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Homogeneous affine hypersurfaces with rank one shape operators

Franki Dillen* and Luc Vrancken**

Katholieke Universiteit Leuven, Departement Wiskunde, Celestijnenlaan 200 B, B-3001 Leuven, Belgium

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1 Introduction

This paper will be the first in a sequence of papers studying nondegenerate affine hypersurfaces of $\mathbb{R}^{n+1}$, which are (locally) homogeneous, in the sense defined below. Homogeneous affine surfaces in $\mathbb{R}^3$ were believed to be classified (see [Nom] or Chap. 12 of [G]). Recently however, Nomizu and Sasaki have discovered a gap in this classification, and succeeded in giving a complete classification, see [NS]. In particular, locally strongly convex (locally) homogeneous surfaces can be classified as follows.

Theorem A. *Let M be a locally strongly convex, locally homogeneous affine surface in $\mathbb{R}^3$. Then M is affine equivalent to either*

(i) *a locally strongly convex quadric, or*
(ii) *the affine surface given by the equation $xyz = 1$, or*
(iii) *the affine surface given by the equation $(x - \frac{1}{2}z^2)^3 y^2 = 1$.*

Notice that the surfaces (i) and (ii), are affine spheres, while in the last case, the rank of the shape operator is 1. Here, we investigate affine homogeneous hypersurfaces whose affine shape operator has rank 1. The Main Theorem that we prove is the following:

Main Theorem. *Let M be a locally strongly convex, locally homogeneous affine hypersurface with* rank $S = 1$ *in $\mathbb{R}^{n+1}$. Then M is affine equivalent to the convex part of the hypersurface with equation*

$$(*)\qquad \left(Z - \frac{1}{2}\sum_{i=1}^{r} X_i^2\right)^{r+2}\left(W - \frac{1}{2}\sum_{j=1}^{s} Y_j^2\right)^{s+2} = 1\,,$$

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** Senior Research Assistant of the National Fund for Scientific Research (Belgium) *where* $r + s = n - 1$ *and* $(X_1, \ldots, X_r, Y_1, \ldots, Y_s, Z, W)$ *are the coordinates of* $\mathbb{R}^{n+1}$.

The hypersurface $(*)$ is not connected, we have to restrict to the "convex part"; this means to the points where both $Z - \frac{1}{2}\sum_{i=1}^{r} X_i^2 > 0$ and $W - \frac{1}{2}\sum_{j=1}^{s} Y_j^2 > 0$.

Finally, we remark that homogeneous affine spheres have been studied by Sasaki in [S], who in the locally strongly convex case obtained a complete classification of the hyperbolic affine spheres which are the orbit of some point under a certain subgroup of the special linear group. In fact, he reduces the classification of those hypersurfaces to the classification of homogeneous convex cones.

2 Preliminaries

Let M be a connected n-dimensional submanifold of the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D and a parallel volume element ω. We allow M to be immersed by an immersion x, but we will not denote the immersion if there is no confusion possible. We assume that M is nondegenerate, such that we can consider the affine normal vector field ξ, as determined by Blaschke. We denote by ∇ the induced affine connection, by h the affine metric and by S the affine shape operator (or Weingarten tensor). We recall the formulas of Gauss and Weingarten

$$D_X Y = \nabla_X Y + h(X, Y)\xi\,,$$

$$D_X \xi = -SX\,.$$

M is called an affine sphere if $S = \lambda I$, proper if $\lambda \neq 0$, improper otherwise. A proper affine sphere has as affine normal $\xi(p) = -\lambda(p - c)$, where c is a constant vector, called the center of M. If h is definite, then we can assume that it is positive definite, and it is known that this is equivalent to M being locally strongly convex. If M is a locally strongly convex affine sphere, we call M elliptic, parabolic or hyperbolic if λ is positive, zero or negative.

The fundamental equations of Gauss, Codazzi and Ricci are given by

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY\,,$$

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z)\,,$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X)\,,$$

$$h(X, SY) = h(SX, Y)\,.$$

Moreover, h and ∇h are related by the apolarity condition

$$\text{trace}_h\{(X, Y) \mapsto (\nabla h)(Z, X, Y)\} = 0\,. \tag{2.1}$$

Since the affine metric h is nondegenerate, h has a Levi Civita connection $\hat{\nabla}$. The difference tensor K is a $(1, 2)$-tensor field defined by

$$K(X, Y) = K_X Y = \nabla_X Y - \hat{\nabla}_X Y\,.$$

It is related to ∇h in the following way.

$$h(K(X, Y), Z) = -\frac{1}{2}(\nabla h)(X, Y, Z)\,.$$

The curvature tensors R and $\hat{R}$ of ∇ and $\hat{\nabla}$ are related by

$$(2.2)\qquad \hat{R}(X, Y)Z = \frac{1}{2}(h(Y, Z)SX - h(X, Z)SY + h(SY, Z)X - h(SX, Z)Y) - [K_X, K_Y]Z .$$

We call M locally homogeneous if for all points p and q of M, there exists a neighbourhood U_p of p in M, and an equiaffine transformation A of $\mathbb{R}^{n+1}$, i.e. $A \in S\ell(n+1, \mathbb{R}) \ltimes \mathbb{R}^{n+1}$, such that $A(p) = q$ and $A(U_p) \subset M$. If $U_p = M$ for all p, then M is called homogeneous.

Remark 1 (a) Every equiaffine transformation leaving M locally invariant, preserves the affine metric h and the induced connection ∇.
(b) If for all points p and q of M, there exist neighbourhoods U_p of p and U_q of q in M, and a diffeomorphism $f: U_p \to U_q$, with $f(p) = f(q)$ such that f preserves both h and ∇, then M is locally homogeneous. Indeed, every such f can be extended to an element of $S\ell(n+1, \mathbb{R}) \ltimes \mathbb{R}^{n+1}$, as follows from the uniqueness theorem for hypersurfaces, see for instance [D].

Now let G be the pseudogroup defined by

$$G = \{A \in S\ell(n+1, \mathbb{R}) \ltimes \mathbb{R}^{n+1} \mid \exists U, \text{open in } M, A(U) \subset M\} ,$$

then M is locally homogeneous if and only if G "acts" transitively on M.

Proposition 1 *If M is a proper affine sphere in $\mathbb{R}^{n+1}$, centered at the origin, then $G \subset S\ell(n+1, \mathbb{R})$.*

Proof. The affine normal ξ is given by $\xi(p) = -\lambda p$ for all $p \in M$. Let $A \in G$. Since the affine normal is an equiaffine invariant, we obtain $\xi(Ap) = A_* \xi(p) = -\lambda A_* p$. On the other hand, $\xi(Ap) = -\lambda Ap$. Hence $A = A_*$, showing that A is linear. □

Before starting the proof of the Main Theorem, we first discuss the example further in detail. So let's consider the affine hypersurface (*). We can take the following parametrization:

$$x(u_1, \ldots, u_r, v_1, \ldots, v_s, t) = \left(u_1, \ldots, u_r, v_1, \ldots, v_s, e^{\lambda_1 t} + \frac{1}{2}\sum_{i=1}^{r} u_i^2, e^{\lambda_2 t} + \frac{1}{2}\sum_{i=1}^{s} v_i^2\right),$$

where λ_1, λ_2 are non-zero real numbers satisfying $(r+2)\lambda_1 + (s+2)\lambda_2 = 0$. Putting

$$c = \left(\frac{-\lambda_1\lambda_2}{(\lambda_2 - \lambda_1)^2}\right)^{\frac{1}{n+2}},$$

a straightforward computation shows that the affine normal is given by

$$\xi = c(0, \ldots, 0, 0, \ldots, 0, e^{\lambda_1 t}, e^{\lambda_2 t}).$$

Hence $S\partial_{u_i} = 0$, $S\partial_{v_j} = 0$ and $S\partial_t = -c\partial_t$, where $i = 1, \ldots, r$; $j = 1, \ldots, s$. So rank $S = 1$. Moreover the hypersurface is affine homogeneous, since it is the orbit of the point $p = (0, \dots, 0, 0, \dots, 0, 1, 1)$ under the action of the group of equiaffine transformations whose matrices are given by

$$\begin{pmatrix} e^{\frac{1}{2}\lambda_1 w} I_{r,r} & 0_{r,s} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} u_1 \\ \vdots \\ u_r \end{matrix} \\ 0_{s,r} & e^{\frac{1}{2}\lambda_2 w} I_{s,s} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} v_1 \\ \vdots \\ v_s \end{matrix} \\ u_1 e^{\frac{1}{2}\lambda_1 w} \dots u_r e^{\frac{1}{2}\lambda_1 w} & 0 \dots 0 & e^{\lambda_1 w} & 0 & \frac{1}{2}\sum_{i=1}^r u_i^2 \\ 0 \dots 0 & v_1 e^{\frac{1}{2}\lambda_2 w} \dots v_s e^{\frac{1}{2}\lambda_2 w} & 0 & e^{\lambda_2 w} & \frac{1}{2}\sum_{j=1}^s v_j^2 \\ 0 \dots 0 & 0 \dots 0 & 0 & 0 & 1 \end{pmatrix}$$

where $I_{p,q}$ (resp. $0_{p,q}$) denotes the identity matrix (resp. the zero matrix) of dimensions (p, q).

Remark 2 The affine metric of M is given by

$$h(\partial_{u_i}, \partial_{u_j}) = \frac{r+2}{r+s+4}\frac{1}{c} e^{-\lambda_1 t}\delta_{ij} ,$$

$$h(\partial_{v_i}, \partial_{v_j}) = \frac{s+2}{r+s+4}\frac{1}{c} e^{-\lambda_2 t}\delta_{ij} ,$$

$$h(\partial_t, \partial_t) = -\frac{1}{c}\lambda_1\lambda_2 ,$$

and all other components are zero. This means that intrinsically, the Riemannian manifold (M, h) is a warped product $\mathbb{R} \times_{\rho_1} \mathbb{R}^r \times_{\rho_2} \mathbb{R}^s$, where $\rho_1(t) = e^{-\frac{1}{2}\lambda_1 t}$ and $\rho_2(t) = e^{-\frac{1}{2}\lambda_2 t}$, and $\mathbb{R}$, $\mathbb{R}^r$ and $\mathbb{R}^s$ are equipped with the appropriate Euclidean metrics.

In general, warped products are defined as follows, see [Nö]. Let $M_0, \dots, M_k$ be Riemannian manifolds with metrics $\langle ., . \rangle_0, \dots, \langle ., . \rangle_k$, and let M be their product $M_0 \times \dots \times M_k$. Let $v = (v_0, \dots, v_k)$ be the canonical decomposition of tangent vectors to M and let $\rho_1, \dots, \rho_k\colon M_0 \to \mathbb{R}_+$ be positive functions. Then

$$\langle v, w \rangle = \langle v_0, w_0 \rangle_0 + \sum_{i=1}^{k} \rho_i^2(p_0)\langle v_i, w_i \rangle_i$$

for $v, w \in T_pM$, $p = (p_0, \dots, p_k)$, defines a Riemannian metric on M. We call $(M, \langle ., . \rangle)$ the warped product $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ of $M_0, \dots, M_k$, and $\rho_1, \dots, \rho_k$ the warping functions.

Remark 3 If we put $s = 0$ in $(*)$, then we get an affine hypersurface given by

$$(**) \qquad \left(Z - \frac{1}{2}\sum_{i=1}^{n-1} X_i^2\right)^{n+1} W^2 = 1 .$$

Both this example and the example $(*)$ can be considered as members of a larger class of homogeneous hypersurfaces. We will discuss this in a separate paper [DV].

3 Proof of the Main Theorem

Throughout this section, we shall assume that M is a locally strongly convex, locally homogeneous affine hypersurface in $\mathbb{R}^{n+1}$ with rank $S = 1$ and we will assume that $n \geqq 3$.

The eigenspaces of S are G-invariant. Since S is symmetric, it can be diagonalized, so it has two orthogonal eigenspaces, a one-dimensional corresponding to the only nonzero eigenvalue, and an $(n-1)$-dimensional, corresponding to 0. So if E_1 is a unit eigenvector (this is always with respect to the affine metric) of S with nonzero eigenvalue μ, then μ is constant and E_1 is G-invariant. Also the distribution $W = \ker S$ is G-invariant. We can always take a local orthonormal frame $\{E_1, U_1, \ldots, U_{n-1}\}$ such that $U_1, \ldots, U_{n-1}$ span W. So $SE_1 = \mu E_1$ and $SU_i = 0$.

Lemma 1 *Let $\{E_1, U_1, \ldots, U_{n-1}\}$, be a local frame as above, then*

(i) *W is involutive and $\nabla_{E_1} W \subset W$. So we can introduce functions a_{ij} such that $\nabla_{E_1} U_i = \sum_{j=1}^{n-1} a_{ij} U_j$,*
(ii) *$\nabla_{U_i} E_1 = b_i E_1$ where $b_i = -\frac{1}{2}(\nabla h)(U_i, E_1, E_1)$,*
(iii) *$\nabla_{U_i} U_j = (a_{ij} + a_{ji})E_1 + \sum_{k=1}^{n-1} d_{ij}^k U_k$ for some functions d_{ij}^k,*
(iv) *$\nabla_{E_1} E_1 = cE_1 + 2(\sum_{i=1}^{n-1} b_i U_i)$, where c is a constant,*
(v) *$d_{ij}^k + d_{ik}^j = d_{ji}^k + d_{jk}^i$.*

Proof. The Codazzi equation for S immediately implies that W is involutive, and that

$$\begin{aligned} 0 &= (\nabla_{E_1} S)U_i - (\nabla_{U_i} S)E_1 \\ &= -S(\nabla_{E_1} U_i) - (\mu - S)(\nabla_{U_i} E_1), \end{aligned}$$

such that $\nabla_{E_1} U_i$ is orthogonal to E_1 and $\nabla_{U_i} E_1$ is proportional to E_1. This proves (i) and (ii). The Codazzi equation for h then gives us

$$\begin{aligned} h(\nabla_{U_i} U_j, E_1) &= -(\nabla h)(U_i, U_j, E_1) - h(\nabla_{U_i} E_1, U_j) \\ &= -(\nabla h)(E_1, U_i, U_j) = a_{ij} + a_{ji}, \end{aligned}$$

which proves (iii). Similarly, we can obtain (iv) and (v) from the Codazzi equation for h, noting that the invariance of E_1 implies that $c = h(\nabla_{E_1} E_1, E_1)$ is constant. □

Lemma 2 $b_i = 0$, $i = 1, \ldots, n-1$.

Proof. Let $p \in M$. Let us suppose that not all the b_i, $i = 1, \ldots, n-1$ vanish at the point p. By changing the orthonormal basis $\{U_1, \ldots, U_{n-1}\}$ of W, it follows from Lemma 1 that we may assume that $b_1 = b \neq 0$ on a connected neighbourhood U of p and $b_2 = \cdots = b_{n-1} = 0$. Then the vector field U_1 is uniquely determined. Hence $b = h(\nabla_{U_1} E_1, E_1)$ and $a_{11} = h(\nabla_{E_1} U_1, U_1)$ are constant. Using that b and c are constants it first follows from the Gauss equation that

$$\begin{aligned} 0 = R(E_1, U_j)E_1 &= \nabla_{E_1} \nabla_{U_j} E_1 - \nabla_{U_j} \nabla_{E_1} E_1 - \nabla_{[E_1, U_j]} E_1 \\ &= \nabla_{E_1}(\delta_{j1} bE_1) - \nabla_{U_j}(cE_1 + 2bU_1) + \delta_{j1} b \nabla_{E_1} E_1 - a_{j1} bE_1 \\ &= \delta_{j1} b(cE_1 + 2bU_1) - c\delta_{j1} bE_1 - 2b\nabla_{U_j} U_1 + \delta_{j1} b(cE_1 + 2bU_1) - a_{j1} bE_1 \\ &= -2b\nabla_{U_j} U_1 + (\delta_{j1} bc - a_{j1} b)E_1 + 4b^2 \delta_{j1} U_1. \end{aligned}$$

Hence, since $b \neq 0$,

$$\nabla_{U_j} U_1 = \left(\delta_{j1}\frac{c}{2} - \frac{a_{j1}}{2}\right)E_1 + 2b\delta_{j1} U_1 .$$

In particular this implies that $d_{j1}^k = d_{11}^j = 0$ for $j > 2$. Lemma 1(v) then shows that $d_{1j}^1 = 0$ for $j > 2$. By a local orthonormal change of $\{U_2, \ldots, U_{n-1}\}$, we may assume that

$$\nabla_{U_j} U_1 = 0, \quad j > 2 .$$

For the functions a_{ij} this means that

$$c = 5a_{11} ,$$

$$a_{12} = -\frac{3}{2}a_{21} ,$$

$$a_{j1} = a_{1j} = 0, \quad j > 2 .$$

Remark that if $a_{12} \neq 0$ at some point $q \in U$, then the vector field U_2 is uniquely determined (and hence G-invariant) on a neighbourhood of q. The fact that M is locally homogeneous then implies that a_{12} is constant on that neighbourhood. Therefore a_{12} is constant on U. From the Gauss equation, we then obtain

$$\begin{aligned}
-\mu E_1 &= R(U_1, E_1)U_1 = \nabla_{U_1}\nabla_{E_1}U_1 - \nabla_{E_1}\nabla_{U_1}U_1 - \nabla_{[U_1, E_1]}U_1 \\
&= \nabla_{U_1}(a_{11}U_1 + a_{12}U_2) - \nabla_{E_1}(2a_{11}E_1 + 2bU_1) \\
&\quad - \nabla_{bE_1 - a_{11}U_1 - a_{12}U_2}U_1 \\
&= a_{11}(2a_{11}E_1 + 2bU_1) + a_{12}\nabla_{U_1}U_2 - 2a_{11}(5a_{11}E_1 + 2bU_1) \\
&\quad - 2b(a_{11}U_1 + a_{12}U_2) - b(a_{11}U_1 + a_{12}U_2) \\
&\quad + a_{11}(2a_{11}E_1 + 2bU_1) + a_{12}\left(-\frac{1}{2}a_{21}\right)E_1 .
\end{aligned}$$

Since $d_{12}^1 = 0$, we find that $0 = -3ba_{11}$, so that $a_{11} = c = 0$. Moreover, we obtain that

$$0 \neq \mu = -a_{12}(a_{12} + a_{21}) + \frac{1}{2}a_{12}a_{21} = -\frac{3}{2}(a_{21})^2 ,$$

$$\nabla_{U_1}U_2 = -\frac{1}{2}a_{21}E_1 + 3bU_2 .$$

The desired contradiction then follows from

$$\begin{aligned}
0 &= R(U_1, U_2)U_1 = \nabla_{U_1}\nabla_{U_2}U_1 - \nabla_{U_2}\nabla_{U_1}U_1 - \nabla_{[U_1, U_2]}U_1 \\
&= -\frac{1}{2}ba_{21}E_1 - 2b\left(-\frac{1}{2}a_{21}E_1\right) - \nabla_{3bU_2}U_1 \\
&= \left(\frac{1}{2}ba_{21} + \frac{3}{2}ba_{21}\right)E_1 . \quad \square
\end{aligned}$$

In particular, Lemma 2 and Lemma 1(ii) imply that

$$h(K_{E_1}U_i, E_1) = -\frac{1}{2}(\nabla h)(E_1, E_1, U_i) = 0\ ,$$

so $K_{E_1}W \subset W$. Since K_{E_1} is symmetric and G-invariant, its eigenspaces are G-invariant and its eigenvalues are constant. Hence we can choose $U_1, \dots, U_{n-1}$ as eigenvectors of K_{E_1}. So with this choice we have

$$K_{E_1}U_i = \frac{1}{2}\mu_i U_i\ , \tag{3.1}$$

where μ_i is a constant real number, $i = 1, \dots, n-1$. In other words,

$$a_{ij} + a_{ji} = -(\nabla h)(E_1, U_i, U_j) = \mu_i\delta_{ij}\ . \tag{3.2}$$

Then the apolarity condition implies that

$$\sum_{k=1}^{n-1} a_{ii} + c = 0\ . \tag{3.3}$$

Now (3.1) and Lemma 1(i) imply that $\hat{\nabla}_{E_1}W = \nabla_{E_1}W - K_{E_1}W \subset W$.

Lemma 3 $\mu\delta_{ij} = c\mu_j\delta_{ij} - a_{ji}\mu_i - a_{ij}\mu_j$.

Proof. From the Gauss equation, we have

$$\begin{aligned}\delta_{ij}\mu &= h(R(E_1, U_i)U_j, E_1) = h(\nabla_{E_1}\nabla_{U_i}U_j - \nabla_{U_i}\nabla_{E_1}U_j - \nabla_{[E_1, U_i]}U_j, E_1)\\ &= E_1(\delta_{ij}\mu_j) + \delta_{ij}\mu_j c - \sum_{k=1}^{n-1} a_{jk}\mu_i\delta_{ik} - \sum_{k=1}^{n-1} a_{ik}\delta_{kj}\mu_j\\ &= c\mu_j\delta_{ij} - a_{ji}\mu_i - a_{ij}\mu_j\ . \quad \square\end{aligned}$$

Putting $i = j$ in Lemma 3, we obtain that

$$\mu = c\mu_i - \mu_i^2\ . \tag{3.4}$$

Therefore, at most 2 of the μ_i are different. From now on we have to consider two different cases.

Case 1 $\mu_i = \lambda$ *for all* i

Then from (3.3) and (3.4) we get

$$c = -\frac{1}{2}(n-1)\lambda \quad \text{and} \quad \mu = -\frac{1}{2}(n+1)\lambda^2 \neq 0\ .$$

Since in this case $K_{E_1} = \frac{1}{2}\lambda I$, there is no restriction on the choice of the U_i. We first make a suitable choice of frame.

Let's fix a point $\bar{p}$ for the remainder of the proof in this case. We have two distributions $W_0 = \text{span}\ E_1$ and W, as defined before. Since W is involutive, it can be integrated, so there is an integral manifold through each point. Let $\tilde{M}$ be an integral manifold through $\bar{p}$. Let $\gamma\colon]-\varepsilon, \varepsilon[\to M$ be the integral curve of E_1 through $\bar{p} = \gamma(0)$. We also know from (3.1) and Lemma 1(i) that

$\hat{\nabla}_{E_1} W = \nabla_{E_1} W - K_{E_1} W \subset W$. Hence we can find a frame $\{E_1, U_1, \ldots, U_{n-1}\}$ such that $\hat{\nabla}_{E_1} U_i = 0$ with arbitrary initial conditions on $\tilde{M}$. For this frame we have $\nabla_{E_1} U_i = \frac{1}{2}\lambda U_i$. From now on we always assume that this choice is made.

Lemma 4 *For all local sections X, Y and Z of W we have that* $(\nabla h)(X, Y, Z) = 0$ *and* $(\nabla h)(E_1, X, Y) = -\lambda h(X, Y)$.

Proof. The second assertion follows immediately from (3.1). It order to prove the first on, we start by

$$\begin{aligned} E_1 h(\nabla_{U_i} U_j, U_k) &= (\nabla h)(E_1, \nabla_{U_i} U_j, U_k) + h(\nabla_{E_1} \nabla_{U_i} U_j, U_k) + h(\nabla_{U_i} U_j, \nabla_{E_1} U_k) \\ &= h(\nabla_{U_i} U_j, U_k)(\nabla h)(E_1, U_k, U_k) + h(R(E_1, U_i) U_j, U_k) \\ &\quad + h(\nabla_{U_i} \nabla_{E_i} U_j, U_k) + h(\nabla_{[E_1, U_i]} U_j, U_k) + \frac{1}{2} \lambda h(\nabla_{U_i} U_j, U_k) \\ &= \frac{1}{2} \lambda h(\nabla_{U_i} U_j, U_k) , \end{aligned}$$

hence

$$E_1 (\nabla h)(U_i, U_j, U_k) = \frac{1}{2} \lambda (\nabla h)(U_i, U_j, U_k) . \tag{3.5}$$

Now let T_1 be the covariant tensor field on $W \oplus W \oplus W$ defined as the restriction of ∇h to $W \oplus W \oplus W$. The G-invariance of W and ∇h implies that the length $h(T_1, T_1)$ of T_1 is constant. But by (3.5) we have

$$0 = E_1(h(T_1, T_1)) = \lambda h(T_1, T_1) .$$

Hence $h(T_1, T_1) = 0$. Since h is definite, this implies the lemma. □

Lemma 5 *In the neighbourhood of* $\bar{p}$, *the Riemannian manifold* (M, h) *is isometric to the warped product* $]-\varepsilon, \varepsilon[\times_{\rho} (\tilde{M}, h)$, *where* $\rho(t) = e^{-\frac{1}{2}\lambda t}$.

Proof. We check Hiepko's condition [H], using the formalism of [Nö, Sect. 3]. In particular we have to check that W_0 is auto-parallel with respect to $\hat{\nabla}$, and that W is spherical. The first assertion is proved by

$$h(\hat{\nabla}_{E_1} E_1, U_i) = h(-K_{E_1} E_1, U_i) = \frac{1}{2} (\nabla h)(E_1, E_1, U_i) = 0 .$$

For the second assertion we first have that

$$\begin{aligned} h(\hat{\nabla}_{U_i} U_j, E_1) &= h(\nabla_{U_i} U_j, E_1) - h(K_{U_i} U_j, E_1) \\ &= \lambda \delta_{ij} + \frac{1}{2} (\nabla h)(E_1, U_i, U_j) \\ &= \lambda \delta_{ij} - \frac{1}{2} \lambda \delta_{ij} = \frac{1}{2} \lambda \delta_{ij} . \end{aligned}$$

This means that W is totally umbilical in M with mean curvature normal $\eta = \frac{1}{2}\lambda E_1$. Since $h(\hat{\nabla}_{U_i} E_1, E_1) = 0$, the mean curvature normal is parallel in the normal bundle, so we get that W is spherical.

Hence (M, h) is isometric to the warped product $]-\varepsilon, \varepsilon[\times_\rho(\tilde{M}, h)$, at least in a neighbourhood of $\bar{p}$. The warping function ρ is determined by

$$\eta = -E_1(\ln\rho)E_1$$

with the initial condition $\rho(0) = 1$. □

So if we know $(\tilde{M}, h)$, we know the intrinsic geometry of (M, h).

Lemma 6 *$(\tilde{M}, h)$ is flat.*

Proof. Let v and w be orthogonal unit vectors tangent to $\tilde{M}$. Then the sectional curvature $\tilde{\kappa}(v, w)$ can be derived from the Gauss equation for the isometric immersion $(\tilde{M}, h) \to (M, h)$:

$$\tilde{\kappa}(v, w) = \hat{R}(v, w, w, v) + \frac{1}{4}\lambda^2 .$$

Now from Lemma 4 we know that $K_v v = K_w w = \frac{1}{2}\lambda E_1$ and that $K_v w = K_w v = 0$, so by (2.2) we have

$$\hat{R}(v, w, w, v) = -h([K_v, K_w]w, v) = -\frac{1}{2}\lambda h(K_v E_1, v) = -\frac{1}{4}\lambda^2 .$$

This proves the lemma. □

Choosing Cartesian coordinates $(u_1, \ldots, u_{n-1})$ on $\tilde{M}$ (such that $\bar{p}$ has coordinates $(0, \ldots, 0)$) we also have coordinates $(u_1, \ldots, u_{n-1}, t)$ on M, such that $E_1 = \partial_t$ and $\partial_{u_1}, \ldots, \partial_{u_{n-1}}$ span W. Then $\nabla_{\partial u_i}\partial_t = 0$ and $h(\partial_{u_i}, \partial_t) = 0$. So the formula of Gauss implies that M is parametrized by

$$x(u_1, \ldots, u_{n-1}, t) = \bar{p} + x_1(u_1, \ldots, u_{n-1}) + x_2(t) ,$$

where $x_1(0) = x_2(0) = 0$. Note that $\tilde{M}$ is given by $\bar{p} + x_1$ and γ is given by $\bar{p} + x_2$.

First we determine the curve γ. It can be checked immediately that the vector fields $e^{-\lambda t}(\lambda E_1 + \xi)$ and $e^{\frac{1}{2}(n+1)\lambda t}(-\frac{1}{2}(n+1)\lambda E_1 + \xi)$ are constant vectors, say c_1 and c_2, along M. Then it can be seen easily that there is a constant vector c_3 such that

$$\gamma(t) = c_3 + \frac{2}{n+3}\left(\frac{1}{\lambda}e^{\lambda t}c_1 + \frac{2}{(n+1)\lambda}e^{-\frac{1}{2}(n+1)\lambda t}c_2\right) .$$

We can take the natural basis $\{e_1, \ldots, e_{n+1}\}$ of $\mathbb{R}^{n+1}$ such that $e_i = \partial_{u_i}(\bar{p})$ for $i = 1, \ldots, n-1$; $e_n = c_1$ and $e_{n+1} = c_2$.

Since ∇h, restricted to W, is identically zero, it follows that the Levi–Civita connection $\tilde{\nabla}$ of $(\tilde{M}, h)$ is the W-component of ∇. So $(\tilde{M}, \tilde{\nabla}) \to \mathbb{R}^{n+1}$ is an affine immersion in the sense of [NP1], with affine normal space span$\{E_1, \xi\}$. The affine second fundamental form α of $(\tilde{M}, \tilde{\nabla})$ is given by

$$\alpha(X, Y) = h(X, Y)(\lambda E_1 + \xi) .$$

From [NP2, Proposition 4] we obtain that $\tilde{M}$ is contained in the n-dimensional linear subspace $\mathbb{R}^n$ of $\mathbb{R}^{n+1}$ through $\bar{p}$ in the direction of $W(\bar{p})$ and $\lambda E_1 + \xi = c_1$. Therefore $(\tilde{M}, \tilde{\nabla})$ is a hypersurface of $\mathbb{R}^n$ with affine normal vector field $\tilde{\xi} = c_1$ and affine second fundamental form h. Since $\tilde{\xi}$ is parallel and $\tilde{\nabla}\tilde{h} = 0$, we obtain that $\tilde{M}$ is a paraboloid, and with the choice of coordinates, we get that $\tilde{M}$ is given by

$$\bar{p} + \left(u_1, \ldots, u_{n-1}, \frac{1}{2}\sum_{i=1}^{n-1} u_i^2, 0\right),$$

such that M is given by

$$c_3 + \left(u_1, \ldots, u_{n-1}, \frac{1}{2}\sum_{i=1}^{n-1} u_i^2 + \frac{2}{n+3}\frac{1}{\lambda}e^{\lambda t}, \frac{4}{(n+1)(n+3)\lambda}e^{-\frac{1}{2}(n+1)\lambda t}\right).$$

So after rescaling the last coordinate and translating c_3 to 0, M is of the form (**).

This finishes the proof of the first case. The second case is similar, so that a lot of details will be omitted.

Case 2 $\mu_i = \lambda_1$ *for* $i = 1, \ldots, r$ *and* $\mu_p = \lambda_2$ *for* $p = r + 1, \ldots, n - 1$

Let $s = n - r - 1$ be the multiplicity of λ_2. Then from (3.3) and (3.4) we get that $c = \lambda_1 + \lambda_2$, $(r + 2)\lambda_1 + (s + 2)\lambda_2 = 0$ and $\mu = \lambda_1\lambda_2 \neq 0$. From now on we take indices $i, j, k \leqq r$ and $p, q, x > r$.

Lemma 7

(i) $a_{ip} = a_{pi} = 0$,

(ii) $d_{ij}^p = d_{pq}^i = 0$.

Proof. The first assertion follows from Lemma 3 and (3.2). In order to prove the second one, consider

$$0 = h(R(U_p, U_i)U_j, E_1) = d_{ij}^p\lambda_2 - d_{pj}^i\lambda_1 - (d_{pi}^j - d_{ip}^j)\lambda_1 . \tag{3.6}$$

From Lemma 1(v) we can obtain that $d_{pi}^j + d_{pj}^i = d_{ip}^j + d_{ij}^p$, so (3.6) reduces to $0 = d_{ij}^p(\lambda_2 - \lambda_1)$. The second equality of (ii) is similar. □

Let's fix a point $\bar{p}$ for the remainder of the proof in this case. We now have three orthogonal distributions $W_0 = \operatorname{span} E_1$, $W_1 = \operatorname{span}\{U_1, \ldots, U_r\}$ and $W_2 = \operatorname{span}\{U_{r+1}, \ldots, U_{n-1}\}$.

From Lemma 7(ii) we find that both W_1 and W_2 are involutive. Let $\tilde{M}_1$ and $\tilde{M}_2$ be the integral manifolds through $\bar{p}$ and let γ: $]-\varepsilon, \varepsilon[\to M$ be the integral curve of E_1 through $\bar{p} = \gamma(0)$.

From Lemma 7(i) we obtain as in Case 1 that $\hat{\nabla}_{E_1} W_1 \subset W_1$ and $\hat{\nabla}_{E_1} W_2 \subset W_2$. Hence we can find a frame $\{E_1, U_1, \ldots, U_{n-1}\}$ such that $U_1, \ldots, U_r$ span W_1 and $U_{r+1}, \ldots, U_{n-1}$ span W_2, and such that $\hat{\nabla}_{E_1} U_i = \hat{\nabla}_{E_1} U_p = 0$ with arbitrary initial conditions on $\tilde{M}_1$ and $\tilde{M}_2$. For this frame we have $\nabla_{E_1} U_i = \frac{1}{2}\lambda_1 U_i$ and $\nabla_{E_1} U_p = \frac{1}{2}\lambda_2 U_p$. From now on we always assume that this choice is made.

Lemma 8 *For all local sections* X, Y *and* Z *of* $W = W_1 \oplus W_2$ *we have that* $(\nabla h)(X, Y, Z) = 0$. *For all local sections* X_1 *and* Y_1 *of* W_1 *and* X_2 *and* Y_2 *of* W_2 *we have* $(\nabla h)(E_1, X_m, Y_n) = -\lambda_m h(X_m, Y_n)$, *where* $m, n = 1, 2$.

Proof. The second assertion is clear. Like in Lemma 4, we have

$$E_1 h(\nabla_{U_i} U_j, U_k) = \frac{1}{2}\lambda_1 h(\nabla_{U_i} U_j, U_k)\,,$$

$$E_1 h(\nabla_{U_p} U_q, U_x) = \frac{1}{2}\lambda_2 h(\nabla_{U_p} U_q, U_x)\,,$$

$$E_1 h(\nabla_{U_p} U_i, U_q) = \frac{1}{2}\lambda_1 h(\nabla_{U_p} U_i, U_q)\,,$$

$$E_1 h(\nabla_{U_i} U_p, U_j) = \frac{1}{2}\lambda_2 h(\nabla_{U_i} U_p, U_j)\,.$$

In combination with Lemma 7, this implies

$$E_1(\nabla h)(U_i, U_j, U_k) = \frac{1}{2}\lambda_1(\nabla h)(U_i, U_j, U_k)\,,$$

$$E_1(\nabla h)(U_p, U_q, U_x) = \frac{1}{2}\lambda_2(\nabla h)(U_p, U_q, U_x)\,,$$

$$E_1(\nabla h)(U_p, U_i, U_q) = \frac{1}{2}\lambda_1(\nabla h)(U_p, U_i, U_q)\,,$$

$$E_1(\nabla h)(U_i, U_p, U_j) = \frac{1}{2}\lambda_2(\nabla h)(U_i, U_p, U_j)\,.$$

Now let T_1 be the covariant tensor field on $W_1 \oplus W_1 \oplus W_1$ defined as the restriction of ∇h to $W_1 \oplus W_1 \oplus W_1$. The G-invariance of W_1 and ∇h imply that the length $h(T_1, T_1)$ is constant. Like in Lemma 4, $h(T_1, T_1) = 0$. Restricting ∇h respectively to $W_1 \oplus W_1 \oplus W_2$, $W_1 \oplus W_2 \oplus W_2$ and $W_2 \oplus W_2 \oplus W_2$ proves the lemma. □

Lemma 9 *In the neighbourhood of $\bar{p}$, the Riemannian manifold (M, h) is isometric to the warped product $]-\varepsilon, \varepsilon[\times_{\rho_1}(\tilde{M}_1, h)\times_{\rho_2}(\tilde{M}_2, h)$, where $\rho_1(t) = e^{-\frac{1}{2}\lambda_1 t}$ and $\rho_2(t) = e^{-\frac{1}{2}\lambda_2 t}$.*

Proof. Following Nölker [Nö, Sect. 3], we have to check that $W_0 \oplus W_1$ and $W_0 \oplus W_2$ are auto-parallel with respect to $\hat{\nabla}$, and that W_1 and W_2 are spherical. This is done as in Case 1. One obtains that W_1 and W_2 are spherical M with mean curvature normals $\eta_1 = \frac{1}{2}\lambda_1 E_1$ and $\eta_2 = \frac{1}{2}\lambda_2 E_1$. The warping functions can be determined as in Case 1. □

The following lemma can be proved similarly as Lemma 6, so we omit the proof.

Lemma 10 *$(\tilde{M}_1, h)$ and $(\tilde{M}_2, h)$ are flat.*

Now we shall finish the proof in the second case. Choosing Cartesian coordinates $(u_1, \ldots, u_r)$ on $\tilde{M}_1$ and $(v_1, \ldots, v_s)$ on $\tilde{M}_2$, we again have coordinates $(u_1, \ldots, u_r, v_1, \ldots, v_s, t)$ on M, such that $E_1 = \partial_t$, $\partial_{u_1}, \ldots, \partial_{u_r}$ span W_1 and $\partial_{v_1}, \ldots, \partial_{v_s}$, span W_2. Clearly $\nabla_{\partial_{u_i}}\partial_t = \nabla_{\partial_{v_j}}\partial_t = 0$; from [Nö, Lemma 2], or simply from the proof of Lemma 9, we get that $\nabla_{\partial_{u_i}}\partial_{v_j} = \hat{\nabla}_{\partial_{u_i}}\partial_{v_j} = 0$. So the formula of Gauss implies that

$$x(u_1, \ldots, u_r, v_1, \ldots, v_s, t) = \bar{p} + x_1(u_1, \ldots, u_r) + x_2(v_1, \ldots, v_s) + x_3(t) ,$$

where $x_1(0) = x_2(0) = x_3(0) = 0$.

First we determine the curve γ. It can be checked immediately that the vector fields $e^{-\lambda_1 t}(\lambda_1 E_1 + \xi)$ and $e^{-\lambda_2 t}(\lambda_2 E_1 + \xi)$ are constant vectors, say c_1 and c_2, along M. So there is a constant vector c_3 such that

$$\gamma(t) = c_3 + \frac{1}{d}\left(\frac{1}{\lambda_1} e^{\lambda_1 t} c_1 - \frac{1}{\lambda_2} e^{\lambda_2 t} c_2\right),$$

where $d = \lambda_1 - \lambda_2$. We take the natural basis $\{e_1, \ldots, e_{n+1}\}$ of $\mathbb{R}^{n+1}$ such that $e_i = \partial_{u_i}(\bar{p})$ for $i = 1, \ldots, r$, $e_{r+j} = \partial_{v_j}(\bar{p})$ for $j = 1, \ldots, s$; $e_n = c_1$ and $e_{n+1} = c_2$.

The same arguments as in Case 1 show that $\tilde{M}_1$ is a paraboloid in $\operatorname{span}\{e_1, \ldots, e_r, e_n\}$, that $\tilde{M}_2$ is a paraboloid in $\operatorname{span}\{e_{r+1}, \ldots, e_{n-1}, e_{n+1}\}$ and that M is given by

$$x(u_1, \ldots, u_s, v_1, \ldots, v_s, t) = c_3 + \left(u_1, \ldots, u_r, v_1, \ldots, v_s, \frac{1}{d\lambda_1} e^{\lambda_1 t} + \frac{1}{2}\sum_{i=1}^{r} u_i^2, -\frac{1}{d\lambda_2} e^{\lambda_2 t} + \frac{1}{2}\sum_{i=1}^{s} v_i^2\right).$$

So after rescaling and translating c_3 to 0, M is of the form (*).

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