s)) .$$

By assumption we find a sequence $s_k \to +\infty$ such that for the usual $L^2$-gradient and $x_k = u(s_k)$ we have

(52) $$\|\Phi'_K(x_k)\|_{L^2} \to 0 .$$

Here let $\Phi'_K$ denote the usual $L^2$-gradient. If $(\|x_k\|_{L^2})$ is bounded we infer from (52) that $(\|x_k\|_{H^{1,2}})$ is bounded, which immediately implies via (52) that $(x_k)$ is precompact in $H^{1,2}(S^1, \mathbb{C}^n)$. Hence without loss of generality we may assume $x_k \to x$ in $H^{1,2}(S^1, \mathbb{C}^n)$, where $x$ solves

(53) $$\dot{x} = X_{K_t}(x), \quad x(0) = x(1) .$$

For solutions of (53) we have a $C^\infty$-apriori estimate from which we can obtain $c_2$. Similarly one constructs $c_1$ by studying $(u(s_k))$ for a suitable sequence $s_k \to -\infty$. So arguing indirectly let us assume that $\|x_k\|_{L^2} \to \infty$ (which we may after taking a subsequence). Define $v_k = x_k/\|x_k\|_{L^2}$ and $\lambda_k = \|x_k\|_{L^2}$. Then

$$-i\frac{d}{dt} v_k - \frac{1}{\lambda_k} K'(t, \lambda_k v_k) \to 0 \quad \text{in } L^2 .$$

If $K'$ is asymptotic to the positive definite matrix $B$ we infer immediately that after taking a subsequence $v_k$ converges in $H^{1,2}$ to some $v$ with $\|v\|_{L^2} = 1$ satisfying

(54) $$-i\dot{v} = Bv \quad \text{with} \quad v(0) = v(1) .$$

By our assumption of (7) the only solution of (54) is $v \equiv 0$ contradicting $\|v\|_{L^2} = 1$. Now define

$$c_1 = \inf \{\Phi_K(x) \mid d\Phi_K(x) = 0\}$$

$$c_2 = \sup\{\Phi_H(x) \mid d\Phi_H(x) = 0\} .$$

Then $-\infty < c_1 \leqq c_2 < +\infty$ by construction and $\Phi_{L(s)}(u(s)) \in [c_1, c_2]$ for all $s \in \mathbb{R}$. □

In order to obtain some sharper estimates we impose some more hypothesis on $L$. Assume $L$ and $J$ satisfy (44)–(47). Moreover suppose there exists a smooth arc $s \to A(s)$ of positive definite matrices $A(s) \in \mathscr{L}_{\mathbb{R}}(\mathbb{C}^n)$ satisfying

(55)
$$\begin{aligned} A(s) &= A(-s_0) && \text{for } s \leqq -s_0 \\ A(s) &= A(s_0) && \text{for } s \geqq s_0 \\ \frac{\partial A}{\partial s}(s) &\geqq 0 && \text{on } \mathbb{R} . \end{aligned}$$

Further $A$ is assumed to satisfy the following "regularity" requirement

(56) If the linear Hamiltonian system $-i\dot{x} = A(\hat{s})x$ on $(0, 1)$ has a nontrivial 1-periodic solution for some $\hat{s} \in \mathbb{R}$ then $\frac{d}{ds} A(s)\Big|_{s=\hat{s}}$ is positive definite.

Finally the data $A$ and $L$ is related by the following requirement

(57)
$$|L'(s,t,u) - A(s)u||u|^{-1} \to 0$$
$$\left|\frac{\partial}{\partial s} L'(s,t,u) - \frac{d}{ds} A(s)u\right||u|^{-1} \to 0$$

uniformly in $(s,t) \in \mathbb{R} \times S^1$ as $|u| \to +\infty$.

**Lemma 10** *Assume $L$, $\hat{J}$ and $A$ satisfy* (44)–(47), *and* (55)–(57). *Then for a given constant $c > 0$ there exists a constant $d > 0$ such that*

(58)
$$\|\Phi'_{L(s)}(x)\|^2_{s,x} + \int_0^1 \frac{\partial L}{\partial s}(s,t,x(t))dt \leqq c$$

*for a number $s \in \mathbb{R}$ and a loop $x \in H^{1,2}(S^1, \mathbb{C}^n)$ implies*

$$\|x\|_{H^{1,2}} \leqq d .$$

*Proof.* Without loss of generality we may assume that $\| \; \|_{s,x}$ is replaced by the usual $L^2$-norm and $\Phi'_{L(s)}$ denotes the usual $L^2$-gradient. (Here we use the properties of $\hat{J}$.) Since $\Phi_{L(s)}$ is independent of $s \in \mathbb{R}$ for $s \leqq -s_0$ and $s \geqq s_0$ we may assume arguing indirectly that there exist sequences $(s_k) \subset \mathbb{R}$, $(x_k) \subset H^{1,2}(S^1, \mathbb{C}^n)$ satisfying

(59)
$$s_k \to s_0, \quad \|x_k\|_{L^2} \to +\infty$$
$$\|\Phi'_{L(s_k)}(x_k)\|_{L^2} \leqq c, \quad \int_0^1 \frac{\partial L}{\partial s}(s_k,t,x_k(t))dt \leqq c .$$

Observe for (59) that the boundedness of $(\|x_k\|_{L^2})$ immediately implies a $H^{1,2}$-bound in view of the properties of $L$ and $\|\Phi'_{L(s_k)}(x_k)\|_{L^2} \leqq c$. Now we define $v_k = x_k/\|x_k\|_{L^2}$ and argue as in Lemma a to find a subsequence of $(v_k)$ converging in $H^{1,2}$ to some $v$ satisfying

(60)
$$-i\dot{v} = A(s_0)v \quad \text{on } (0,1)$$
$$v(0) = v(1)$$
$$\|v\|_{L^2} = 1 .$$

Hence we must have $\left(\frac{d}{ds}A\right)(s_0) \geqq \varepsilon \mathrm{Id}$ for a suitable $\varepsilon > 0$. By assumption we have

(61)
$$c \geqq \int_0^1 \frac{\partial L}{\partial s}(s_k,t,x_k(t))dt .$$

Moreover we compute

(62)
$$\begin{aligned}\frac{\partial L}{\partial s}(s,t,u) &= \int_0^1 \left\langle \frac{\partial L'}{\partial s}(s,t,\gamma u), u \right\rangle d\gamma + \frac{\partial L}{\partial s}(s,t,o)\\ &= \int_0^1 \left\langle \frac{\partial L'}{\partial s}(s,t,\gamma u) - \frac{dA}{ds}(s)\gamma u, u \right\rangle d\gamma\\ &\quad + \frac{1}{2}\left\langle \frac{dA}{ds}(s)u, u\right\rangle + \frac{\partial L}{\partial s}(s,t,o) .\end{aligned}$$

For every $\tau > 0$ there exists a constant $c_\tau > 0$ such that

(63) $$\left\| \frac{\partial L'}{\partial s}(s, t, \gamma u) - \frac{d}{ds} A(s)(\gamma u) \right\| \leqq \tau |\gamma u| + c_\tau$$

for all $(s, t, u)$ and $\gamma$. This implies for $k$ large enough

$$\begin{aligned} c &\geqq \int_0^1 \frac{\partial L}{\partial s}(s_k, t, x_k(t)) dt \\ &\geqq \frac{1}{4} \varepsilon \|x_k\|_{L^2}^2 - \frac{1}{2} \tau \|x_k\|_{L^2}^2 - c_\tau \|x_k\|_{L^2} - \sup_{s \in \mathbb{R}} \int_0^1 \frac{\partial L}{\partial s}(s, t, o) dt \\ &\geqq \left( \frac{\varepsilon}{4} - \frac{\tau}{2} \right) \|x_k\|_{L^2}^2 - c_\tau \|x_k\|_{L^2} - \hat{c} \,. \end{aligned}$$

Taking $\tau = \frac{\varepsilon}{4}$ we obtain a $L^2$-bound on $(x_k)$ contradicting the fact that

$$\|x_k\|_{L^2} \longrightarrow +\infty \,.$$

This completes the proof of Lemma 10. □

**Proposition 11** *Assume $L$, $\hat{J}$ and $A$ are as described above (see Lemma 10). Given $\delta > 0$ there exists a constant $c_\delta \in \mathbb{R}$ such that for every solution of (48) satisfying*

(64) $$-\infty < \inf_{s \in \mathbb{R}} \Phi_{L(s)}(u(s), \sup_{s \in \mathbb{R}} \Phi_{L(s)}(u(s)) < +\infty$$

*we have*

(65) $$[|u|]_\delta \leqq c_\delta \,.$$

*Proof.* From Lemma 9 we know the existence of numbers $-\infty < c_1 \leqq c_2 < +\infty$ such that $\Phi_{L(s)}(u(s)) \in [c_1, c_2]$. Hence for every pair of numbers $-\infty < a \leqq b < +\infty$

(66) $$\begin{aligned} &\int_a^b \left[ \|\Phi'_{L(s)}(u(s))\|_{s, u(s)}^2 + \int_0^1 \frac{\partial L}{\partial s}(s, t, u(s, t)) dt \right] ds \\ &\leqq c_2 - c_1 =: \bar{c} \,. \end{aligned}$$

Define $\hat{s}_k = k \frac{\delta}{4}$ and put $\tau = \frac{1}{16} \delta$. Then using (66) we find $s_k \in [\hat{s}_k - \tau, \hat{s}_k + \tau]$ such that $x_k = u(s_k)$ satisfies

$$\|\Phi'_{L(s_k)}(x_k)\|_{s_k, x_k}^2 + \int_0^1 \frac{\partial L}{\partial s}(s_k, t, x_k(t) dt \leqq 8 \bar{c} \delta^{-1} \,.$$

In view of Lemma 10 we find a constant $\tilde{d}(\delta)$ such that $\|x_k\|_{H^{1,2}} \leqq \tilde{d}(\delta)$. Using the compact embedding $H^{1,2}(S^1, \mathbb{C}^n) \hookrightarrow C^0(S^1, \mathbb{C}^n)$ we find a constant $c_\delta$ such that

(67) $$\|x_k\|_{C^0} \leqq c_\delta \,.$$

Now observe that $\frac{1}{8} \delta \leqq s_{k+1} - s_k \leqq \frac{6}{16} \delta$. Hence

$$[|u|]_\delta \leqq c_\delta \,.$$ □

*2.3 A $C^0$-estimate*

Our aim is to derive a $C^0$-estimate for solutions of

$$u_s - \hat{J}(s, t, u)\, u_t - (\nabla_J L)(s, t, u) = 0$$
$$u\colon Z \to \mathbb{C}^n$$

under suitable assumptions on $\hat{J}, L$ and $u$. We list our assumptions. $\hat{J}$ is a $(s, t) \in \mathbb{R} \times S^1$-depending smooth $\omega$-calibrated almost complex structure such that

(68) $$\hat{J}(s, t, u) = i \text{ for } |u| \text{ large}$$

uniformly in $(s, t)$ and

(69) $$\hat{J}(s, t, u) = \hat{J}(s_0, t, u) \quad \text{for } s \geqq s_0$$
$$\hat{J}(s, t, u) = \hat{J}(-s_0, t, u) \quad \text{for } s \leqq -s_0 .$$

Moreover

(70) $$\frac{\partial L}{\partial s}(s, t, u) \geqq 0$$
$$H(t, u) = L(s, t, u) \quad \text{for } s \leqq -s_0$$
$$K(t, u) = L(s, t, u) \quad \text{for } s \geqq s_0 .$$

There exists a regular arc $s \to A(s)$ as described in (55) and (56) such that

(71) $$|L'(s, t, u) - A(s)u|\,|u|^{-1} \to 0$$
$$\left|\frac{\partial L'}{\partial s}(s, t, u) - \frac{d}{ds} A(s)\, u\right| |u|^{-1} \to 0$$

uniformly in $(s, t)$ as $|u| \to +\infty$. Moreover we have the estimates

(72) $$\left|\frac{\partial}{\partial t} L'(s, t, u)\right| \leqq c(1 + |u|)$$
$$|L''(s, t, u)\, h| \leqq c|h|$$

for all $(s, t, u) \in \mathbb{R} \times S^1 \times \mathbb{C}^n$ and $h \in \mathbb{C}^n$.

Our main result in this section is

**Theorem 12** *Assuming* (68)–(72) *there exists a constant* $d := d(L, \hat{J}) > 0$ *such that every solution u of*

$$u_s - \hat{J}(s, t, u) u_t - (\nabla_J L)(s, t, u) = 0$$

*with*

$$\inf_{s \in \mathbb{R}} \Phi_{L(s)}(u(s)) > -\infty , \quad \sup_{s \in \mathbb{R}} \Phi_{L(s)}(u(s)) < +\infty$$

*satisfies*

$$\sup_{(s,t) \in Z} |u(s, t)| \leqq d .$$

*Proof.* Pick a smooth map $\varphi:\mathbb{R}\to[0,+\infty)$ satisfying

$$\varphi(s)=0 \quad \text{for } s\leqq R^2+1$$

$$\varphi''(s)>0 \quad \text{for } R^2+1<s<R^2+2$$

$$\varphi'(s)=1 \quad \text{for } s\geqq R^2+2 .$$

We find a constant $c_1>0$ such that

$$\varphi'(s)s\leqq\varphi(s)+c_1 \quad \text{for all } s\in\mathbb{R} . \tag{73}$$

Define $\alpha(s,t)=\varphi(|u(s,t)|^2)$ for $(s,t)\in Z$ and let

$$\Gamma=\{(s,t)\in Z \mid |u(s,t)|>R\} .$$

On $\Gamma$ we have for a solution $u$ of (21)

$$u_s-iu_t+L'(s,t,u)=0 .$$

Hence

$$\begin{aligned} &u_{ss}-iu_{st}-\frac{\partial}{\partial s}L'(s,t,u)-L''(s,t,u)u_s=0 \\ &iu_{st}+u_{tt}-i\frac{\partial}{\partial t}L'(s,t,u)-iL''(s,t,u)u_t=0 . \end{aligned} \tag{74}$$

Adding up the two equations in (74) gives on $\Gamma$ the estimate

$$|\Delta u|\leqq c_2(1+|u|+|u_t|+|u_s|)$$

for a suitable constant $c_2>0$ independent of $u$. Hence we have on $Z$ for a suitable constant $c_3>0$

$$\begin{aligned} &|\varphi'(|u|^2)\langle u,\Delta u\rangle| \\ &\quad\leqq\varphi'(|u|^2)(c_2(1+|u|+|u_s|+|u_t|))|u| \\ &\quad\leqq\varphi'(|u|^2)(c_3(1+|u|^2)+|\nabla u|^2) . \end{aligned}$$

Using (73) gives on $Z$

$$|\varphi'(|u|^2)\langle u,\Delta u\rangle|\leqq c_3\alpha+c_3c_1+c_3+\varphi'(|u|^2)\,|\nabla u|^2 . \tag{75}$$

Next we compute

$$\begin{aligned} \Delta\alpha&=4\varphi''(|u|^2)(\langle u,u_s\rangle^2+\langle u,u_t\rangle^2)+2\varphi'(|u|^2)|\nabla u|^2+2\varphi'(|u|^2)\langle u,\Delta u\rangle \\ &\geqq 2\varphi'(|u|^2)|\nabla u|^2-2\varphi'(|u|^2)\,|\nabla u|^2-2c_3\alpha-2(c_3c_1+c_3) . \end{aligned}$$

Hence for $a:=2(c_3c_1+c_3)$, a constant only depending on $L$ and $\hat{J}$ we infer

$$-\Delta\alpha-2c_3\alpha\leqq a \quad \text{on } Z , \tag{76}$$

where $2c_3$ is also a constant only depending on $L$ and $\hat{J}$. Let $\delta>0$ be so small that

$$2c_3<\pi^2\delta^{-2} . \tag{77}$$

In view of Proposition 11 we find a constant $c_\delta$ such that

$$[|u|]_\delta \leqq c_\delta \tag{78}$$

for every solution $u$ satisfying the hypotheses. This gives a constant $b > 0$ such that

$$[\varphi(|u|^2)]_\delta \leqq b \tag{79}$$

for every such $u$. Now we apply Proposition 8 and find a constant $c = c(a, b, 2c_3, \delta) > 0$ only depending therefore on $L$ and $\hat{J}$ s.t.

$$\sup_{(s,t)\in Z} \varphi(|u(s,t)|^2) \leqq c .$$

Hence for a suitable $d > 0$

$$\sup_{(s,t)\in Z} |u(s,t)| \leqq d$$

as required. □

The previous method of obtaining $C^0$-bounds in a noncompact symplectic manifold has many more applications, [6]. We thank K. Cieliebak for many stimulating discussions.

### *2.4 $C^\infty$-estimates*

This section is quite standard provided one has the $C^0$-estimates from the previous section. We include it for the convenience of the reader and allow ourselves to be somewhat sketchy. The key point is a bubbling off analysis going back in the harmonic map case to Sacks and Uhlenbeck, [37], and in the pseudoholomorphic curve case to Gromov, [26].

We consider the partial differential equation

$$u_s - \hat{J}(s, t, u)u_t - (\nabla_{\hat{J}} L)(s, t, u) = 0 , \tag{80}$$

where $\hat{J}$ and $L$ are as described in the previous section. In view of theorem we find a constant $d > 0$ such that for every solution of (80) satisfying

$$\inf_{s\in\mathbb{R}} \Phi_{L(s)}(u(s)) > -\infty , \quad \sup_{s\in\mathbb{R}} \Phi_{L(s)}(u(s)) < +\infty \tag{81}$$

we have the $C^0$-estimate

$$\sup\{|u(s,t)| \mid (s,t)\in Z\} \leqq d . \tag{82}$$

The crucial estimate for finding uniform $C^\infty$-estimates is a uniform estimate for $|\nabla u|$. We need the following useful topological lemma from [30].

**Lemma 13** *Let $(X, d)$ be a metric space. Equivalent are the following statements.*

i) *$(X, d)$ is complete*
ii) *For every continuous map $\phi: X \to [0, +\infty)$, a given point $x \in X$ and a number $\varepsilon > 0$ there exist $x' \in X$ and $\varepsilon' > 0$ such that*
   (a) *$\varepsilon' \leqq \varepsilon$ and $\phi(x')\varepsilon' \geqq \phi(x)\varepsilon$*
   (b) *$d(x, x') \leqq 2\varepsilon$*
   (c) *$2\phi(x') \geqq \phi(y)$ for all $y\in X$ with $d(y, x) \leqq \varepsilon'$.* □

The main result of this section is

**Theorem 14** *There exists a constant* $\hat{d} > 0$ *such that for every solution of* (80) *satisfying* (81) *we have*

$$|\nabla u(s, t)| \leqq \hat{d} \quad \text{for } (s, t) \in Z .$$

*Proof.* Arguing indirectly we find a sequence $(z_k) = ((s_k, t_k))$ and a sequence $(u_k)$ such that $u_k$ solves (80) and (81) and

$$|u_k(s_k, t_k)| \to +\infty \ . \tag{83}$$

We know from Theorem 12 that

$$|u_k(s, t)| \leqq d \quad \text{for } (s, t) \in Z \tag{84}$$

and from Lemma 9

$$\Phi_{L(s)}(u_k(s)) \in [c_1, c_2] \quad \text{for } s \in \mathbb{R} \ . \tag{85}$$

From (85) we deduce for every solution of (80), (81)

$$\int_{-\infty}^{+\infty} \int_0^1 g_{\hat{J}}(s, t, u(s, t))(u_s(s, t), u_s(s, t)) ds dt \leqq c_2 - c_1 \ . \tag{86}$$

From this we deduce using the properties of $\hat{J}$

$$\int_Z |u_s|^2 ds dt \leqq \bar{c} \tag{87}$$

for a suitable constant $\bar{c} > 0$ independent of $u$. Choose a sequence $(\varepsilon_k) \subset (0, +\infty)$ satisfying

$$\varepsilon_k \to 0 , \quad \varepsilon_k |\nabla u_k(s_k, t_k)| \to +\infty \ . \tag{88}$$

In view of the topological Lemma 13 we may slightly modify the data and may therefore assume in addition

$$|\nabla u_k(s, t)| \leqq 2 |\nabla u_k(s_k, t_k)| \tag{89}$$

for all $(s, t) \in \mathbb{R}^2$ with $|(s, t) - (s_k, t_k)| \leqq \varepsilon_k$, where we consider $u_k$ periodically extended over $\mathbb{R}^2$ in the second variable. Now we define

$$v_k(s, t) = u_k((s_k, t_k) + \lambda_k^{-1}(s, t))$$

$$\lambda_k = |\nabla u_k(s_k, t_k)| \ .$$

We define $R_k = \varepsilon_k \lambda_k$ and observe that $R_k \to +\infty$. Then for $|(s, t)| \leqq R_k$

$$\begin{aligned} |\nabla v_k(s, t)| &= \lambda_k^{-1} |(\nabla u_k)((s_k + \lambda_k^{-1} s), (t_k + \lambda_k^{-1} t))| \\ &\leqq 2 \lambda_k^{-1} \lambda_k = 2 \ . \end{aligned} \tag{90}$$

Moreover for $k$ large enough

$$\int_{B_{R_k}} \left| \left( \frac{\partial}{\partial s} v_k \right)(s, t) \right|^2 ds dt \leqq \bar{c} \ . \tag{91}$$

Further

$$(92)\qquad 0=\frac{\partial}{\partial s}v_k-\hat{J}(s_k+\lambda_k^{-1}s,t_k+\lambda_k^{-1}t,v_k)\frac{\partial}{\partial t}v_k$$
$$-\lambda_k^{-1}(\nabla_{\hat{J}}L)(s_k+\lambda_k^{-1}s,t_k+\lambda_k^{-1}t,v_k)\ .$$

Without loss of generality we may assume $t_k\to t_0$. Since $\hat{J}$ is independent of $s$ for large $|s|$ we may also assume that in the $\hat{J}$ expression $s_k\to s_0$ for some $s_0\in\mathbb{R}$, as well as $t_k\to t_0$. Using the gradient bound in (90) it follows from standard linear theory (and a perturbation argument) that $(v_k)$ is $C^\infty_{\mathrm{loc}}$-bounded. Hence after taking a subsequence we may assume

$$(93)\qquad \begin{aligned} v_k&\to v \quad \text{in } C^\infty_{\mathrm{loc}}\\ |\nabla v(0,0)|&=1\\ |\nabla v(s,t)|&\leqq 2 \quad \text{for } (s,t)\in\mathbb{R}\oplus\mathbb{R}\cong\mathbb{C}\\ |v(s,t)|&\leqq d \quad \text{for } (s,t)\in\mathbb{R}\oplus\mathbb{R}\end{aligned}$$

and

$$(94)\qquad v_s-\hat{J}(s_0,t_0,v)\,v_t=0 \quad \text{on } \mathbb{R}^2\ .$$

Now (91) implies that in addition to (93) and (94)

$$(95)\qquad \int_{\mathbb{R}^2}|v_s|^2\,ds\,dt<\infty\ .$$

In view of (94) and (95) a removable singularity theorem applies, see [26]. Hence we obtain a smooth map $v:S^2\to\mathbb{C}^n$, $S^2=\mathbb{C}\cup\{+\infty\}$, which is nonconstant and satisfies

$$(Tv)i=-\hat{J}(s_0,t_0,v)\,Tv \quad \text{on } S^2\ .$$

Since $\omega\circ(\mathrm{Id}\times\hat{J}(s_0,t_0,*))=:g$ defines a Riemannian metric we infer that $0>\int_{S^2}v^*\omega$. On the other hand by Stokes' theorem for a suitable 1-form $\lambda$

$$\int_{S^2}v^*\omega=\int_{S^2}d(v^*\lambda)=\int_{\emptyset}v^*\lambda=0\ .$$

This contradiction proves our assertion. □

Rudimentary linear theory gives now using Theorem 14, – see [17, 19, 28] –

**Theorem 15** *Under our standing assumption there exists for every multiindex $\alpha$ a number $d_\alpha>0$ such that for every solution $u:Z\to\mathbb{C}^n$ of* (80) *satisfying* (81) *the following holds*

$$|(D^\alpha u)\,(s,t)|\leqq d_\alpha \quad \text{for all} \quad (s,t)\in Z\ .$$

## 3 Transversality and compactness

### *3.1 Regular Hamiltonians*

Recall that the set of Hamiltonians $\mathscr{H}$ consists of all smooth maps $H\colon S^1\times\mathbb{C}^n\to\mathbb{R}$ having the following properties:

$$(96)\qquad H\mid(S^1\times\bar{U})<0$$

for some open set $U \subset \mathbb{C}^n$. Moreover there exists a $\langle *, * \rangle$-positive definite matrix $A$ such that

$$(97) \qquad |H'(t,u) - Au|\,|u|^{-1} \to 0 \quad \text{as } |u| \to +\infty$$

uniformly for $t \in S^1$ and

$$(98) \qquad |H''(t,u)h| \leqq c(h)$$

$$\left|\frac{\partial}{\partial t} H(t,u)\right| \leqq c(1 + |u|)$$

for a suitable constant $c > 0$ and $(t, u) \in S^1 \times \mathbb{C}^n$ and $h \in \mathbb{C}^n$. Further the matrix $A$ is required to satisfy

(99) The ordinary differential equation $-i\dot{v} = Av$ on $(0, 1)$ has no nontrivial solution satisfying $v(0) = v(1)$.

As a corollary of Lemma 10 we have

**Lemma 16** *Given $H \in \mathscr{H}$ there exists a constant $c(H) > 0$ such that every 1-periodic solution of $\dot{x} = X_{H_t}(x)$ satisfies $|x(t)| \leqq c(H)$ for $t \in S^1$.*

Following a construction already used in [17] we denote by $C_\varepsilon^\infty(S^1 \times \mathbb{C}^n, \mathbb{R})$ the Banach space of all smooth maps $\Delta\colon S^1 \times \mathbb{C}^n \to \mathbb{R}$ such that

$$(100) \qquad \|\Delta\|_\varepsilon := \sum_{k=0}^{\infty} \varepsilon_k \|h\|_{C^k(S^1 \times \mathbb{C}^n, \mathbb{R})} < \infty \; .$$

Here $\varepsilon = (\varepsilon_k)$ is a sequence of positive numbers and

$$\|h\|_{C^k(S^1 \times \mathbb{C}^n, \mathbb{R})} = \sum_{|\alpha| \leqq k} \sup_{(t,u)} (D^\alpha h(t,u)) \, .$$

If $(\varepsilon_k)$ converges sufficiently fast to zero the space consisting of all restrictions of maps $u$ in $C_\varepsilon^\infty$ to a ball $B_R$ is dense in $L^2(B_R)$, see [17].

Now let $\varepsilon = (\varepsilon_k)$ be a sufficiently fast decreasing sequence. If $H_0 \in \mathscr{H}$ and $\Delta \in C_\varepsilon^\infty$ then $H_0 + \Delta \in \mathscr{H}$.

**Proposition 17** *There exists for given $H_0 \in \mathscr{H}$ a residual set $\Gamma \subset C_\varepsilon^\infty$ such that for every $H \in H_0 + \Gamma$ all 1-periodic solutions are nondegenerate, i.e. $H$ is regular.*

*Proof.* Consider the separable Banach space

$$\mathscr{B} := H^{1,2}(S^1, \mathbb{C}^n) \times C_\varepsilon^\infty(S^1 \times \mathbb{C}^n, \mathbb{R}) \, .$$

We define

$$\Sigma := \{(x, \Delta) \mid \dot{x} = X_{H_{o,t}} + X_{\Delta_t}(x)\}$$

and

$$F\colon \mathscr{B} \to L^2(S^1, \mathbb{C}^n)$$

$$F(x, \Delta) = -\dot{x} + X_{H_{o,t}}(x) + X_{\Delta_t}(x) \, .$$

Then for fixed $\Delta \in C_\varepsilon^\infty$, the map $F(*, \Delta)$ is a nonlinear proper Fredholm operator of index zero. Using the density assertion for $C_\varepsilon^\infty$ it is an easy exercise to show that

$DF(x, \Delta): \mathscr{B} \to L^2(S^1, \mathbb{C}^n)$ is onto for every choice $(x, \Delta) \in \mathscr{B}$. Moreover for $(x, \Delta) \in \Sigma$ the kernel Kern$(DF(x, \Delta))$ splits. Therefore $\Sigma$ is a smooth submanifold of $\mathscr{B}$. The projection map $\pi$: $\Sigma \to C_\varepsilon^\infty$: $(x, \Delta) \to \Delta$ is a smooth nonlinear Fredholm map of index zero. Using the Sard–Smale-theorem there exists a residual set $\Gamma$ of regular values for $\pi$. Trivially for $\Delta \in \Gamma$ the map $H_o + \Delta$ is regular. □

Assume $\Delta_k \in \Gamma$ such that $\|\Delta_k\|_\varepsilon \to 0$. Then there exists $c > 0$ such that for every 1-periodic solution of some $\dot{x} = X_{H_{o,t}}(x) + X_{\Delta_{k,t}}(x)$ we have $|x(t)| \leq c$ for all $t \in S^1$. Using this fact we have the following corollary:

**Corollary 18** *Given $H_o \in \mathscr{H}$ there exists a constant $c > 0$ and a sequence $(\Delta_k) \subset C_\varepsilon^\infty$ with* supp$(\Delta_k) \subset S^1 \times B_c$ *such that $\|\Delta_k\|_\varepsilon \to 0$ and $H_o + \Delta_k$ is regular for every $k \in \mathbb{N}$.*

For the following we introduce the space $\mathscr{H}_{\text{reg}}$ by

**Definition 19** $\mathscr{H}_{\text{reg}}$ is the subset of $\mathscr{H}$ consisting of all regular Hamiltonians.
In view of Corollary 18 $\mathscr{H}_{\text{reg}}$ is dense for the strong $C^\infty$-Whitney topology.

### *3.2 Regular trajectory spaces*

Assume $H \in \mathscr{H}_{\text{reg}}$. We study for given $J \in \mathscr{J}$ the partial differential equation

$$u_s - J(t, u)u_t - (\nabla_J H)(t, u) = 0$$
$$u: Z \to \mathbb{C}^n$$
$$u(s) \to x^\pm \quad \text{as } s \to \pm\infty ,$$

where $x^\pm \in \mathscr{P}_H = \{x \mid x \text{ is a 1-periodic solution of } \dot{x} = X_{H_t}(x)\}$. For $p > 2$ we define $\mathscr{B}^{1,p}(x^-, x^+)$ by

$$\mathscr{B}^{1,p}(x^-, x^+) := u_0 + W^{1,p}(Z, \mathbb{C}^n) ,$$

where $u_0$: $Z \to \mathbb{C}^n$ is a smooth map satisfying

$$u_0(s, t) = x^-(t) \quad \text{for } s \leqq -s_0$$
$$u_0(s, t) = x^+(t) \quad \text{for } s \geqq s_0$$

for a suitable $s_0$. Then we define $\partial_{H,J}$: $\mathscr{B}^{1,p}(x^-, x^+) \to L^p(Z, \mathbb{C}^n)$ by

$$(\partial_{H,J} u)(s, t) = u_s(s, t) - J(t, u(s, t))\, u_t(s, t) - (\nabla_J H)(t, u(s, t)) .$$

From results in [15, 17, 18], see also [34, 38], we know that $\partial_{H,J}$ is a smooth Fredholm operator, where the index is given by the difference of the Conley–Zehnder index of the periodic orbits $x^-, x^+$. This we will explain later.

Our aim is to show that for given $H \in \mathscr{H}_{\text{reg}}$ a generic choice of $J$ will result in the fact that for all pairs $(x^-, x^+)$ zero will be a regular value for $\partial_{H,J}$.

Consider the manifold $\mathscr{E}$ consistig of all real linear $j$: $\mathbb{C}^n \to \mathbb{C}^n$ such that $\omega \circ (\text{Id} \times j) = g_j$ is a Riemannian metric and $j^2 = -1$. Let $j_0 \in \mathscr{E}$ and denote by $\mathscr{S}_{j_0}$ the set of all $\mathbb{R}$-linear maps $\phi$: $\mathbb{C}^n \to \mathbb{C}^n$ such that

$$\omega(\phi h, k) + \omega(h, \phi k) = 0 \tag{101}$$
$$j_0\phi + \phi j_0 = 0 .$$

Then $\mathscr{S}_{j_0}$ is a vectorsubspace of $\mathscr{L}_{\mathbb{R}}(\mathbb{C}^n)$ and for $\phi \in \mathscr{S}_{j_0}$ the map

$$j = j_0 \exp(-j_0\phi) \tag{102}$$

belongs to $\mathscr{E}$. In fact for every $j$ close to $j_0$ we have the representation (102) for some $\phi$ close to zero.

Now let $J \in \mathscr{J}$ be given. Then $J(t, u)$ is smoothly depending on $(t, u)$ and $J(t, u) = i$ for $|u|$ large. Assume $X: S^1 \times \mathbb{C}^n \to \mathscr{L}_{\mathbb{R}}(\mathbb{C}^n)$ is a smooth map with compact support such that $|X(t, u)|$ is small and

$$\omega(X(t, u)h, k) + \omega(h, X(t, u)k) = 0$$

$$J(t, u)X(t, u) + X(t, u)J(t, u) = 0 \, .$$

Then $\tilde{J}(t, u) = J(t, u) \exp(-J(t, u)X(t, u))$ will be a new structure in $\mathscr{J}$. Again we choose a quickly decreasing sequence $\varepsilon = (\varepsilon_k)$ of positive numbers and define for given $R > 0$ the Banach space $\mathscr{S}^J_\varepsilon(R)$, consisting of all smooth $X$ as above with $X(t, u) = 0$ for all $t \in S^1$ and $|u| \geqq R$ with

$$\|X\|_\varepsilon := \sum_{k=0}^{\infty} \varepsilon_k \|X\|_{C^k} < +\infty \, .$$

We denote by $U^J_\delta(R)$ for $\delta > 0$ sufficiently small the set of all $\tilde{J}$ defined by

$$\tilde{J}(t, u) = J(t, u) \exp(-J(t, u)X(t, u)) \, ,$$

where $\|X\|_\varepsilon \leqq \delta$ and $X(t, u) = 0$ for $|u| \geqq R$. $U^J_\delta$ is an open subset of a Banach manifold. It has been proved in [23]

**Theorem 20** *Let $H \in \mathscr{H}_{\text{reg}}$ and $J \in \mathscr{J}$. Let $R > 0$ such that*

$$|x(t)| \leqq R - 1 \quad \text{for all} \quad t \in S^1$$

*for every $x \in \mathscr{P}_H$. For $\delta > 0$ sufficiently small put $U := U^J_\delta(R)$ as defined above. Then there exists a residual set $\Gamma \subset U$ such that for every $\tilde{J} \in \Gamma$ the operator $\partial_{H,\tilde{J}}: \mathscr{B}^{1,p}(x^-, x^+) \to L^p(Z, \mathbb{C}^n)$ for $p > 2$ has zero as a regular value for every choice $x^-, x^+ \in \mathscr{P}_H$.*

For the next result let $J \in \mathscr{J}$ and $H \in \mathscr{H}_{\text{reg}}$. For $x \in \mathscr{P}_H$ we consider

$$gr(x) = \{(t, x(t)) \mid t \in S^1\} \, .$$

For $\tau > 0$ let $U_\tau(x)$ be the $\tau$-neighbourhood of $gr(x)$ in $S^1 \times \mathbb{C}^n$. For $\tau > 0$ sufficiently small we have $\bar{U}_\tau(x) \cap \bar{U}_\tau(g) = \phi$ for $x \neq y$, $x, y \in \mathscr{P}_H$. Let $U_\tau(H) = \bigcup_{x \in \mathscr{P}_H} U_\tau(x)$.

Consider the Banach space

$$C^\infty_\varepsilon(U_\tau(H), \mathbb{R}) = \{\Delta \in C^\infty_\varepsilon(S^1 \times \mathbb{C}^n, \mathbb{R}) \mid \Delta \mid U_\tau(H) \equiv 0,$$
$$\operatorname{supp}(\Delta) \subset S^1 \times B^{2n}(R)\} \, .$$

Let $R > 0$ be chosen in such a way that $x \in \mathscr{P}_H$ satisfies the apriori estimate $|x(t)| \leqq R - 1$. We find a $\delta > 0$ such that $\|\Delta\|_\varepsilon \leqq \delta$ and $\Delta \in C^\infty_\varepsilon(U_\tau(H), \mathbb{R})$ implies that $\mathscr{P}_{H+\Delta} = \mathscr{P}_H$.

Denote by $\mathscr{V}_\delta$ the open $\delta$-neighbourhood in $C^\infty_\varepsilon(U_\tau(H), R)$ of zero. We have

**Theorem 21** *Let $J \in \mathscr{J}$, $H \in \mathscr{H}_{\text{reg}}$, $R > 0$, and $\mathscr{V}_\delta$ be as described above. Then there exists a residual subset $\Gamma \subset \mathscr{V}_\delta$ such that for every $\Delta \in \Gamma$ the pair $(H + \Delta, J)$ is regular, i.e. belongs to $\mathscr{N}_{\text{reg}}$.*

Theorem 21 is proved with the help of 3.1 in [23] in [40] for a related case.

Summing up there is a plentiful supply of data in $\mathcal{N}_{\text{reg}}$ for which we have the following theorem.

**Theorem 22** *Given any regular pair* $(H, J) \in \mathcal{N}_{\text{reg}}$ *and* $x^-, x^+ \in \mathcal{P}_H$ *the set* $\mathcal{M}(x^-, x^+, H, J)$ *defined by*

$$\mathcal{M}(x^-, x^+, H, J) = \{u\colon Z \to \mathbb{C}^n \mid u(s, *) \to x^\pm \quad \text{as} \quad s \to \pm\infty$$
$$\text{in } C^\infty(S^1, \mathbb{C}^n) \text{ and } u_s - J(t, u)u_t - (\nabla H)(t, u) = 0\}$$

*carries in a natural way the structure of a finite dimensional manifold induced from* $\mathcal{B}^{1,p}(x^-, x^+)$, $p > 2$. *This structure does not depend on the choice of p by elliptic regularity theory. Moreover*

$$\dim \mathcal{M}(x^-, x^+; H, J) = \operatorname{Ind}(x^-, H) - \operatorname{Ind}(x^+, H)\,.$$

*3.3 Monotone homotopies*

Let $(H, J)$ and $(K, \tilde{J})$ be elements of $\mathcal{N}_{\text{reg}}$. We define a partial ordering by

(103) $$(H, J) \leqq (K, \tilde{J})\colon \Leftrightarrow H(t, x) \leqq K(t, x) \quad \text{for all} \quad (t, x) \in S^1 \times \mathbb{C}^n\,.$$

Given two pairs satisfying (103) we call a pair $(L, \hat{J})$ a monotone homotopy between $(H, J)$ and $(U, J)$ provided $L\colon R \times S^1 \times \mathbb{C}^n \to \mathbb{R}$ is a smooth map satisfying for suitable $s_0 > 0$

(104) $$L(s, t, u) = H(t, u) \quad \text{for } s \leqq -s_0$$
$$L(s, t, u) = K(t, u) \quad \text{for } s \geqq s_0$$
$$\frac{\partial L}{\partial s}(s, t, u) \geqq 0 \qquad \text{for all } (s, t, u)\,.$$

Moreover there exists a smooth map $A\colon \mathbb{R} \to \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n)$ associating to a number $s$ a positive definite (for $\langle *, * \rangle$) matrix satisfying

(105) If $-i\dot{x} = A(\hat{s})x$ and $x(0) = x(1)$ for some non zero $x$ and $\hat{s} \in \mathbb{R}$ then $\left.\frac{d}{ds} A(s)\right|_{s=\hat{s}}$ is positive definite.

Further the data $A$ and $L$ satisfies

(106) $$|L'(s, t, u) - A(s)u|\; |u|^{-1} \to 0 \text{ uniformly as } |u| \to +\infty$$
$$\left|\frac{\partial}{\partial s} L'(s, t, u) - \frac{d}{ds} A(s)u\right| |u|^{-1} \to 0 \text{ uniformly as } |u| \to +\infty$$
$$\left|\frac{\partial}{\partial t} L'(s, t, u)\right| \leqq c(1 + |u|)$$
$$|L''(s, t, u)h| \leqq c|h|$$

for all $(s, t, u)$ and $h$. Moreover the map $(s, t, u) \to \hat{J}(s, t, u)$ associating to $(s, t, u) \in R \times S^1 \times \mathbb{C}^n$ a $\omega$-calibrated structure is smooth and satisfies

(107) $$\hat{J}(s, t, u) = i \qquad \text{for } |u| \text{ large}$$
$$\hat{J}(s, t, u) = J(t, u) \quad \text{for } s \leqq -s_0$$
$$\hat{J}(s, t, u) = \hat{J}(t, u) \quad \text{for } s \geqq s_0\,.$$

In view of Sect. 2 the solutions of the partial differential equation associated to $(L, \hat{J})$, see the introduction, satisfies a uniform $C^\infty$-bound.

Similarly to Theorem 20 we have the following transversality theorem which is in fact somewhat easier.

**Theorem 23** *Given a monotone homotopy $(L, \hat{J})$ there exist constants $R > 0, s_0 > 0$, and a sequence $(\hat{J}_k)$ with*

(108)
$$\begin{aligned} \hat{J}_k(s, t, u) &= i && \text{for } |u| \geqq R \\ \hat{J}_k(s, t, u) &= J(t, u) && \text{for } s \leqq -s_0 \\ \hat{J}_k(s, t, u) &= \tilde{J}(t, u) && \text{for } s \geqq s_0 \end{aligned}$$

*with*

$$\hat{J}_k \to J \quad \text{in } C^\infty$$

*and $(L, \hat{J}_k)$ is regular for every $k \in \mathbb{N}$.*

*Regular of course means that the first order elliptic operator $\partial_{(H, \hat{J}_k)}$ has zero as a regular value for every choice of asymptotic boundary conditions $x \in \mathscr{P}_H$, $y \in \mathscr{P}_K$. We denote by $\mathscr{N}_{\text{reg}}(H, J; K, \tilde{J})$ (for $(H, J), (K, \tilde{J}) \in \mathscr{N}_{\text{reg}}$ and $(H, J) \leqq (K, \tilde{J})$ ) the set of regular monotone homotopies between the two pairs.*

We note that we could also formulate a stronger version of Theorem 23 similar to 24. Namely there exists a neighbourhood $U(\hat{J})$ containing a residual set $\Gamma$ such that $(L, \hat{J}) \in \mathscr{N}_{\text{reg}}(H, J; \tilde{J})$ for $\hat{L} \in \Gamma$.

Similar as in Subsect. 3.2 we arrive at

**Theorem 24** *Given $(L, \hat{J}) \in \mathscr{N}_{\text{reg}}(H, J; K, \tilde{J})$ there exists for given $x \in \mathscr{P}_H$ and $y \in \mathscr{P}_K$ a natural finite dimensional manifold*

$$\begin{aligned} \mathscr{M}(x, y; L, \hat{J}) = \{u\colon Z \to \mathbb{C}^n \mid\ & u \text{ is smooth } u(s, *) \to x \text{ as } s \to -\infty \\ & u(s, *) \to y \text{ as } s \to +\infty \text{ in } C^\infty(S^1, \mathbb{C}^n) \\ & \text{and } u_s - \hat{J}(s, t, u) u_t - (\nabla_{\hat{J}} L)(s, t, u) = 0\} \end{aligned}$$

*Moreover*

$$\dim \mathscr{M}(x, y;\ L, \hat{J}) = \operatorname{Ind}(x, H) - \operatorname{Ind}(y, K)$$

*and necessarily if $\mathscr{M}(x, y, L, \hat{J}) \neq \phi$ the inequalities*

$$\operatorname{Ind}(x, H) \geqq \operatorname{Ind}(y, K), \quad \Phi_H(x) \geqq \Phi_K(y)$$

*hold.*

### *3.4 Compactness*

Let $(H, J) \in \mathscr{N}_{\text{reg}}$ and consider $\mathscr{M}(x^-, x^+; H, J)$ for a choice of $x^-, x^+ \in \mathscr{P}_H$. We assume that

(109)
$$0 \leqq \operatorname{Ind}(x^-, H) - \operatorname{Ind}(x^+, H) \leqq 2\,.$$

Hence $\mathscr{M} := \mathscr{M}(x^-, x^+; H, J)$ (assuming $\mathscr{M} \neq \phi$) is a manifold of dimension 0, 1 or 2. We have a natural $\mathbb{R}$-action on $\mathscr{M}$ defined by

(110)
$$(\rho * u)(s, t) = u(s + \rho, t)$$

for $s \in \mathbb{R}$ and $u \in \mathcal{M}$. If $x^- = x^+$ the space $\mathcal{M}$ consists precisely of the point $[(s,t) \to x^-(t)]$ and $\mathbb{R}$ acts trivially. If $x^- \neq x^+$ then the $\mathbb{R}$-action is free and the quotient space $\hat{\mathcal{M}} = \mathcal{M}/\mathbb{R}$ has dimension one less than $\mathcal{M}$:

$$\dim(\mathcal{M}) = \dim(\hat{\mathcal{M}}) + 1 \quad \text{if } \dim(\mathcal{M}) \geqq 1 \,. \tag{111}$$

Having the $C^\infty$-bounds from Sect. 2 the analysis in [16, 17, 19], see also [34, 38], leads to the following conclusion.

**Theorem 25** *Let $(H, J) \in \mathcal{N}_{\text{reg}}$. If* $\dim \mathcal{M} = 1$ *then $\hat{\mathcal{M}}$ is a compact manifold, i.e. consists of finitely many points.*

Next consider the case $\dim \mathcal{M} = 2$. Then $\hat{\mathcal{M}}$ is 1-dimensional and decomposes into components either diffeomorphic to $S^1$ or $(0, 1)$. The crucial point is to understand the meaning of its ends in geometrical terms. This has been analysed in [16, 17, 19]. We follow the description in [34]. (We oppress the data $(H, J) \in \mathcal{N}_{\text{reg}}$ in the notation.)

**Theorem 26** *Let $x, y, z \in \mathcal{P}_H$ with* $\text{Ind}(x, H) = \text{Ind}(y, H) + 1 = \text{Ind}(z, H) + 2$. *Then there exists a local diffeomorphism $\#$ from an open subset $\mathcal{O}$ of $\hat{\mathcal{M}}(x, y) \times \hat{\mathcal{M}}(y, z) \times \mathbb{R}$ into $\hat{\mathcal{M}}(x, z)$ such that*

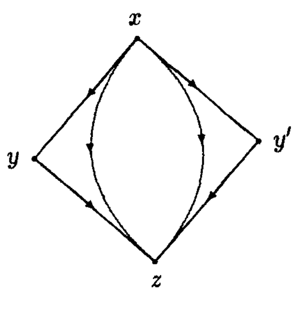
i) *Given a compact subset $K$ of $\hat{\mathcal{M}}(x, y) \times \hat{\mathcal{M}}(y, z)$ there exists a number $\sigma(K)$ such that $K \times [\sigma(K), \infty) \subset \mathcal{O}$*
ii) *There are lifts*

$$\#_1, \#_2 \colon \mathcal{O} \to \mathcal{M}(x, z)$$

*such that for every pair $(u_1, u_2) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z)$ with $\Phi_H(u_1(0)) = \frac{1}{2}(\Phi_H(x) + \Phi_H(y))$ and $\Phi_H(u_2(0)) = \frac{1}{2}(\Phi_H(y) + \Phi_H(z))$ the maps $\#_i(\hat{u}_1, \hat{u}_2, \rho)$ converge in $C^\infty_{\text{loc}}$ to $u_i$ as $\rho \to +\infty$. (Here $\hat{u}$ is the element in $\hat{\mathcal{M}}$ corresponding to $u$ in $\mathcal{M}$).*

iii) *Let $\hat{\mathcal{M}} = \bigcup \hat{\mathcal{M}}(x, y)$, where the union is taken over all pairs $(x, y) \in \mathcal{P}_H \times \mathcal{P}_H$ satisfying* $\text{Ind}(x, H) - \text{Ind}(y, H) = 2$. *Let $\hat{\mathcal{O}} \subset \hat{\mathcal{M}}$ be the image of all maps $\#$ as constructed above. Then $\hat{\mathcal{M}} \setminus \hat{\mathcal{O}}$ is compact.*

Geometrically Theorem 26 identifies the ends of the components in $\hat{\mathcal{M}}(x, z)$ diffeomorphic to $(0, 1)$ with broken trajectories from $x$ to $z$ "factorizing" over $y$. We have the following figure explaining this.



The maps $\#$ are not natural, however given orientations of $\hat{\mathcal{M}}(x, y)$, $\hat{\mathcal{M}}(y, z)$ they induce a natural orientation of $\mathcal{M}(x, z)$ which is not depending on the choice of $\#$ as long as $\#$ is obtained from the glueing procedure as explained in [17, 19].

In order to obtain symplectic homology with arbitrary coefficients we have to orient the $\mathcal{M}(x, y)$'s so that the orientations are preserved and compatible with the glueing procedure. This is done in the next section.

## 4 Symplectic homology

### *4.1 Coherent orientations*

Let $(L, \hat{J}) \in \mathcal{N}_{\mathrm{reg}}((H, J); (K, \tilde{J}))$. Note that this includes the special case $(H, J) \in \mathcal{N}_{\mathrm{reg}}$. For $x \in \mathcal{P}_H$ and $y \in \mathcal{P}_K$ and $p > 2$ we consider the previously introduced Fredholm map

$$\partial_{L, \hat{J}} : \mathcal{B}^{1,p}(x, y) \to L^p(Z, \mathbb{C}^n),$$

where the space $\mathcal{B}^{1,p}$ has been previously constructed. If $u \in \mathcal{M}(x, y;\ L, \hat{J})$ denote by

$$\partial'_{L, \hat{J}}(u) \colon H^{1,p}(Z, \mathbb{C}^n) \to L^p(Z, \mathbb{C}^n)$$

the linearization at $u$. Since $(L, \hat{J}) \in \mathcal{N}_{\mathrm{reg}}$ the linearization is a surjection and the tangent space of $\mathcal{M}(x, y; L, \hat{J})$ at $u$ is precisely the kernel of $\partial'_{L, \hat{J}}(u)$, i.e.

$$T_u \mathcal{M}(x, y;\ L, \hat{J}) = \{u\} \times \mathrm{kern}(\partial'_{L, \hat{J}}(u)).$$

For a finite dimensional $\mathbb{R}$-vectorspace $E$ let $\wedge^{\max} E := E \wedge \ldots \wedge E$ ($\dim E$-times) with the convention $\wedge^{\max}\{0\} = \mathbb{R}$. With the definition of the determinant of a Fredholm map, see [22], we have

$$\begin{aligned}
&\mathrm{Det}\,(\partial'_{L, \hat{J}}(u)) \\
&\quad := (\wedge^{\max} \mathrm{kern}(\partial'_{L, \hat{J}}(u))) \otimes (\wedge^{\max} \mathrm{cokern}\,(\partial'_{L, \hat{J}}(u)))^* \\
&\quad = (\wedge^{\max} \mathrm{kern}\,(\partial'_{L, \hat{J}}(u)) \otimes \mathbb{R}^* \\
&\quad = (\wedge^{\max}(T_u \mathcal{M}(x, y;\ L, \hat{J}))) \otimes \mathbb{R}^* \\
&\quad \simeq \wedge^{\max} T_u \mathcal{M}(x, y; L, \hat{J})).
\end{aligned}$$

Letting $u$ vary over $\mathcal{M}(x, y;\ L, \hat{J})$ we obtain the determinant bundle of the family of linear Fredholm operators $u \to \partial'_{L, \hat{J}}(u)$, which is isomorphic to the maximal wedge of the tangent bundle $T\mathcal{M}(x, y; L, \hat{J})$. Hence an orientation of

$$\mathrm{Det}(\partial'_{L, \hat{J}}(*)) \to \mathcal{M}(x, y;\ L, \hat{J})$$

is equivalent to an orientation of $\mathcal{M}(x, y; L, \hat{J})$.

To recall the results in [21] (we note that we studied in [21] $\bar{\partial}$-operators and here $\partial$-operators, which of course does not make any difference) let us define the asymptotic operators of $\partial'_{L, \hat{J}}(u)$. In order to simplify notation we put $T := \partial'_{L, \hat{J}}(u)$. Then the asymptotic operators $T^{\pm}$ are defined by

$$T^{\pm} \colon H^{1,2}(S^1, \mathbb{C}^n) \to L^2(S^1, \mathbb{C}^n)$$

$$\begin{aligned}
(T^{\pm} h)(t) = &- \hat{J}(\pm\infty, t)\, h_t(t) - (D_3 \hat{J}(\pm\infty, t, x^{\pm}(t))\, h(t))\, x_t^{\pm}(t) \\
&- D_3(\nabla_{\hat{J}} L)(\pm\infty, t, x^{\pm}(t))\, h(t),
\end{aligned}$$

where $x^- = x$ and $x^+ = y$. The operators $T^\pm$ are selfadjoint unbounded operators in $L^2(S^1, \mathbb{C}^n)$ for suitable $L^2$-inner products. Namely

$$(h, k)^\pm = \int_0^1 g_{\bar{J}}(\pm\infty, t, x^\pm(t))(h(t), k(t))dt .$$

The operators of "type" $T$ belong to the class $\Sigma$ introduced in [21]. To make this precise consider the two point compactification $\bar{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ equipped with the unique differentiable structure turning the homeomorphism

$$h: \bar{\mathbb{R}} \to [-1, 1]$$

defined by

$$h(s) = \begin{cases} \pm 1 & \text{for } s = \pm\infty \\ s(1+s^2)^{-\frac{1}{2}} & \text{for } s \in \mathbb{R} \end{cases}$$

into a diffeomorphism. Let us put $\bar{Z} = \bar{\mathbb{R}} \times S^1$. For $J(z)$ being an $\omega$-calibrated complex structure depending smoothly on $z \in \bar{Z}$ we have asymptotic structures $(J(\pm\infty, *))$ depending on $t \in S^1$. Associated to $J$ there is an inner product $(*, *)_J$ on $L^2(Z, \mathbb{C}^n)$ given by

$$(h, k)_J = \int_Z \omega \circ (h(s, t), J(s, t)\, k(s, t))dsdt . \tag{112}$$

Moreover we have asymptotic inner products given by

$$(h, k)_J^\pm = \int_{S_1} \omega(h(t), J(\pm\infty, t)\, k(t))dt . \tag{113}$$

$\Sigma$ consists now of all first order partial differential operators defined by

$$(Tu)(s, t) = u_s(s, t) - J(s, t)\, u_t(s, t) - A(s, t)u(s, t) , \tag{114}$$

where $(s, t) \in Z$. Here $J$ is as described above and the map $\bar{Z} \ni (s, t) \to A(s, t) \in \mathscr{L}_{\mathbb{R}}(\mathbb{C}^n)$ is smooth such that the asymptotic operators defined by

$$(T^\pm h)(t) = -J(\pm\infty, t)h_t(t) - A(\pm\infty, t)h(t) \tag{115}$$

satisfy

$$\operatorname{kern}(T^\pm) = \{0\} \tag{116}$$

$$(T^\pm h, k)_J^\pm = (h, T^\pm k)_J^\pm$$

and are selfadjoint operators in $L^2(Z, \mathbb{C}^n)$ with domain $H^{1,2}(Z, \mathbb{C}^n)$ (for the particular inner products). Here one should note that the operators $\bar{\partial}'_{L,\bar{J}}(u)$ introduced previously induce operators in $L^2(Z, \mathbb{C}^n)$.

Fixing asymptotic operators $\alpha, \beta$ we consider the set $\theta_{\alpha,\beta}$ consisting of all operators $T$ in $\Sigma$ with $T^- = \alpha$, $T^+ = \beta$, equipped with the topology induced from $\mathscr{L}_{\mathbb{R}}(H^{1,2}(Z, \mathbb{C}^n), L^2(Z, \mathbb{C}^n)$. The natural determinant bundle over $\theta_{\alpha,\beta}$ (all $T$ in $\theta_{\alpha,\beta}$ are Fredholm operators) is trivial and hence orientable, see [21] for more details.

Given asymptotic operators $\alpha, \beta, \gamma$ and having orientation for $\mathrm{Det}(\theta_{\alpha,\beta})$ and $\mathrm{Det}(\theta_{\beta,\gamma})$ there is a naturally induced orientation for $\mathrm{Det}(\theta_{\alpha,\gamma})$ obtained by glueing, see [21]. Formally

(117) $$o_{\alpha,\beta} \# o_{\beta,\gamma} = o_{\alpha,\gamma} .$$

A coherent orientation is a choice of orientation for every $\theta_{\alpha,\beta}$ such that the formula (117) holds for every triplet of asymptotic operators. Consider the group $\Gamma$ consisting of all maps $f$ associating to a pair $(\alpha, \beta)$ of asymptotic operators a number $f(\alpha, \beta) \in \{-1, 1\}$ such that

(118) $$f(\alpha, \beta) f(\beta, \gamma) = f(\alpha, \gamma)$$

holds for all triplets $(\alpha, \beta, \gamma)$. It has been shown that the group $\Gamma$ acts freely and transitively on the set of coherent orientations by

$$(f\sigma)(\alpha, \beta) = f(\alpha, \beta)\sigma(\alpha, \beta) ,$$

where $\sigma(\alpha, \beta)$ is the (coherent) orientation for $\theta_{\alpha,\beta}$.

Having fixed a coherent orientation $\sigma$ and taking a pair $(L, \hat{J}) \in \mathcal{N}_{\mathrm{reg}}(H, J; K\tilde{J})$ the manifold $\mathcal{M}(x, y; L, \hat{J})$ for $x \in \mathcal{P}_H$, $y \in \mathcal{P}_K$ carries an orientation in view of the previous discussion.

First assume $(H, J) \in \mathcal{N}_{\mathrm{reg}}$ is given and a coherent orientation $\sigma$. Let $x^{\pm} \in \mathcal{P}_H$ with $\mathrm{Ind}(x^-, H) - \mathrm{Ind}(x^+, H) = 1$. Then

$$\dim \hat{\mathcal{M}}(x^-, x^+; H, J) = 0$$
$$\# \hat{\mathcal{M}}(x^-, x^+, H, J) < \infty .$$

For every $\hat{u} \in \hat{\mathcal{M}}(x^-, x^+; H, J)$ the corresponding component of $\mathcal{M}(x^-, x^+; H, J)$ is given by $\{\rho * u \mid \rho \in \mathbb{R}\}$ and carries the orientation $\sigma(u) = \sigma(x^-, x^+; H, J)$ and a natural orientation given by $\left[\frac{\partial u}{\partial s}\right]$. We define a number $\tau_\sigma(u) \in \mathbb{Z}$ by

(119) $$\tau_\sigma(u)\,[u_s] = \sigma(u) .$$

If we take the coherent orientations $f\sigma$ with $f \in \Gamma$ we have

$$\tau_{f\sigma}(u)[u_s] = f_u(x^-, x^+; H, J)\tau_\sigma(u)[u_s] .$$

Hence

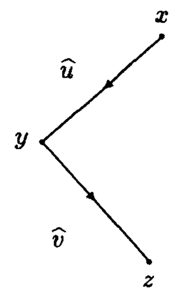
(120) $$\tau_{f\sigma}(u) = f_u(x^-, x^+; H, J)\tau_\sigma(u) .$$

Moreover given $(L, \hat{J}) \in \mathcal{N}_{\mathrm{reg}}(H, J; K, \tilde{J})$ and $x \in \mathcal{P}_H$, $y \in \mathcal{P}_k$ with $\mathrm{Ind}(x, H) - \mathrm{Ind}(y, K) = 0$ the set of connecting trajectories is a compact 0-dimensional manifold, see [17, 19], and every $w \in \mathcal{M}(x, y; L, \hat{J})$ carries an orientation $1 \otimes 1^*$ where $1^*(1) = 1$. We define a number $\tau(w) \in \{-1, 1\}$ by

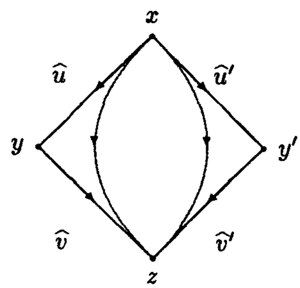
(121) $$\sigma(w) = \tilde{\tau}_\sigma(w)[1 \otimes 1^*] .$$

*4.2 Construction of chain complexes*

Let us assume $\hat{u} \in \hat{\mathcal{M}}(x, y; H, J)$ and $\hat{v} \in \hat{\mathcal{M}}(y, z; H, J)$ such that $\mathrm{Ind}(x, H) = \mathrm{Ind}(y, H) + 1 = \mathrm{Ind}(z, H) + 2$, where $(H, J) \in \mathcal{N}_{\mathrm{reg}}$. We have the associated figure:




The glueing construction [17, 19], precisely explained in [34, 38], shows that the ends of $\hat{\mathcal{M}}(x, z)$ correspond bijectively to broken trajectories. This gives the following picture




The glueing construction shows immediately that the glued orientation for $u, v, u', v'$ must be related as follows

(122) $$[u_s] \# [v_s] = -[u_s'] \# [v_s'] .$$

For any coherent orientation $\sigma$ we have

$$\sigma(u) \; \# \sigma(v) = \sigma(u') \# \sigma(v') .$$

Hence using the definition of $\tau_\sigma$ we obtain the formula

(123) $$\tau_\sigma(u)\tau_\sigma(v) + \tau_\sigma(u') \, \tau_\sigma(v') = 0 .$$

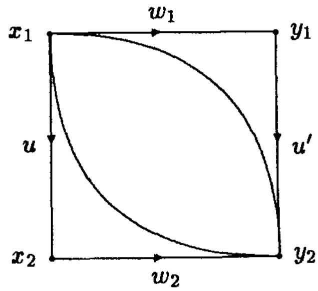
Next assume $(L, \hat{J}) \in \mathcal{N}_{\text{reg}}(H, J; K, \tilde{J})$ is given. Then let $x_1, x_2 \in \mathcal{P}_H$, $y_1, y_2 \in \mathcal{P}_K$ with

(124) $$\operatorname{Ind}(x_i, H) = \operatorname{Ind}(y_i, K) \quad \text{for } i = 1, 2$$

$$\operatorname{Ind}(x_1, H) - \operatorname{Ind}(x_2, H) = 1$$

$$\operatorname{Ind}(y_1, K) - \operatorname{Ind}(y_2, K) = 1 .$$

Again the ends of $\mathcal{M}(x_1, y_2; L, \hat{J})$ correspond to broken trajectories. See the picture




Glueing gives

$$\sigma(w_1)\#\sigma(u') = \sigma(u)\#\sigma(w_2)$$

and

$$[1 \otimes 1^*]\#[u_s'] = [u_s]\#[1 \otimes 1^*] .$$

Hence using the definition of $\tau_\sigma$ we obtain the formula

$$\begin{aligned}(125)\qquad \tilde{\tau}_\sigma(w_1)\,\tau_\sigma(u')\,[1 \otimes 1^*]\#[u_s'] &= \sigma(w_1)\#\sigma(u') \\ &= \sigma(u)\#\sigma(w_2) \\ &= \tau_\sigma(u)\tilde{\tau}_\sigma(w_2)\,[u_s]\#[1 \otimes 1^*] \\ &= \tau_\sigma(u)\tilde{\tau}_\sigma(w_2)[1 \otimes 1^*]\#[u_s'] .\end{aligned}$$

Hence

$$(126)\qquad \tilde{\tau}_\sigma(w_1)\tau_\sigma(u') - \tau_\sigma(u)\,\tilde{\tau}_\sigma(w_2) = 0 .$$

Now let $a \in (-\infty, +\infty]$. We define for $(H, J) \in \mathcal{N}_{reg}$ and $k \in \mathbb{Z}$ the free Abelian group $C_k^a(H, J)$ by

$$(127)\qquad C_k^a(H, J) = \bigoplus\{\mathbb{Z}x \mid x \in \mathscr{P}_H, \Phi_H(x) < a\ , \operatorname{Ind}(x, H) = k\}$$

and the graded Abelian group $C_*^a(H, J)$ by

$$(128)\qquad C_*^a(H, J) = \bigoplus_{k \in \mathbb{Z}} C_k^a(H, J) .$$

For a fixed coherent orientation $\sigma$ we define a boundary operator $\partial^\sigma$ by

$$(129)\qquad \partial^\sigma\colon C_k^a(H, J) \to C_k^a(H, J)$$

$$\partial^\sigma x = \sum_{\operatorname{Ind}(y,H) = \operatorname{Ind}(x,H) - 1} \tau_\sigma(x, y)\,y\ ,$$

where

$$(130)\qquad \tau_\sigma(x, y) := \sum_{\hat{u} \in \hat{\mathscr{M}}(x,y;H,J)} \tau_\sigma(u) \in \mathbb{Z} .$$

We observe that

$$(131)\qquad \tau_{f\sigma}(x, y) = f(x, y; H, J)\,\tau_\sigma(x, y)\ ,$$

where $f(x, y) \in \{-1, 1\}$ and $f(x, y) f(y, z) = f(x, z)$. (In fact $f \in \Gamma$, the precise notation should be $f(\alpha, \beta)$ where $\alpha$ is the asymptotic operator at $x$, etc.) Given different orientations we may apriori obtain different theories. To study this question let $\sigma' = f\sigma$. We write $f$ in the form

$$f(\alpha, \beta) = \rho(\alpha)\, \rho(\beta)\,, \tag{132}$$

where $\rho$ is a function associating to an asymptotic operator a number in $\{-1, 1\}$. Given $H \in \mathscr{H}_{\text{reg}}$ and $J \in \mathscr{J}$ we denote by $\alpha(x, H, J)$ the asymptotic operator determined by $(H, J)$ over the periodic solution $x \in \mathscr{P}_H$. Having $(H, J) \in \mathscr{N}_{\text{reg}}$ fixed we shall often write $\alpha(x)$ instead of $\alpha(x, H, J)$. We observe that in (132) $\rho$ is determined uniquely up to sign. Given a $\rho$ as in (132) we define a map

$$\lambda_\rho\colon (C_*^a(H, J), \partial^\sigma) \to (C_*^a(H, J), \partial^{\sigma'})$$

by

$$x \to \rho(\alpha(x))\, x \tag{133}$$

for $x \in \mathscr{P}_H$. We compute using (131) and the fact $f(x, y; H, J) = \rho(\alpha(x))\, \rho(\alpha(y))$ with $\rho(x) := \rho(\alpha(x))$

$$\begin{aligned} \partial^{\sigma'} \lambda_\rho(x) &= \sum_y \tau_{\sigma'}(x, y)\, \rho(x)\, y \\ &= \sum_y \rho(x)^2 \rho(y)\, \tau_\sigma(x, y) y \\ &= \sum_y \rho(y)\, \tau_\sigma(x, y) y \\ &= \lambda_\rho \partial^\sigma(x)\,. \end{aligned} \tag{134}$$

Hence $\lambda_\rho$ defines a chain isomorphism. Due to the ambiguity of the choice of the signs we have the situation

$$(C_*^a(H, J), \partial^\sigma) \rightrightarrows (C_*^a(H, J), \partial^{\sigma'}) \tag{135}$$

saying that between two objects there are precisely two isomorphisms. The "double arrow" is natural. Of course what we would like to have is precisely one arrow between two different objects. In order to do so we have to construct in some sense a "double covering" of the category with different lifts for the two arrows.

In order to follow the above scheme we introduce the capped half cylinder $Z_c := D \cup ([0, +\infty) \times S^1)$, where $\partial D \cong S^1$ is identified with $S^1 \cong \{0\} \times S^1$. We equip $Z_c$ with a complex structure denoted by $i$ which on $[0, +\infty) \times S^1$ is precisely the one induced from $Z$, i.e. $\frac{\partial}{\partial s} \to \frac{\partial}{\partial t}$, $\frac{\partial}{\partial t} \to -\frac{\partial}{\partial s}$, where $(s, t)$ are the coordinates on $[0, +\infty) \times S^1$. We assume $Z_c$ to be a $C^\infty$-manifold. We write $\bar{Z}_c$ for $Z_c \cup (\{+\infty\} \times S^1)$. For a $\omega$-calibrated almost complex structure $J$ for $\bar{Z}_c \times \mathbb{C}^n \to \bar{Z}_c$ we define a vectorbundle $X_J \to \bar{Z}_c$ as follows. The fibre over $z \in \bar{Z}_c$ consists of all real linear maps $\phi\colon T_z \bar{Z}_c \to \mathbb{C}^n$ such that

$$J(z)\phi - \phi i = 0\,. \tag{136}$$

Hence $X_{J, z} \subset \mathscr{L}_{\mathbb{R}}(T_z \bar{Z}_c, \mathbb{C}^n)$. So we may assume $X_J$ is a subbundle of $\mathscr{L}_{\mathbb{R}}(T\bar{Z}_c, \bar{Z}_c \times \mathbb{C}^n)$. $X_J$ splits $\mathscr{L}_{\mathbb{R}}$, where the splitting is smoothly depending on

$X_J$ (it is the canonical splitting into complex linear and complex antilinear maps.) We introduce the Hilbert spaces $H^{1,2}(Z_c \times \mathbb{C}^n)$ and $L^2(X_J)$ of sections in the obvious way. Observe that $L^2(X_J)$ is a subspace of $L^2(\mathscr{L}_{\mathbb{R}}(TZ_c, Z_c \times \mathbb{C}^n))$. We introduce a class $\Sigma_c$ of linear operators $L$ as follows. $L \in \mathscr{L}(H^{1,2}(Z_c \times \mathbb{C}^n), L^2(\mathscr{L}_{\mathbb{R}}(TZ_c, Z_c \times \mathbb{C}^n)))$ is said to belong to $\Sigma_c$ if there exists a $J$ as above and a smooth section $A\colon \bar{Z}_c \to \mathscr{L}_{\mathbb{R}}(\mathbb{C}^n, X_J)$ such that

$$(Lu)(z) = Tu(z) - J(z)\, Tu(z)i + A(z)u(z)$$

for $z \in Z_c$. Moreover over $Z_+ = [0, +\infty) \times S^1$ we have that

$$\begin{aligned}(Lu)(z)\left(\frac{\partial}{\partial s}\right) &= \frac{\partial u}{\partial s}(s,t) - J(s,t)\,\frac{\partial u}{\partial t}(s,t) - (A(s,t)u(s,t))\left(\frac{\partial}{\partial s}\right)\\ &=: \frac{\partial u}{\partial s}(s,t) - J(s,t)\,\frac{\partial u}{\partial t}(s,t) - \hat{A}(s,t)\,u(s,t)\end{aligned}$$

is the restriction of an operator $\tilde{L}$ of class $\Sigma$ to $Z_+$. We have following [19]

**Theorem 27** *Every $L \in \Sigma_c$ is Fredholm. L has the asymptotic operator*

$$h \to -J(+\infty, t)\, h(t) - \hat{A}(+\infty, t)\, h(t)\,.$$

Having a fixed asymptotic operator $\alpha$ we denote by $\theta^c_\alpha$ the collection of all those $L$ and put on them the topology induced from $\mathscr{L}(H^{1,2}(Z_c \times \mathbb{C}^n), L^2(\mathscr{L}_{\mathbb{R}}(TZ_c, Z_c \times \mathbb{C}^n)))$. As in [21], one shows that $\theta^c_\alpha$ is a contractable space. We denote by $o(\alpha)$ an orientation of the canonical determinant bundle over $\theta^c_\alpha$, see [21]. Using the glueing construction for linear operators as explained in [21], we have an almost natural procedure to glue an operator $L \in \theta^c_\alpha$ with an operator $T \in \theta_{\alpha,\beta}$ to obtain some operator $L \# T \in \theta^c_\beta$. Although the glueing is not naturally the orientation $o(\beta)$ induced from an orientation $o(\alpha)$ of $\theta^c_\alpha$ and $o(\alpha, \beta)$ of $\theta_{\alpha,\beta}$ by the glueing construction does not depend on the glueing parameters.

Consider now pairs $(\sigma, \theta^c_{\alpha_0})$ consisting of a coherent orientation $\sigma$ and an oriented class of operators $\theta^c_{\alpha_0}$ with orientation $o(\alpha_0)$. Assume $(\sigma', \theta^c_{\alpha'_0})$ is a second such pair. $\sigma$ and $o(\alpha_0)$ determine via glueing an orientation $\sigma(\alpha_0) \# \sigma(\alpha_0, \alpha'_0)$ of $\theta^c_{\alpha'_0}$. We have $f\sigma = \sigma'$ and choose a potential $\rho$ for $f$ according to the following requirements

$$\begin{aligned} o(\alpha'_0) &= \rho(\alpha'_0)\, o(\alpha_0) \# \sigma(\alpha_0, \alpha'_0)\\ f(\alpha, \beta) &= \rho(\alpha)\, \rho(\beta)\,. \end{aligned}$$

We observe that $\rho$ is uniquely determined by these requirements. Let us denote this $\rho$ by $\rho_{(\sigma', \alpha'_0, \sigma, \alpha_0)}$. Assume a third pair is given, say $(\sigma'', \alpha''_0)$. We have

$$\begin{aligned} o(\alpha''_0) &= \psi(\alpha''_0) o(\alpha_0) \# \sigma(\alpha_0, \alpha''_0)\\ (gf)(\alpha, \beta) &= \psi(\alpha)\psi(\beta) \end{aligned}$$

with $\sigma'' = g\sigma'$, $\sigma' = f\sigma$. Then

$$\begin{aligned} o(\alpha''_0) &= \psi(\alpha''_0)\, o(\alpha_0) \# \sigma(\alpha_0, \alpha'_0) \# \sigma(\alpha'_0, \alpha''_0)\\ &= \psi(\alpha''_0)\; \rho(\alpha'_0) o(\alpha'_0) \# \sigma(\alpha'_0, \alpha''_0)\,. \end{aligned}$$

If $\hat{\rho} = \rho_{(\sigma'', \alpha_0''; \sigma' = \alpha_0')})$, $\rho = \rho_{(\sigma', \alpha_0'; \sigma, \alpha_0)}$ we deduce

$$\hat{\rho}(\alpha_0'')o(\alpha_0')\# \sigma'(\alpha_0', \alpha_0'') = \psi(\alpha_0'')\, \rho(\alpha_0')\, o(\alpha_0')\, \# \sigma(\alpha_0', \alpha_0'')$$

$$= \psi(\alpha_0'')\, \rho(\alpha_0') o(\alpha_0') \# (f(\alpha_0', \alpha_0'')\, \sigma'(\alpha_0', \alpha_0'')) .$$

Hence

$$\hat{\rho}(\alpha_0'') = \psi(\alpha_0'')\, \rho(\alpha_0')\, \rho(\alpha_0')\, \rho(\alpha_0'')$$

which is equivalent to

$$\psi(\alpha_0'') = \hat{\rho}(\alpha_0'')\, \rho(\alpha_0'') .$$

This implies immediately that

$$\rho \hat{\rho} = \psi$$

or in other words

$$\rho_{(\sigma'', \alpha_0''; \sigma, \alpha_0)} = \rho_{(\sigma'', \alpha_0''; \sigma', \alpha_0')}\, \rho_{(\sigma', \alpha_0'; \sigma, \alpha_0)} .$$

Now we define for a pair $(\sigma, \alpha_0)$ the chain complex

$$C_*^a(H, J; (\sigma, \alpha_0)) := (C_*^a(H, J), \partial^\sigma)$$

and between two different chain complexes a chain isomorphism $\lambda_{(\sigma', \alpha_0'; \sigma, \alpha_0)} := \lambda_{\rho(\sigma', \alpha_0'; \sigma, \alpha_0)}$ via the formula (133). Then

$$\lambda_{(\sigma', \alpha_0'; \sigma, \alpha_0)} \colon C_*^a(H, J; \sigma, \alpha_0) \to C_*^a(H, J, \sigma', \alpha_0')$$

defines a connected simple system.

### *4.3 Monotonicity homotopies*

In view of the naturality of the construction of the chain complexes in respect to the chosen data $(\sigma, \alpha_0)$ we may assume for the following that $(\sigma, \alpha_0)$ is fixed.

Suppose now $(H, J) \leqq (K, \tilde{J})$ are both belonging to $\mathcal{N}_{\text{reg}}$ and $(L, \hat{J}) \in \mathcal{N}_{\text{reg}}(H, J;\ K, \tilde{J})$. For $a \in (-\infty, \infty]$ and dropping $(\sigma, \alpha_0)$ in our notation we obtain an induced chain map

$$\Psi_{(L, \hat{J})} \colon C_k^a(H, J) \to C_k^a(K, \tilde{J})$$

via the formula

$$\Psi_{(L, \hat{J})}(x) = \sum_{\substack{y \in \mathcal{P}_K \\ \operatorname{Ind}(y, K) = \operatorname{Ind}(x, H) = k}} \tilde{\tau}(x, y) ,$$

where

$$\tilde{\tau}(x, y) := \tilde{\tau}_\sigma(x, y) = \sum \tilde{\tau}_\sigma(w)$$

with $\tilde{\tau}_\sigma$ defined in (121) and the sum being taken over all $w \in \mathcal{M}(x, y; L, \hat{J})$. In view of (126) $\Psi_{(L, \hat{J})}$ defines a chain homomorphism.

A crucial point we have to discuss is the following. Given another regular monotone homotopy the induced map is naturally chain homotopic. This will allow ourselves to regard $C_*^a$ as a covariant functor associating to $(H, J) \in \mathcal{N}_{\text{reg}}$ a graded Abelian group and to an "inequality sign" a chain homotopy class of maps. So let $(L_0, \hat{J}_0)$ and $(L_1, \hat{J}_1)$ be elements of $\mathcal{N}_{\text{reg}}(H, J; K, \tilde{J})$. Arguing as in Sect. 3 we can construct a regular homotopy of monotone homotopies

$\lambda \to (L_\lambda, \hat{J}_\lambda)$, $\lambda \in \mathbb{R}$ such that

$$(L_s, \hat{J}_s) = (L_0, \hat{J}_0) \quad \text{for } s \leqq 0$$

$$(L_s, \hat{J}_s) = (L_1, \hat{J}_1) \quad \text{for } s \geqq 1 ,$$

so that the operator (for $p > 2$)

$$\mathscr{B}\colon \mathbb{R} \times \mathscr{B}^{1,p}(x, y) \to L^p(Z, \mathbb{C}^n)\colon (\lambda, u) \to \bar{\partial}_{(L_\lambda, \hat{J}_\lambda)}(u)$$

has zero as a regular value. We are mainly interested in the cases

$$\mathrm{Ind}(x, H) - \mathrm{Ind}(y, K) \in \{-1, 0\} .$$

Then $\mathrm{Ind}(T) \in \{0, 1\}$ in view of the additional $\lambda$-parameter and our index formula. Our aim is to prove the formula

$$\Psi_{(L_1, \hat{J}_1)} - \Psi_{(L_0, \hat{J}_0)} = \partial \Lambda - \Lambda \partial \tag{137}$$

for a suitable homotopy operator $\Lambda$. Clearly (137) implies the desired conclusion. We shall proceed as follows. Firstly we give an explicit definition for $\Lambda$ and then secondly we derive the equality (137). In order to proceed it is useful to define a coherent orientation for a somewhat larger class of operators. Consider operators

$$\hat{T}\colon \mathbb{R} \times H^{1,2}(Z, \mathbb{C}^n) \to L^2(Z, \mathbb{C}^n)$$

$$\hat{T}(\lambda, H) = Th + \lambda a ,$$

where $T$ is of class $\Sigma$ as introduced in Subsect. 4.1 and $a \in L^2(Z, \mathbb{C}^n)$. We have an exact sequence

$$0 \to \mathrm{kern}(T) \xrightarrow{d_1} \mathrm{kern}(\hat{T}) \xrightarrow{d_2} \mathbb{R} \xrightarrow{d_3} \mathrm{cokern}(T) \xrightarrow{d_4} \mathrm{cokern}(\hat{T}) \to 0 , \tag{138}$$

where

$$\mathrm{d}_1(h) = (o, h)$$

$$d_2(\lambda, h) = \lambda$$

$$d_3(\lambda) = \lambda a + R(T)$$

$$d_4(h + R(T)) = h + R(\hat{T}) .$$

A simple algebraic lemma, see [21], shows that (138) induces a natural isomorphism

$$\Lambda^{\max} \mathrm{kern}(T) \otimes \mathbb{R} \otimes \Lambda^{\max} \mathrm{cokern}(\hat{T}) \xrightarrow{\sim} \Lambda^{\max} \mathrm{kern}(\hat{T}) \otimes \Lambda^{\max} \mathrm{cokern}(T) .$$

Multiplying by $(\Lambda^{\max} \mathrm{cokern}(\hat{T}))^* \otimes (\Lambda^{\max} \mathrm{cokern}(T))^*$ and using the natural isomorphisms $E \otimes F \simeq F \otimes E$, $E \otimes E^* \simeq \mathbb{R}$, $E \otimes \mathbb{R} \simeq E$ we obtain a natural homomorphism

$$\mathrm{Det}(T) \xrightarrow{\sim} \mathrm{Det}(\hat{T}) . \tag{139}$$

Independent of the choice of $a \in L^2$ an operator $\hat{T}_a$ has an asymptotics given of course by the asymptotics of $T$. For given asymptotic operators $\alpha$, $\beta$ we denote by $\hat{\theta}_{\alpha,\beta}$ the set consisting of all $\hat{T}_a$, with $T \in \theta_{\alpha,\beta}$, $a \in L^2$. We have a glueing procedure, see [21], which allows to construct for an orientation $\hat{o}_{\alpha,\beta}$ of $\hat{\theta}_{\alpha,\beta}$ and an orientation

$\hat{o}_{\beta,\gamma}$ of $\hat{\theta}_{\beta,\gamma}$ an orientation $o_{\alpha,\beta}\#\hat{o}_{\beta,\gamma}$ of $\hat{\theta}_{\alpha,\gamma}$ and similarly for $\hat{o}_{\hat{\theta}\alpha,\beta}$ of $\theta_{\alpha,\beta}$ and $o_{\beta,\gamma}$ of $\theta_{\beta,\gamma}$ a $\hat{o}_{\alpha\beta}\# o_{\beta\gamma}$ of $\hat{\theta}_{\alpha\gamma}$. The isomorphism in (139) produces from an $o_{\alpha\beta}$ a $\hat{o}_{\alpha\beta}$. The following formulae are easily established given a coherent orientation $\sigma$

$$
\begin{aligned}
(\sigma(\alpha,\beta)\#\sigma(\beta,\gamma))^{\wedge} &= \sigma(\alpha,\beta)^{\wedge}\#\sigma(\beta,\gamma)\\
&= \sigma(\alpha,\beta)\ \#\sigma(\beta,\gamma)^{\wedge}\ .
\end{aligned}
$$

Here $\sigma(\alpha,\beta)$ stands for the orientation of the determinant bundle over $\theta_{\alpha\beta}$ etc.

Now let $x\in\mathscr{P}_H$ and $y\in\mathscr{P}_K$ with $\operatorname{Ind}(x,H)-\operatorname{Ind}(y,K)=-1$. Then the Fredholm index of the nonlinear operator

$$
\begin{aligned}
B:\mathbb{R}\times\mathscr{B}^{1,p}(x,y)&\to L^p(2,\mathbb{C}^n)\\
B(\lambda,u)&=\bar{\partial}_{(H_\lambda,\tilde{J}_\lambda)}(u)
\end{aligned}
$$

is zero. In our generic case the linearization $B'(\lambda,w)$: $\mathbb{R}\times H^{1,p}(Z,\mathbb{C}^n)\to L^p(Z,\mathbb{C}^n)$ is an isomorphism at every zero $(\lambda,w)$ of $B$. We note that $B'(\lambda,w)$ also induces an isomorphism $\mathbb{R}\times H^{1,2}(Z,\mathbb{C}^n)\to L^2(Z,\mathbb{C}^n)$. For simplifying notation we denote the regular homotopy $\lambda\to(L_\lambda,\hat{J}_\lambda)$ by $\lambda\to M(\lambda)$. For $x$ and $y$ as above we define

$$
\begin{aligned}
\mathscr{M}(x,y;\ M(*)) = \{(\lambda,w)\,|\,w:Z\to\mathbb{C}^n \text{ smooth}, \lambda\in\mathbb{R}\\
B(\lambda,w)=0,\ w(s,*)\to x \text{ as}\\
s\to-\infty,\ w(s,*)\to y, \text{ as } s\to+\infty\}\ .
\end{aligned}
$$

Under the previous index assumption the manifold $\mathscr{M}(x,y;\ M(*))$ consists of finitely many points[1], say $(\lambda_j,w_j)_{j=1,\dots,l}$ at which $B'(\lambda_j,w_j)$ is an isomorphism. We take as orientation of $\operatorname{Det}(B'(\lambda_j,w_j))$ the orientation $[1\otimes 1^*]$. The coherent orientation $\sigma$ gives an orientation $\hat{\sigma}(B'(\lambda_j,w_j))$. We define a number $\hat{\tau}(\lambda_j,w_j)\in\{-1,1\}$ by

(140) $$\hat{\sigma}(B'(\lambda_j,w_j))=\hat{\tau}(\lambda_j,w_j)[1\otimes 1^*]\ .$$

Finally we define $\hat{\tau}(x,y)\in\mathbb{Z}$ by

(141) $$\hat{\tau}(x,y):=\sum_{j=1}^{l}\hat{\tau}(\lambda_j,w_j)\ .$$

Then we put

(142) $$
\begin{aligned}
\Lambda(x) &= \sum\hat{\tau}(x,y)y\\
\Lambda\colon C_k^a(H,J) &\to C_{k+1}^a(K,\tilde{J})\,,
\end{aligned}
$$

where the sum is taken over all $y\in\mathscr{P}_K$ with $\operatorname{Ind}(x,H)-\operatorname{Ind}(y,K)=-1$. Our aim is to verify the formula (137). The proof of (137) is based on a study of $\mathscr{M}(x,z;\ M(*))$ for $x\in\mathscr{P}_H$, $y\in\mathscr{P}_K$ with $\operatorname{Ind}(x,H)=\operatorname{Ind}(y,K)$. Since the data is assumed to be generic a compactness argument(in the spirit of those previously used) shows that $\mathscr{M}(x,z;\ M(\cdot))$ decomposes into finitely many components, which are(taking orientation preserving maps) diffeomorphic to $[0,1]$, $(0,1]$, $[0,1)$, $(0,1)$ or $S^1$, if we consider only the parts of the components in $[0,1]\times\mathscr{B}^{1,p}(x,z)$. This is illustrated by Fig. 1.

[1] By a compactness argument and regularity

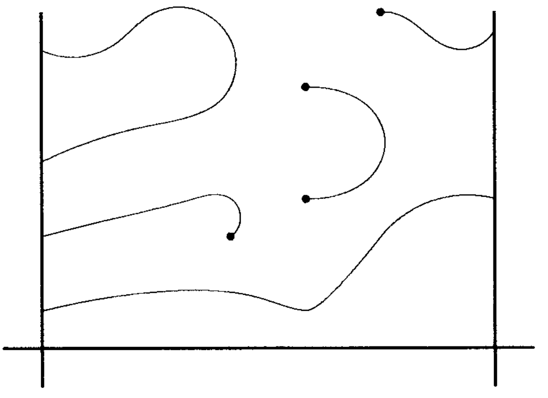



**Fig. 1.**

Let us write $\Psi_i = \Psi_{(L_i, \tilde{J}_i)}$ for $i = 0, 1$. By construction

$$(\Psi_1 - \Psi_0)(x) = \sum (\tilde{\tau}_1(x, z) - \tilde{\tau}_0(x, z))z ,$$

where $\tilde{\tau}_i(x, z) = \sum \tilde{\tau}_i(w)$ with $\tilde{\tau}_i(w)$ defined in(121) for $i = 0, 1$.

Through every $(0, w)$ or $(1, w)$ where $w$ is as above there goes exactly one component as depicted in Fig. 1. Let us show that in the formula for $\Psi_1 - \Psi_0$ only those $w$ count for which $(0, w)$ or $(1, w)$ is contained in a component of type $[0, 1)$ or $(0, 1]$. So let us assume for example $(0, w)$ lies on a component of type $[0, 1]$. By definition

$$\sigma(w) = \tilde{\tau}_0(w)\ [(1 \otimes 1^*)_w] .$$

The other end of the component corresponds to a $(\varepsilon, w')$ for $\varepsilon \in \{0, 1\}$. Let $T$: $[0, 1] \times \mathscr{B}^{1,p}(x, z) \to L^p$ with $p > 2$ be the obvious maps. Using elliptic regularity theory we always consider the linearization of $T$ as an operator in a $L^2$-set up. The first case is $\varepsilon = 0$ We know that $\hat{\sigma}(DT(0, w))$ and $\hat{\sigma}(DT(0, w'))$ are related by continuation and also $([1 \otimes 1^*]$ at $(0, w))$ and $([-1 \otimes 1^*]$ at $(0, w'))$. Hence

$$\tilde{\tau}_0(w) = -\tilde{\tau}_0(w') .$$

If $\varepsilon = 1$ we infer similarly

$$\tilde{\tau}_0(w) = \tilde{\tau}_1(w') .$$

Let $\Gamma_\varepsilon(z) = \{(\varepsilon, w) \mid$ There exists a component of type $[0, 1)$ or $(0, 1)$ through $(\varepsilon, w)\}$.

We have shown

$$(\Psi_1 - \Psi_0)(x) = \sum_z \left( \sum_{(1, w) \in \Gamma_1(z)} \tilde{\tau}_1(w) \right) z - \left( \sum_{(0, w) \in \Gamma_0(z)} \tilde{\tau}_0(w) \right) z .$$

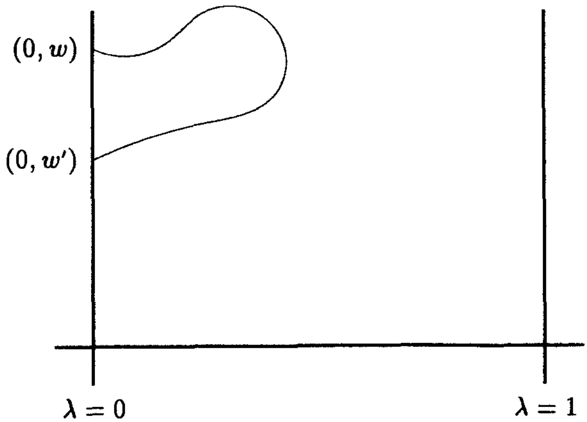
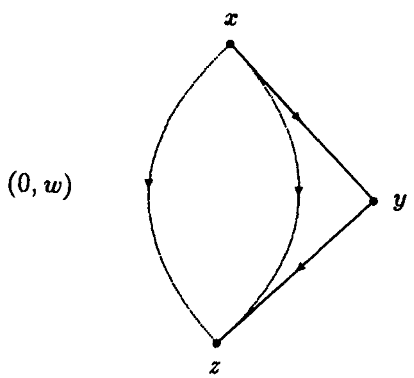



**Fig. 2.**

Next let $(0, w)\in\Gamma_0$. The component through $(0, w)$ has a non compact end which corresponds via glueing and the usual compactness analysis to a broken trajectory




There are two cases to be considered. The broken trajectory has the form $(\lambda', w')$ followed by some $u'\in\mathcal{M}(y, z, K, \tilde{J})$ or $u\in\mathcal{M}(x, y, H, J)$ followed by some $(\lambda', w')\in\mathcal{M}(y, z, M(*))$. Consider the first case.

The orientation $\hat{\sigma}(DT(0, w))$ and the glued orientation

$$\hat{\sigma}(DT(\lambda', w'))\#\sigma(\partial'_{K,\tilde{J}}(u'))$$

on a connecting orbit(near the broken one) are related by continuation. Also the orientation given by $(-1\otimes 1^*)\in\operatorname{kern}(DT(0, w))\otimes\mathbb{R}^*$ and the orientation obtained by glueing $1\otimes 1^*$ above $(\lambda', w')$ and $u'_s$ above $u'$ correspond. Hence

$$\begin{aligned}
-[(1\otimes 1^*)_{(0,w)}] &= [(1\otimes 1^*)_{(\lambda',w')}]\#[u'_s]\\
&= \hat{\tau}(\lambda', w')\hat{\sigma}(DT(\lambda', w'))\#\tau(u')\sigma(u')\\
&= \hat{\tau}(\lambda', w')\tau(u')\hat{\sigma}(DT(0, w))\\
&= \hat{\tau}(\lambda', w')\tau(u')\tilde{\tau}_0(w)[(1\otimes 1^*)_{(0,w)}]\,.
\end{aligned}$$

Therefore we obtain the formula

(143)
$$\tilde{\tau}_0(w) = -\hat{\tau}(\lambda', w')\tau(u') \quad \text{for} \quad (0, w) \rightsquigarrow \begin{array}{l} \bullet \\ \downarrow\, (\lambda', w') \\ \bullet \\ \downarrow\, u' \\ \bullet \end{array}$$

Similarly

(144)
$$\tilde{\tau}_0(w) = \tau(u)\hat{\tau}(\lambda', w') \quad \text{for} \quad (0, w) \rightsquigarrow \begin{array}{l} \bullet \\ \downarrow\, u \\ \bullet \\ \downarrow\, (\lambda', w') \\ \bullet \end{array}$$

Further

$$\tilde{\tau}_1(w') = \hat{\tau}(\lambda, w)\tau(u') \quad \text{for} \quad (0, w') \rightsquigarrow \begin{array}{l} \bullet \\ \downarrow\, (\lambda, w) \\ \bullet \\ \downarrow\, u' \\ \bullet \end{array}$$

(145) and

$$\tilde{\tau}_1(w') = -\tau(u)\hat{\tau}(\lambda, w) \quad \text{for} \quad (1, w') \rightsquigarrow \begin{array}{l} \bullet \\ \downarrow\, u \\ \bullet \\ \downarrow\, (\lambda, w) \\ \bullet \end{array}$$

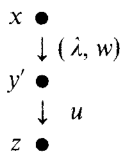
Hence we may write $\Psi_1(x) - \Psi_0(x)$ as follows:

$$\begin{aligned} \Psi_1(x) - \Psi_0(x) &= \sum_z \Bigg( \sum_{(1, w) \in \Gamma_1(z)} \tilde{\tau}_1(w) \Bigg) z - \sum_z \Bigg( \sum_{(0, w) \in \Gamma_0(z)} \tilde{\tau}_0(w) \Bigg) z \\ &= -\sum_z \Bigg( \sum_{(u, (\lambda, w))} \tau(u)\hat{\tau}(\lambda, w) \Bigg) z \\ &\quad + \sum_z \Bigg( \sum_{((\lambda, w), u)} \hat{\tau}(\lambda, w)\tau(u) \Bigg) z \,. \end{aligned}$$

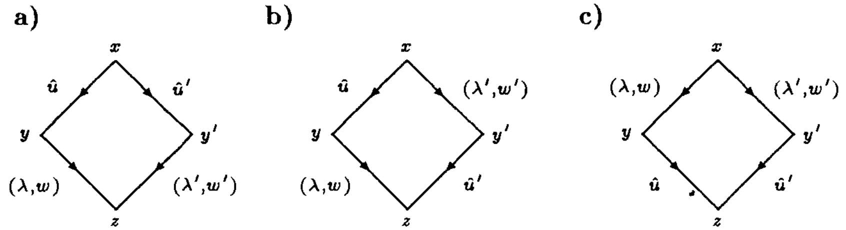
Here in the first sum over all $z$ with $\operatorname{Ind}(z, K) = \operatorname{Ind}(x, H)$ the pairs $(u, (\lambda, w))$ vary over all broken trajectories of the type

$$\begin{array}{rl} x & \bullet \\ & \downarrow\, u \\ y & \bullet \\ & \downarrow\, (\lambda, w) \\ z & \bullet \end{array}$$

*with* $\mathrm{Ind}(y, H) = \mathrm{Ind}(x, H) - 1$, $\mathrm{Ind}(z, K) - \mathrm{Ind}(y, H) = -1$, corresponding to an end of a component of type $[0, 1)$ or $(0, 1]$. In the second sum over $z$ the $z$-coefficient is obtained by taking the sum over all broken trajectories of the type




*with* $y' \in \mathscr{P}_K$, beeing the "end" of a component of type $[0, 1)$ or $(0, 1]$. Consider finally components of the form $(0, 1)$. We have to consider the following cases in identifying their ends with broken trajectories




We obtain the following formulae

a) $\tau(u)\hat{\tau}(\lambda, w) = -\tau(u')\hat{\tau}(\lambda', w')$

$y, y' \in \mathscr{P}_H$

b) $\tau(u)\hat{\tau}(\lambda, w) = \hat{\tau}(\lambda', w')\tau(u')$

$y \in \mathscr{P}_H$, $y' \in \mathscr{P}_K$

c) $\hat{\tau}(\lambda, w)\tau(u) = -\hat{\tau}(\lambda', w')\tau(u')$

$y, y' \in \mathscr{P}_K$ .

Using the above discussion we may write

$$\begin{aligned}\Psi_1(x) - \Psi_0(x) &= \sum_z \left( \sum_{(u, (\lambda, w))} \tau(u)\hat{\tau}(\lambda, w) \right) z - \sum_z \left( \sum_{((\lambda, w), u)} \hat{\tau}(\lambda, w)\, \tau(u) \right) z \\ &= -\sum_z n_1(x, z) z + \sum_z n_0(x, z) .\end{aligned}$$

Here $n_1(x, z)$ is obtained by taking the sum over all broken trajectories of the type $(u, (\lambda, w))$ and $n_0(x, z)$ by taking all broken trajectories between $x$ and $z$ of type $((\lambda, w), u)$.

Consider the expression $\partial \Lambda(x) - \Lambda\partial(x)$. We have

$$\partial \Lambda(x) = \sum_z \left( \sum_{((\lambda, w), u)} \hat{\tau}(\lambda, w)\tau(u) \right) z$$

$$\Lambda\partial(x) = \sum_z \left( \sum_{(u, (\lambda, w))} \tau(u)\hat{\tau}(\lambda, w) \right) z ,$$

where the second $\sum$ in each row is taken over the indicated set of broken trajectories from $x$ to $z$. Hence

$$\Lambda\partial(x) - \partial\Lambda(x) = \sum_z n_1(x, z)z - \sum_z n_0(x, z)z .$$

This proves

$$\Psi_1 - \Psi_0 = \partial\Lambda - \Lambda\partial .$$

Implying that $\Psi_1$ and $\Psi_0$ induce the same map in homology.

*4.4 A general construction*

In the previous sections we have constructed a covariant functor on $(\mathcal{N}_{\text{reg}}, \leqq)$ associating to a pair $(H, J)$ and a number $a \in (-\infty, +\infty]$ a chain complex $(C^a(H, J), \partial)$. We drop the dependence on the choice of a pair $(\sigma, \alpha_0)$ in view of our previous discussion. Moreover to an inequality $(H, J) \leqq (K, \tilde{J})$ we have a natural chain homotopy class denoted by

$$C_*^a(H, J) \to C_*^a(K, \tilde{J}) , \tag{146}$$

which is compatible with the natural inclusion

$$C_*^a(H, J) \to C_*^b(H, J) \tag{147}$$

for $a \leqq b$. For an open bounded $U \subset \mathbb{C}^n$ we denote by $\mathcal{N}_{\text{reg}}(U)$ the subset of $\mathcal{N}_{\text{reg}}$ consisting of all $(H, J)$ such that $H \mid (S^1 \times \bar{U}) < 0$. For $\alpha = (H, J) \in \mathcal{N}_{\text{reg}}(U)$ and $-\infty < a \leqq b \leqq +\infty$ we define

$$C_*^{[a,b)}(\alpha) := (C_*^b(\alpha)/C_*^a(\alpha), \partial) .$$

In view of our transversality results we have

**Proposition 28** *$(\mathcal{N}_{\text{reg}}(U), \leqq)$ is a directed set, i.e. given $\alpha, \beta \in \mathcal{N}_{\text{reg}}(U)$ there exists $\gamma \in \mathcal{N}_{\text{reg}}(U)$ such that*

$$\alpha \leqq \gamma, \quad \beta \leqq \gamma .$$

Define $S_*^{[a,b)}(\alpha) = H_*(C_*^{[a,b)}(\alpha))$. Then

$$(S_*^{[a,b)}(\alpha))_{\alpha \in (\mathcal{N}_{\text{reg}}(U), \leqq)}$$

is a directed system of Abelian groups. We note here that we could tensor $C_*^{[a,b)}(\alpha)$ with any Abelian group $G$ before taking homology. In that case we write

$$S_*^{[a,b)}(\alpha; G) := H_*(C_*^{[a,b)}(\alpha) \otimes G) .$$

Hence we have

**Theorem 29**

$$(S_*^{[a,b)}(\alpha; G))_{\alpha \in \mathcal{N}_{\mathrm{reg}}(U)}$$

*is a directed system of graded Abelian groups.*

Now we are able to define the symplectic homology of $U$ as a subset of $\mathbb{C}^n$.

**Definition 30** Let $-\infty < a \leqq b \leqq +\infty$ and $U$ be an open bounded set and $G$ be an Abelian group. The $[a, b)$-symplectic homology group of the set $U$ in dimension $k$ with coefficients in $G$ is the Abelian group

$$S_k^{[a,b)}(U; G) := \varinjlim ((S_k^{[a,b)}(\alpha;\ G))_{\alpha \in (\mathcal{N}_{\mathrm{reg}}(U), \leqq)}) .$$

Consider intervals $[a, b)$ with $-\infty < a \leqq b \leqq +\infty$ and write

$$[a, b) \leqq [a', b') \quad \text{if} \quad a \leqq a' ,\ b \leqq b' .$$

For $[a, b) \leqq [a', b')$ we have a natural chain map

$$C_*^{[a,b)}(\alpha;\ G) \rightarrow C_*^{[a',b')}(\alpha;\ G) . \tag{148}$$

This natural map gives rise to an exact sequence if a triplet of numbers $-\infty < a \leqq b \leqq c \leqq +\infty$ is given

$$0 \rightarrow C_*^{[a,b)}(\alpha; G) \rightarrow C_*^{[a,c)}(\alpha;\ G) \rightarrow C_*^{[b,c)}(\alpha;\ G) \rightarrow 0 . \tag{149}$$

Passing to homology we have an exact homology triangle $\Delta_{a,b,c}(\alpha;\ G)$

$$\begin{array}{ccc} S_*^{[a,b)}(\alpha; G) & \longrightarrow & S_*^{[a,c)}(\alpha; G) \\ {\scriptstyle \partial_*}\nwarrow & & \swarrow \\ & S_*^{[b,c)}(\alpha; G) & \end{array} \tag{150}$$

with $\partial_*$ being of degree $-1$. We sum up this result by

**Theorem 31** *Given an open bounded set $U \subset \mathbb{C}^n$ and a triplet of numbers $-\infty < a \leqq b \leqq c < +\infty$ we obtain an exact triangle $\Delta_{a,b,c}(U;\ G)$.*

*Proof.* Observe that $\varinjlim$ preserves exactness.

**Corollary 32** *Given triplets $-\infty < a \leqq b \leqq c \leqq +\infty$ and $+\infty < a' \leqq b' \leqq c' \leqq +\infty$ with $[a, b) \leqq [a', b')$ and $[b, c) \leqq [b', c')$ we have a natural map between exact triangles*

$$\Delta_{a,b,c}(U;\ G) \rightarrow \Delta_{a',b',c'}(U;\ G) . \tag{151}$$

Next assume $U$ and $V$ are bounded open subsets of $\mathbb{C}^n$ and $U \subset V$. Then we have a natural inclusion

$$\mathcal{N}_{\mathrm{reg}}(V) \hookrightarrow \mathcal{N}_{\mathrm{reg}}(U)$$

and an induced natural map

$$S_*^{[a,b)}(V;\ G) \xrightarrow{mm} S_*^{[a,b)}(U;\ G)\,, \tag{152}$$

where "$mm$" stands for monotonicity map. This gives also

$$\Delta_{a,b,c}(V;\ G) \xrightarrow{mm} \Delta_{a',b',c'}(U,G)\,. \tag{153}$$

The maps "$\longrightarrow$" in (151) and "$\xrightarrow{mm}$" commute. Next let us denote by $\mathscr{D}$ the group of compactly supported symplectic diffeomorphism in $\mathbb{C}^n$. Given $\alpha = (H, J) \in \mathscr{N}_{\text{reg}}$ and $\Psi \in \mathscr{D}$ we define $\alpha_\Psi \in \mathscr{N}_{\text{reg}}$ by

$$\begin{aligned} H_\Psi(t,u) &= H(t, \Psi^{-1}(u)) \\ J_\Psi(t,u) &= T\Psi(\Psi^{-1}(u))\, J(t, \Psi^{-1}(u))\, T\Psi^{-1}(u)\,. \end{aligned} \tag{154}$$

If $u$ is a solution of the $PDE$ associated to $\alpha$ then $\Psi(u)$ solves the $PDE$ associated to $\alpha_\Psi$. If $\alpha \in \mathscr{N}_{\text{reg}}(U)$ then $\alpha_\Psi \in \mathscr{N}_{\text{reg}}(\Psi(U))$. $\Psi$ induces therefore an isomorphism, denoted by $\Psi_{\#\#}$

$$\Psi_{\#\#}: S_*^{[a,b)}(U;G) \xrightarrow{\sim} S_*^{[a,b)}(\Psi(U);G)\,. \tag{155}$$

$\Psi_{\#\#}$ is obtained in the direct limit by the maps $\Psi_{\#\#}(x) = \Psi(x), x \in \mathscr{P}_H$, $\alpha = (H, J)$. We observe that $\Psi_{\#\#}$ can be considered as isomorphism between exact triangles $\Delta_{a,b,c}(U;\ G)$ and $\Delta_{a,b,c}(\Psi(U);G)$. Using the monotonicity maps we define for $\Psi \in \mathscr{D}$ with $\Psi(U) \subset V$ the induced morphism $\Psi^*$: $S^{[a,b)}(V;G) \to S^{[a,b)}(U;\ G)$ by the factorization

$$\begin{array}{ccc} S^{[a,b)}(V;G) & \xrightarrow{\quad\Psi^*\quad} & S^{[a,b)}(U;\ G) \\ & {\scriptstyle mm}\searrow \qquad \nearrow{\scriptstyle \Psi_{\#\#}^{-1}} & \\ & S^{[a,b)}(\Psi(U);\ G) & \end{array} \tag{156}$$

This gives also a morphism $\Psi^*$ between exact triangles

$$\Delta_{a,b,c}(V;\ G) \xrightarrow{\Psi^*} \Delta_{a,b,c}(U;\ G)\,. \tag{157}$$

From the construction it follows immediately that $\Psi^*$ commutes with maps of type (151).

**Theorem 33** *Let $U$, $V$, $W$ be bounded open subsets of $\mathbb{C}^n$ and $\Phi$, $\Psi \in \mathscr{D}$ such that*

$$\Phi(U) \subset V, \quad \Psi(V) \subset W\,.$$

*Then we have the commutative diagram*

$$\begin{array}{ccc} \Delta_{a,b,c}(W) & \xrightarrow{(\Psi\circ\Phi)^*} & \Delta_{a,b,c}(U) \\ & {\scriptstyle \Psi^*}\searrow \qquad \nearrow{\scriptstyle \Phi^*} & \\ & \Delta_{a,b,c}(V) & \end{array} \tag{158}$$

*Proof.* Let us denote by $m_{UV}$: $S^{[a,b)}(V) \to S^{[a,b)}(U)$ the monotonicity map for $U \subset V$. We have to show

$$(\Psi \circ \Phi)^{-1}_{\#\#} m_{\Psi \circ \Phi(U), W} = \Phi^{-1}_{\#\#} m_{\Phi(U), V} \Psi^{-1}_{\#\#} m_{\Psi(V), W} .$$

We have $m_{\Psi \circ \Phi(U), W} = m_{\Psi \circ \Phi(U), \Psi(V)} m_{\Psi(V), W}$. Hence it suffices to show that

$$\Psi^{-1}_{\#\#} m_{\Psi \circ \Phi(U), \Psi(V)} = m_{\Phi(U), V} \Psi^{-1}_{\#\#} ,$$

using that $(\Psi \circ \Phi)^{-1}_{\#\#} = \Phi^{-1}_{\#\#} \Psi^{-1}_{\#\#}$. Given $\alpha = (H, J) \in \mathcal{N}_{\text{reg}}(\Psi(V))$ we have for $\beta \geqq \alpha_{\Psi^{-1}}$ the commutative diagram

(159)
$$\begin{array}{ccc} S^{[a,b)}_*(\alpha) & \xrightarrow[\sim]{\Psi^{-1}_{\#\#}} & S^{[a,b)}_*(\alpha_{\Psi^{-1}}) \\ \downarrow & & \downarrow \\ S^{[a,b)}_*(\beta_\Psi) & \xrightarrow[\sim]{\Psi^{-1}_{\#\#}} & S^{[a,b)}_*(\beta) \end{array}$$

For this observe that we just apply to the first vertical the map $\Psi^{-1}$ also transforming the complex giving the monotonicity morphism to obtain a monotonicity morphism depicted by the second vertical arrow (recall that the choice of monotone homotopy does not matter). Passing to the limit in (159) gives

(160)
$$\begin{array}{ccc} S^{[a,b)}_*(\Psi(V)) & \xrightarrow[\sim]{\Psi^{-1}_{\#\#}} & S^{[a,b)}_*(V) \\ mm \downarrow & & \downarrow mm \\ S^{[a,b)}_*(\Psi \circ \Phi(U) & \xrightarrow[\sim]{\Psi^{-1}_{\#\#}} & S^{[a,b)}_*(\Phi(U)) \end{array}$$

(We dropped the $G$-dependence in our notation). The above diagram in (160) implies the desired conclusion.

### 4.5 *Isotopy invariance*

We start with some notation. Let $L$: $\mathbb{R} \times S^1 \times \mathbb{C}^n \to \mathbb{R}$ be a smooth map satisfying for suitable constants $c$ and $R > 0$

(161)
$$\begin{aligned} |L'(s,t,u)| &\leqq c(1+|u|) && \text{for all} \quad (s,t,u) \\ |L''(s,t,u)h| &\leqq c|h| && \text{for all} \quad (s,t,u),\, h \in \mathbb{C}^n \\ L(s,t,u) &= L(o,t,u) && \text{for all} \quad (s,t,u) \text{ with } |u| \geqq R \\ L(s,t,u) &=: H(t,u) && \text{for} \quad s \leqq s_0 \\ L(s,t,u) &=: K(t,u) && \text{for} \quad s \geqq s_0 , \end{aligned}$$

where $H, K \in \mathscr{H}_{\text{reg}}$. By our assumption necessarily $H - K$ has compact support. We call $L$ a homotopy between $H$ and $K$. We define a number $d(L) \in [0, +\infty)$ by

(162)
$$d(L) = \int_{-\infty}^{+\infty} \left( \int_0^1 \left( \max_{x \in \mathbb{C}^n} \left| \frac{\partial L}{\partial s}(\tau, t, u) \right| \right) dt \right) d\tau .$$

Moreover we put for $H, K \in \mathscr{H}$ with $H - K$ compactly supported

(163) $\quad d(H,K) = \inf\{d(L) \mid L \text{ is a homotopy between } H \text{ and } K\} .$

Next let $H \in \mathscr{H}_{\text{reg}}$ and $a \in (-\infty, +\infty]$. We define $\mathscr{P}_H(a)$ by

$$\mathscr{P}_H(a) = \{x \in \mathscr{P}_H \mid \Phi_H(x) < a\} .$$

For $-\infty < a \leqq b \leqq +\infty$ we put

$$\mathscr{P}_H([a,b)) = \mathscr{P}_H(b) \backslash \mathscr{P}_H(a) .$$

Given $H \in \mathscr{H}_{\text{reg}}$ and $a, b$ as above we define the "gap" $g(H, [a,b))$ by

$$g(H, [a,b)) = \inf\{|\Phi_H(x) - \Phi_H(y)| \quad |x \in \mathscr{P}_H([a,b)) ,\\ y \notin \mathscr{P}_H \backslash (\mathscr{P}_H([a,b))\} .$$

We have

**Lemma 34** *Let $H \in \mathscr{H}_{\text{reg}}$ and $-\infty < a \leqq b \leqq +\infty$, then*

$$g(H, [a,b)) \in (0, +\infty] .$$

*Define for $H, K \in \mathscr{H}_{\text{reg}}$*

$$g(H, K, [a,b)) = \inf\{g(H, [a,b)), g(K, [a,b))\} .$$

A crucial result is the following proposition

**Proposition 35** *Given $(H, J), (K, \tilde{J}) \in \mathscr{N}_{\text{reg}}$ with $H - K$ compactly supported and*

$$d(H, K) < g(H, K; [a,b))$$

*there exists a natural map*

$$\phi_{(K,H)}: S^{[a,b)}(H, J) \longrightarrow S^{[a,b)}(K, \tilde{J})$$

*given by the meanwhile usual construction by any homotopy $L$ between $H$ and $K$ satisfying*

$$d(L) < g(H, K, [a,b))$$

*and an associated regular almost complex structure having a $(s,t)$-dependence in the usual way. Moreover if $H_1, H_2, H_3$ are given with*

$$d(H_i, H_{i+1}) < \frac{1}{2} g(H_i, H_{i+1}, [a,b)) \quad \text{for } i = 1, 2$$

*then*

$$\phi_{(H_3,H_2)} \phi_{(H_2,H_1)} = \phi_{(H_3,H_1)} .$$

*Moreover*

$$\phi(H, H) = \text{Id} .$$

*Proof.* Let $u$ be a solution of

(164)
$$u_s - \hat{J}(s,t,u)u_t - (\nabla_{\hat{J}} L)(s,t,u) = 0$$
$$u(s,*) \longrightarrow x \quad \text{as } s \longrightarrow -\infty$$
$$u(s,*) \longrightarrow y \quad \text{as } s \longrightarrow +\infty ,$$

where $x \in \mathscr{P}_H(c)$ for either $c = a$ or $c = b$, respectively. Assume $d(L) < g(H, K, [a,b))$ and $(L, \hat{J})$ is a regular pair (with the obvious meaning). Then

$$\frac{d}{ds} \Phi_{L(s)}(u(s)) = -\left\| \Phi'_{L(s)}(u(s)) \right\|^2_{s,u(s)} - \int_0^1 \frac{\partial L}{\partial s}(s,t,u(s,t))dt$$
$$\leqq \int_0^1 \left[ \sup_{x \in \mathbb{C}^n} \left| \frac{\partial L}{\partial s}(s,t,u) \right| \right] dt .$$

*Hence*

$$\Phi_K(y) - \Phi_H(x) \leqq d(L) < g(H, K; [a, b)) .$$

Therefore

$$\Phi_K(y) \in (-\infty, c + d(L)] ,$$

which implies by the definition of the gap and our hypotheses

$$\Phi_K(y) \in (-\infty, c)$$

i.e. $y \in \mathscr{P}_K(c)$. Hence we may use the combinatorics of the solutions of (164) to define a chain map

$$C_*^{[a,b)}(H, J) \to C_*^{[a,b)}(K, \tilde{J}) .$$

If we take another homotopy with the same properties we deduce that the induced map is chain homotopic. For this the same argument works as in the section concerning monotone homotopies. (In fact we construct a homotopy between the homotopies satisfying for every parameter $\lambda \in [0, 1]$ the inequality between $d$ and $g$. Then we choose a generic $\tilde{J}$ making the pair a regular homotopy between homotopies).

Next let $L^1$ be a homotopy between $H^1$ and $H^2$ and $L^2$ between $H^2$ and $H^3$. Using $L^1$ and $L^2$ we can construct a homotopy between $H^1$ and $H^3$ denoted by $L^1 \# L^2$ satisfying

$$d(L^1 \# L^2) < g(H^1, H^3, [a, b))$$

so that via a glueing argument

$$\phi_{(H^3,H^2)}\phi_{(H^2,H^1)} = \phi_{(H^3,H^1)} .$$

see [38] for a discussion of formulae of the above type, or [19]. Obviously

$$\phi_{(H,H)} = \mathrm{Id} . \qquad \square$$

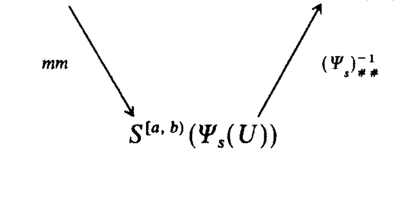
We shall refer to Proposition 35 as the Stability theorem. It is a crucial ingredient of the proof of

**Theorem 36** *Let* $(\Psi_s)_{s \in [0,1]} \subset \mathscr{D}$ *be a smooth arc and* $U, V \subset \mathbb{C}^n$ *bounded open subsets such that* $\Psi_s(U) \subset V$ *for all* $s \in [0, 1]$. *For given numbers* $-\infty < a \leqq b \leqq +\infty$ *we have that the induced morphism*

$$(\Psi_s)^*: S^{[a,b)}(V) \to S^{[a,b)}(U)$$

*is independent of* $s \in [0, 1)$.

It suffices to show that for every $s_0 \in [0, 1]$ there exists an open neighbourhood $U(s_0)$ in $[0, 1]$ such that for $s \in U(s_0)$ we have $(\Psi_s)^* = (\Psi_{s_0})^*$. By definition we have



Let $(H, J)\in\mathcal{N}_{\text{reg}}(U)$ and consider $(H_{\Psi_s}, J_{\Psi_s})\in\mathcal{N}_{\text{reg}}\,(\Phi_s(U))\mathcal{N}_{\text{reg}}(V)$. We observe that given any $(K, \tilde{J})\in\mathcal{N}_{\text{reg}}\,(V)$ we can always find $(H, J)\in\mathcal{N}_{\text{reg}}\,(U)$ such that

$$(H_{\Psi_s}, J_{\Psi_s}) \geqq (K, \tilde{J}) \quad \text{for all } s\in[0, 1]\ .$$

Hence we have the monotonicity maps from

$$S^{[a,b)}(K, \tilde{J}) \xrightarrow{mm} S^{[a,b)}(H_{\Psi s}, J_{\Psi s})\ .$$

We observe that for $s_2 - s_1 \to 0$ we have $d(H_{\Psi s2}, H_{\Psi s1}) \to 0$. Hence for $|s_2 - s_1|$ small we have the commutative diagram

$$\begin{array}{ccc} S^{[a,b)}(K; \tilde{J}) & \xrightarrow{\quad mm \quad} & S^{[a,b)}(H_{\Psi s2}, J_{\Psi s2}) \\ & {\scriptstyle mm}\searrow \qquad \nearrow{\scriptstyle \Phi(H_{\Psi s2}, H_{\Psi s1})} & \\ & S^{[a,b)}(H_{\Psi s1}, J_{\Psi s1}) & \end{array}$$

If we can show that

$$(165) \qquad \phi(H_{\Psi s2}, H_{\Psi s1}) = (\Psi_{s_2} \circ \Psi_{s_1}^{-1})_{\#\#}$$

the proof of the theorem is complete in view of the definition of $\Phi_s^*$. Hence it remains to prove (135). For doing so we may assume $\tau \to \Psi_\tau$ is a smooth arc in $\mathscr{D}$ with $\Psi_0 = \text{Id}$. Let $(H, J)\in\mathcal{N}_{\text{reg}}$ and $-\infty < a \leqq b \leqq +\infty$. We have to show that $(\Psi_\tau)_{\#\#} = \phi$ for $\tau$ close to zero, where $\phi$ is the "small distance isomorphism". Consider the partial differential equation

$$(166) \qquad v_s - J(t, v)v_t - (\nabla_J H)(t, v) = 0$$

with asymptotic boundary conditions. We are interested in solutions which connect data with the same index. Since the data is generic we have necessarily

$$v(s, t) = x(t), \quad x\in\mathscr{P}_H\ .$$

Next let $\beta: \mathbb{R} \to [0, 1]$ be a smooth map such that $\beta(s) = 0$ for $s \leqq 0$ and $\beta(s) = 1$ for $s \geqq 1$, $\beta'(s) > 0$ for $s\in(0, 1)$. For $\varepsilon > 0$ smooth, define

$$(167) \qquad H^\varepsilon(s, t, u) = H(t, \Psi_{\varepsilon\beta(s)}^{-1}(u))$$

$$J^\varepsilon(s, t, u) = D\Psi_{\varepsilon\beta(s)}(\Psi_{\varepsilon\beta(s)}^{-1}(u))\, J(t, \Psi_{\varepsilon\beta(s)}^{-1}(u))\, D\Psi_{\varepsilon\beta(s)}^{-1}(u)\ .$$

Then $(H^\varepsilon, J^\varepsilon)$ can be considered as a homotopy between $(H, J)$ and $(H_{\Psi_\varepsilon}, J_{\Psi_\varepsilon})$. We shall see shortly that $(H^\varepsilon, J^\varepsilon)\in\mathcal{N}_{\text{reg}}(H, J;\, H_{\Psi_\varepsilon}\, J_{\Psi_\varepsilon})$ for $\varepsilon > 0$ small. We solve

$$(168) \qquad 0 = w_s - J^\varepsilon(s, t, w)w_t - (\nabla_{J^\varepsilon} H^\varepsilon)(s, t, w)\ ,$$

connecting two critical points of the same Morse index. We define

$$w(s, t) = \Psi_{\varepsilon\beta(s)}(u(s, t)) =: \theta(s, u(s, t))\ .$$

Then

$$(169)\qquad 0 = \frac{\partial\theta}{\partial s}(s,u) + D_2\theta(s,u)u_s - J^\varepsilon(s,t,\theta(s,u))\,D_2\theta(s,u)u_t$$
$$- (\nabla_{J^\varepsilon}H^\varepsilon)(s,t,\theta(s,u))$$
$$= D_2\theta(s,u)\left[(u_s - J(t,u)u_t - (\nabla_J H)(t,u)) + D_2\theta(s,u)^{-1}\frac{\partial\theta}{\partial s}(s,u)\right]$$
$$=: D_2\theta(s,u)[(u_s - J(t,u)u_t - (\nabla_J H)(t,u)) + \Gamma_\varepsilon(s,u)]\ .$$

We observe that $\Gamma_\varepsilon(s,u) = 0$ for $|s| \geqq 1$ and $\Gamma_\varepsilon(s,u) \to 0$ as $\varepsilon \to 0$. $\Gamma_\varepsilon$ is a compact perturbation. If $\varepsilon \to 0$ it follows quite easily that the solutions $u$ of (169) converge (if the Morse index difference is zero) to $v$, $v(s,t) = x(t)$, for a suitable $x \in \mathscr{P}_H$, where the convergence is in $\mathscr{B}^{1,p}(x,x)$. If we assume $(H,J) \in \mathscr{N}_{\text{reg}}$ it follows from the implicit function theorem that given any $x \in \mathscr{P}_H$ and putting $x_\varepsilon = \Psi_\varepsilon(x)$ there exists a unique solution $w_\varepsilon$ of (168) connecting $x$ with $x_\varepsilon$ such that the associated $u_\varepsilon$ given by (169) is close to $v$. And in view of the compactness these are all solutions. Hence $(H^\varepsilon, J^\varepsilon)$ is regular for $\varepsilon > 0$ small and the chain map given by

$$C^a(H,J) \to C^a(H_{\Psi_\varepsilon}, J_{\Psi_\varepsilon})$$

by the map $x \to \Psi_\varepsilon(x)$, $x \in \mathscr{P}_H$ coincides with the "small distance isomorphism" chain map. Hence

$$(\Psi_\tau)_{\#\#} = \phi \quad \text{for } |\tau| \text{ small}\ . \qquad \square$$

### 4.6 Some remarks

*Products.* Let $U \subset \mathbb{C}^l$, $V \subset \mathbb{C}^m$ be open and put $n = l + m$. Let $(H,i) \in \mathscr{N}_{\text{reg}}(U)$ and $(K,i) \in \mathscr{N}_{\text{reg}}(V)$. In view of our discussion in previous chapters we know that pairs of the above type are cofinal so that $S^{[a,b)}(U)$ or $S^{[a,b)}(V)$ could be defined by taking only limits over these classes of pairs. We define $H \oplus K\colon \mathbb{C}^n \to \mathbb{R}$ in the obvious way and obtain $(H \oplus K, i) \in \mathscr{N}_{\text{reg}}(U \times V)$. Strictly speaking $H \oplus K$ does not belong to the admissible class of Hamiltonians. However, the analysis works as well here. Obviously

$$C^a_*(H \times K, i) = \bigcup_{c \in \mathbb{R}} (C^{a-c}(H,i) \otimes C^c(K,i))$$
$$\subset C^\infty(H,i) \otimes C^\infty(K,i)\ .$$

This formula can be used to study the symplectic homology of a product if the chain complexes for the factors are known. This will be used in [24], in computing the symplectic homology of polydisks.

*Closed characteristics.* If $U$ is a bounded open set with smooth boundary one can start with a Hamiltonian $H\colon \mathbb{C}^n \to \mathbb{R}$ which satisfies $H \mid \bar{U} < 0$ and having $\partial U$ as a regular level surface. The neighbourhood of $\partial U$ will be foliated by other level sufaces of $H$. If $H$ grows fast enough outside of $\bar{U}$ it will follow that the 1-periodic solutions of $\dot{x} = X_M(x)$ will be close to $\partial U$. Taking now a small $t$-dependent perturbation in a suitable way a nondegenerate 1-periodic solution of the autonomous system will split into 2 nondegenerate 1-periodic solutions of the perturbed system with Conley-Zehnder index differing by 1.

In some sense our limiting process of $S^{[a,b)}(H,J)$ with $(H,J) \in \mathscr{N}_{\text{reg}}(U)$ can be understood as an approximation of the group "$S^{[a,b)}(H_U)$" with $H_U(x) = 0$ for

$x \in \bar{U}$ $H(x) = +\infty$ , $x \notin \bar{U}$. In this way one can understand $S^{[a,b)}(U)$ for nice $U$ in the following way. For every closed characteristic $P$ (perhaps an iterated one) consider symbols $P^-, P^+$ with numerical values $A(P) := |k \int \lambda \mid P|$, $d\lambda = \omega$, i.e. the action (multiplicity is $k$) and a Conley-Zehnder index $\mu(P^-)$, $\mu(P^+)$ with $\mu(P^+) - \mu(P^-) = 1$. For $a \in (-\infty, +\infty]$ we define

$$C_k^a = \bigoplus \mathbb{Z} B$$

with $B \in \{P^+, P^- \mid P\}$ and $A(B) < a \mu(B) = k$. The limiting process defines then a boundary operator on $C^a$. This kind of approximation has been constructed in [10] in order to define symplectic capacities. So the symplectic homology is in some sense partially generated by closed characteristics, where each closed characteristic gives a contribution in two consecutive dimensions. This will be made more precise in [24].

*Outlook.* The construction we used here takes advantage of some of the special features of $\mathbb{C}^n$. Replacing $\mathbb{C}^n$ by some symplectic manifold with some assumptions on $\omega$ and $c_1$ one can modify the construction in several ways leading to different, however closely related theories. The techniques are comparable to those which occurred here. In [22] we will present some of the possible constructions. They will be illustrated in [25].

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