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Toeplitz operators and algebras

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Introduction

In the past two decades many papers have appeared which are concerned with generalised Toeplitz operators defined relative to an ordered or partially ordered group, and with the corresponding Toeplitz algebras. The most directly relevant to this paper are [2, 3, 5, 6]. Interest has increased in recent years with the application of K-theory, see for example [8, 9]. The key idea of the approach to Toeplitz operators taken in these papers is to study the individual operators by means of the C*-algebras that they generate. The aim is to try to extend the results of Coburn, Widom et al. on Toeplitz operators with continuous symbols on the circle group **T** (see [1, 4]). In this paper we also pursue this goal, and one of the main results is an index theorem similar to the classical Krein-Widom-Devinatz Theorem (for a statement of the latter theorem see Sect. 1).

Given an ordered group G there is associated to it a C*-algebra $T(G)$, the Toeplitz algebra. We consider the question of when $T(G)$ is Type I, and show that if G has finite rank n this is equivalent to $G = \mathbf{Z}^n$, where $\mathbf{Z}^n$ has the lexicographic order. This surprising result is obtained by means of a quotient-factorisation theorem, Theorem 2.3, which is the key result of the paper and which enables us to prove our index theorem in Sect. 3. We begin in Sect. 1 by defining our terms and setting the scene.

1 Toeplitz operators and Toeplitz algebras

An ordered group is a pair $(G, \leqq)$ where G is a discrete abelian group, $\leqq$ is a total order on G, and for all $x, y, z \in G$ we have $x \leqq y \Rightarrow x + z \leqq y + z$. We set $G^+ = \{x \in G | 0 \leqq x\}$. Obvious examples are subgroups of **R** with the induced order. We call these the ordered subgroups of **R**. It is easy to characterise when an abelian group G admits a total order making it an ordered group. This can happen if and only if G is torsion-free, or equivalently its Pontryagin dual group $\hat{G}$ is connected. We shall see more examples of ordered groups below. For the elements of the theory of these groups see [7], which may also be referred to for the theory of generalised H^p-spaces which we now need to briefly mention.

Helson and Lowdenslager generalised the classical theory of H^p-spaces by replacing the ordered group **Z** and its dual **T** by an arbitrary ordered group G and its dual $\hat{G}$. For $p \in [1, \infty]$ they define $H^p(G)$ to be the set of all $f \in L^p(\hat{G})$ for which $\hat{f}(x) = 0$ $(x \in G, x < 0)$. Here, as usual, $\hat{f}$ denotes the Fourier transform of f.

One can now define Toeplitz operators relative to G. Let P denote the orthogonal projection of the Hilbert space $L^2(\hat{G})$ onto the closed vector subspace $H^2(G)$. If $\varphi \in L^\infty(\hat{G})$ define the operator $T_\varphi \in B(H^2(G))$ by setting $T_\varphi(f) = P(\varphi f)\,(f \in H^2(G))$. We call T_φ the *Toeplitz* operator with *symbol* φ. Many results of the classical theory of Toeplitz operators on the circle extend to this setting with little modification. For instance it is shown in [6] that $\sigma(\varphi) \subseteq \sigma(T_\varphi) \subseteq \operatorname{hull} \sigma(\varphi)$, where $\sigma(\varphi)$ is the essential range of φ (its spectrum in the algebra $L^\infty(\hat{G})$), and hull $\sigma(\varphi)$ is the closed convex hull of $\sigma(\varphi)$. These inclusions are generalisations of classical results of Hartman-Wintner and Brown-Halmos respectively (see [4]). However not everything carries over in a straightforward way like this, as we shall see presently in relation to index theory.

We define the *Toeplitz algebra* of G to be the C*-algebra $T(G)$ generated by all T_φ where $\varphi \in C(\hat{G})$. The properties of these algebras are analysed in [6], and we shall need many results from that paper, among them a characterisation of $T(G)$ in terms of a very useful universal property which we now discuss.

An *isometric homomorphism* from G^+ to a unital C*-algebra B is a map $W: G^+ \to B$, $x \mapsto W_x$, for which $W_x^* W_x = 1$ and $W_{x+y} = W_x W_y$ for all $x, y \in G^+$. If $x \in G$ define $\varepsilon_x \in C(\hat{G})$ by setting $\varepsilon_x(\gamma) = \gamma(x)$. Then the map V: $G^+ \to T(G)$, $x \mapsto V_x = T_{\varepsilon_x}$, is an isometric homomorphism, and is in fact the universal one, for if W: $G^+ \to B$ is an arbitrary isometric homomorphism then there exists a unique $*$-homomorphism β: $T(G) \to B$ such that $\beta(V_x) = W_x (x \in G^+)$. Moreover if W_x is non-unitary for all $x > 0$ then β is injective.

From the above remarks it follows that there is a unique surjective $*$-homomorphism $\pi: T(G) \to C(\hat{G})$ such that $\pi(V_x) = \varepsilon_x (x \in G^+)$. In [6] it is shown that the kernel of π is the commutator ideal $KT(G)$ of $T(G)$. It is also shown that $T(G)$ acts irreducibly on $H^2(G)$, so in particular $T(G)$ is primitive.

Let us remark that (as is well known) the $(\varepsilon_x)_{x \in G}$ form an orthonormal basis for $L^2(\hat{G})$, and it is easily checked that $(\varepsilon_x)_{x \in G^+}$ form one for $H^2(G)$. Also $V_x (x \in G^+)$ generate $T(G)$.

The algebra $T(G)$ was first studied by Douglas [3] in the special case that G is an ordered subgroup of **R**. He showed that in this case $KT(G)$ is simple. If we combine this with Theorem 4.4 of [6] we see that if G is a non-cyclic ordered subgroup of **R** then $KT(G) \cap K(H^2(G)) = 0$. It follows in this case that if $\varphi \in C(\hat{G})$ never vanishes and T_φ is Fredholm then T_φ is invertible. This is surprising when we compare it to what happens when $G = \mathbf{Z}$, for if $\varphi \in C(\mathbf{T})$ then T_φ is Fredholm if and only if φ never vanishes, and in this case the Fredholm index of T_φ is minus the winding number of φ with respect to the origin. This is the simplest of index theorems and is due to Krein-Widom-Devinatz (for a proof see [4]). Thus we have a classical result which does not extend in any straightforward manner. For related index results see for example [2] and [5]. We shall return to index theory in Sect. 3, but we shall first have to analyse further the algebra $T(G)$, which we do in the next section.

2 A quotient-factorisation theorem

If $G_1, \ldots, G_n$ are ordered groups we denote by $G_1 * \cdots * G_n$ the group $G_1 \times \cdots \times G_n$ (pointwise operation) endowed with the *lexicographic order* $\leqq$. This order is defined by specifying that $(x_1, \ldots, x_n) < (y_1, \ldots, y_n)$ if $x \neq y$ and at the first index i for which $x_i \neq y_i$ we have $x_i < y_i$. We call $G_1 * \cdots * G_n$ the *lexicographic product* of the groups $G_1, \ldots, G_n$, and note that it is an ordered group.

We define the ordered group $\mathbf{Z}^n$ to be the lexicographic product of n factors, all of which are equal to the ordered group $\mathbf{Z}$.

If G_1 and G_2 are ordered groups a *positive homomorphism* from G_1 to G_2 is a group homomorphism $\theta: G_1 \to G_2$ such that $\theta(G_1^+) \subseteq G_2^+$ (equivalently, θ is an increasing map). If θ is a bijective positive homomorphism we call it an *order isomorphism*. In this case θ^{-1} is a positive homomorphism also.

An ordered group G is *archimedean* if for every pair of non-zero positive elements x, y of G there exists $n \in \mathbf{N}$ such that $x \leqq ny$. All ordered subgroups of $\mathbf{R}$ are of course archimedean, and up to order isomorphism, these are all the archimedean ordered groups (see [7, p. 194] for details).

An *ideal* in an ordered group G is a subgroup I such that $0 \leqq x \leqq y \in I^+$ implies that $x \in I$ for all $x \in G$. It is readily verified that G is achimedean if and only if its only ideals are 0 and G itself.

If x is an element of an ordered group G we define $|x| \in G^+$ in the obvious way. We set

$$F(G) = \{x \in G \mid \forall y \in G, y > 0, \exists n \in \mathbf{N}, |x| \leqq ny\} .$$

Then $F(G)$ is an archimedean ideal in G contained in every non-zero ideal of G. There exists non-zero ordered groups for which $F(G) = 0$, but if G is finitely generated and non-zero then $F(G)$ is non-zero, and this will be important in the sequel. For these elementary results, see [6].

These remarks have been leading up to the following factorisation theorem. The result will be used below, and is also of independent interest. It may well be known, but the author has no reference for it.

Theorem 2.1 *If G is a finitely generated ordered group then there exist archimedean ordered groups $G_1, \ldots, G_n$ such that $G = G_1 * \cdots * G_n$.*

Proof. We prove the result by induction on the rank $m = \operatorname{rank}(G)$ of G. Obviously a finitely generated ordered group is a finite-rank free abelian group. Thus if $m = 0$ then $G = 0$, so the result holds trivially. Suppose now that $m > 0$ and that the result holds for all ordered groups of rank less than m. Since G is non-zero, $F(G)$ is a non-zero ideal of G. Now $G/F(G)$ is an ordered group when endowed with the obvious quotient order, and is finitely generated as G is. Hence $G/F(G)$ is a finite-rank free abelian group. It follows that there exists a subgroup I of G such that $G = I \oplus F(G)$. In fact we have $G = I * F(G)$, using the fact that $F(G)$ is an ideal in G. The order on I is of course the one induced from G. Now $\operatorname{rank}(G) = \operatorname{rank}(I) + \operatorname{rank}(F(G)) > \operatorname{rank}(I)$, as $F(G) \neq 0$, so by the inductive hypothesis there exist archimedean ordered groups $G_1, \ldots, G_{n-1}$, such that $I = G * \cdots * G_{n-1}$. If we now set $G_n = F(G)$ then we have $G = G_1 * \cdots * G_n$, and the induction is completed. $\square$

If G is an ordered group we define $FT(G)$ to be the closed ideal of $T(G)$ generated by the projections $1 - V_x V_x^*(x \in F(G)^+)$. It was shown in [6] that $FT(G)$ is simple, and therefore it is contained in every non-zero closed ideal of $T(G)$ because $T(G)$ is primitive. Since V_x is non-unitary for $x > 0$ it follows that $FT(G)$ is non-zero if and only if $F(G)$ is non-zero.

Theorem 2.2 *If G is an ordered group then $FT(G)$ is Type* I *if and only if $F(G)$ is cyclic.*

Proof. If $F(G) = 0$ then $FT(G) = 0$, and therefore $FT(G)$ is trivially Type I. If $F(G)$ is non-zero cyclic then $F(G) = \mathbf{Z}x$ for a least positive element x of $F(G)$. It follows that x is in fact the least positive element of G, and from this one easily checks that the projection $1 - V_x V_x^*$ has range $\mathbf{C}_{\varepsilon_0}$. Thus $1 - V_x V_x^*$ is a non-zero element of $FT(G) \cap K(H^2(G))$. This implies that $T(G)$ contains $K(H^2(G))$, since $T(G)$ acts irreducibily on $H^2(G)$. Hence $FT(G) = K(H^2(G))$, and therefore $FT(G)$ is Type I.

Now suppose conversely that G is a non-zero ordered group for which $FT(G)$ is Type I. Since $FT(G)$ acts irreducibily on $H^2(G)$ it must contain $K(H^2(G))$, by the Type I condition. Hence $FT(G) = K(H^2(G))$, by simplicity of $FT(G)$. It follows that there exists a positive element x of $F(G)$ such that $1 - V_x V_x^*$ has least rank. Consequently x is the least positive element of $F(G)$, and therefore by the archimedean property of $F(G)$ we must have $F(G) = \mathbf{Z}x$. This proves the theorem. □

If I is an ideal in an ordered group G we define $T(G, I)$ to be the closed ideal in $T(G)$ generated by all $1 - V_x V_x^*(x \in I^+)$. (Thus $T(G, F(G)) = FT(G)$.)

If G is a lexicographic product of two ordered groups G_1 and G_2, $G = G_1 * G_2$, there does not appear to be any 'factorisation' of $T(G)$ into algebras related to G_1 and G_2. However the next result does give a factorisation of the quotient algebra $T(G)/T(G, G_2)$ (it is readily verified that G_2 is an ideal in G). This result is very important for the sequel. We shall use the symbol $\otimes$ to denote the (spatial) C*-tensor product.

Theorem 2.3 *Let G_1 and G_2 be ordered groups and let $G = G_1 * G_2$. Then there is a unique $*$-isomorphism $\beta: T(G)/T(G, G_2) \to T(G_1) \otimes C(\hat{G}_2)$ such that for all $(x, y) \in G^+$ we have*

$$\beta(V_{(x,y)} + T(G, G_2)) = V_x \otimes \varepsilon_y .$$

Proof. Put $I = T(G, G_2)$ and $Z = T(G_1) \otimes C(\hat{G}_2)$.

The map

$$G^+ \to Z, \qquad (x, y) \mapsto V_x \otimes \varepsilon_y ,$$

is an isometric homomorphism, and therefore induces a $*$-homomorphism $\alpha: T(G) \to Z$ such that $\alpha(V_{(x,y)}) = V_x \otimes \varepsilon_y$ for all $(x, y) \in G^+$. Since the $V_x(x \in G_1^+)$ generate $T(G_1)$ and the $\varepsilon_y(y \in G_2)$ generate $C(\hat{G}_2)$, it follows that the elements $V_x \otimes \varepsilon_y((x, y) \in G^+)$ generate Z, and therefore α is surjective.

If $y \in G_2^+$ then $\alpha(1 - V_{(0,y)} V_{(0,y)}^*) = 1 - (1 \otimes \varepsilon_y)(1 \otimes \varepsilon_y)^* = 1 - 1 \otimes \varepsilon_y \varepsilon_y^* = 0$. Hence $\alpha(I) = 0$, and so α induces a surjective $*$-homomorphism β: $T(G)/I \to Z$ given by $\beta(b + I) = \alpha(b)$.

The map

$$U: G_2^+ \to T(G)/I, \qquad y \mapsto V_{(0,y)} + I\,,$$

is obviously an isometric homomorphism, but since each U_y is actually unitary (by the definition of $I = T(G, G_2)$) it follows that U extends to a unitary representation $U: G_2 \to T(G)/I$. Since $C(\hat{G}_2)$ is the group C*-algebra of G_2 there is therefore a unique $*$-homomorphism $\delta_2: C(\hat{G}_2) \to T(G)/I$ such that $\delta_2(\varepsilon_y) = U_y (y \in G_2)$.

The map

$$G_1^+ \to T(G)/I, \qquad x \mapsto V_{(x,0)} + I\,,$$

is an isometric homomorphism, so there exists a unique $*$-homomorphism δ_1: $T(G_1) \to T(G)/I$ such that $\delta_1(V_x) = V_{(x,0)} + I (x \in G_1^+)$. If $x \in G_1^+$ and $y \in G_2^+$ then $\delta_1(V_x)$ and $\delta_2(\varepsilon_y)$ obviously commute, and since $\delta_2(\varepsilon_y)$ is unitary it follows that $\delta_1(V_x)$ and $\delta_2(\varepsilon_y)^*$ also commute. Hence every element of $\delta_1(T(G_1))$ commutes with every element of $\delta_2(C(\hat{G}_2))$, and therefore there exists a unique $*$-homomorphism δ: $Z \to T(G)/I$ such that $\delta(a_1 \otimes a_2) = \delta_1(a_1)\delta_2(a_2)$ for all $a_1 \in T(G_1)$ and $a_2 \in C(\hat{G}_2)$.

Let $(x, y) \in G^+$ and write $y = y_1 - y_2$ where $y_1, y_2 \in G_2^+$. Then

$$\begin{aligned} \delta\beta(V_{(x,y)} + I) &= \delta\alpha(V_{(x,y)}) \\ &= \delta(V_x \otimes \varepsilon_y) \\ &= \delta_1(V_x)\delta_2(\varepsilon_y) \\ &= V_{(x,0)} V_{(0,y_1)} V^*_{(0,y_2)} + I \\ &= V_{(x,y)} + I\,. \end{aligned}$$

Hence $\delta\beta = \mathrm{id}$, so β is injective, and therefore a $*$-isomorphism.

Uniqueness of β follows from the fact that the elements $V_{(x,y)} + I$, where $(x, y) \in G^+$, generate $T(G)/I$. □

For the next result we define $\mathbf{Z}^0 = 0$.

Theorem 2.4 *Let G be an ordered group of finite rank n. Then $T(G)$ is Type* I *if and only if G is order isomorphic to $\mathbf{Z}^n$.*

Proof. Suppose firstly that $T(G)$ is Type I. We shall prove $G = \mathbf{Z}^n$ by induction on the rank n of G. If $n = 0$ then $G = 0$, and so the result trivially holds. Suppose then $n > 0$, and the result holds for all ordered groups of rank less than n. By Theorem 2.1 $G = G_1 * \cdots * G_r$ for some archimedean ordered groups $G_1, \ldots, G_r$, and it is readily verified that G_r is necessarily $F(G)$. If we let $I = G_1 * \cdots * G_{r-1}$, then $G = I * F(G)$. Since $T(G)$ is assumed to be Type I it follows that $FT(G)$ is Type I, and so by Theorem 2.2 we have $F(G) = \mathbf{Z}y$ for some positive element y of G.

The map

$$I^+ \to T(G), \qquad x \mapsto V_{(x,0)}\,,$$

is an isometric homomorphism with $V_{(x,0)}$ non-unitary for $x > 0$, so there exists an isometric $*$-homomorphism β: $T(I) \to T(G)$ such that $\beta(V_x) = V_{(x,0)}$ for all $x \in I^+$. As $\beta(T(I))$ is a C*-subalgebra of $T(G)$ it is also Type I, and since it is $*$-isomorphic to $T(I)$, so $T(I)$ is Type I. But $\mathrm{rank}(I) = \mathrm{rank}(G) - \mathrm{rank}(F(G)) = n - 1$, so by the inductive hypothesis I is order isomorphic to $\mathbf{Z}^{n-1}$. Hence $G = I*(\mathbb{Z}y)$ is order isomorphic to $\mathbf{Z}^n$. This completes the induction.

Now suppose conversely that G is order isomorphic to $\mathbf{Z}^n$ and we shall show that $T(G)$ is Type I. Since $T(G)$ is $*$-isomorphic to $T(\mathbf{Z}^n)$ we may suppose that $G = \mathbf{Z}^n$. We again prove the result by induction on n, and note that it is trivially true when $n = 0$ or 1. Suppose then $n > 1$ and that $T(\mathbf{Z}^m)$ is Type I for all $m < n$.

Let x be the element of $\mathbf{Z}^n$ which has all of its entries 0 except for the last one, which we assume to be 1. Then $F(G) = \mathbf{Z}x$, and it follows, as in the proof of Theorem 2.2, that $FT(G) = K(H^2(G))$. Since $G = (\mathbf{Z}^{n-1}) * \mathbf{Z}$ it is a consequence of Theorem 2.3, that $T(G)/FT(G)$ is $*$-isomorphic to $T(\mathbf{Z}^{n-1}) \otimes C(\mathbf{T})$. By the inductive hypothesis $T(\mathbf{Z}^{n-1})$ is Type I, and therefore $T(\mathbf{Z}^{n-1}) \otimes C(\mathbf{T})$ is Type I. Thus $FT(G)$ and the quotient $T(G)/FT(G)$ are Type I, and therefore so is $T(G)$. This completes the induction and the proof. □

Since the 'best-behaved' C*-algebras are the Type I C*-algebras, it seems plausible that this case is a good starting point for the study of the index theory of Toeplitz operators, and this is what we examine in the next section.

3 An index theorem

For $n \geqq 1$ set $T_n = T(\mathbf{Z}^n)$, $F_n = FT(\mathbf{Z}^n)$ and $C_n = C(\hat{\mathbf{Z}}^n) = C(\mathbf{T}^n)$. From Sect. 2 we know that $F_n = K(H^2(\mathbf{Z}^n))$. If x denotes the element of $\mathbf{Z}^n$ with all entries zero except the last, which is 1, then $F(\mathbf{Z}^n) = \mathbf{Z}x$, and we have $\mathbf{Z}^n = (\mathbf{Z}^{n-1}) * F(\mathbf{Z}^n)$. It follows from Theorem 2.3 that there is a unique $*$-isomorphism β: $T_n/F_n \to T_{n-1} \otimes C_1$ such that

$$(*) \qquad \beta(V_{(x,m)} + F_n) = V_x \otimes \varepsilon_m, \quad (x,m) \in (\mathbf{Z}^n)^+ .$$

We canonically identify $C_{n-1} \otimes C_1$ with C_n by means of the unique $*$-isomorphism which maps $\varepsilon_x \otimes \varepsilon_m$ to $\varepsilon_{(x,m)}$ for all $(x, m) \in \mathbf{Z}^n$.

Theorem 3.1 *Let $\varphi \in C_{n-1}$ and $\psi \in C_1$. Then $T_{\varphi \otimes \psi}$ is Fredholm if and only if T_φ is invertible and ψ never vanishes on* $\mathbf{T}$. *In this case the Fredholm index of $T_{\varphi \otimes \psi}$ is equal to minus the winding number of ψ about the origin.*

Proof. It follows from equation (∗) that

$$\beta(T_{\varepsilon_{(x,m)}} + F_n) = T_{\varepsilon_x} \otimes \varepsilon_m, \quad (x,m) \in \mathbf{Z}^n .$$

An elementary computation shows that if φ and ψ are trigonometric polynomials in C_{n-1} and C_1 respectively then

$$(**) \qquad \beta(T_{\varphi \otimes \psi} + F_n) = T_\varphi \otimes \psi .$$

(A *trigonometric polynomial* is a linear combination of the εs). By density of the trigonometric polynomials in C_{n-1} and C_1 we deduce that equation (∗∗) holds for arbitrary $\varphi \in C_{n-1}$ and $\psi \in C_1$.

It was shown in [6] that for any ordered group G the map

$$C(\hat{G}) \to T(G)/KT(G), \qquad \varphi \mapsto T_\varphi + KT(G) ,$$

is a $*$-isomorphism. Using this, and the fact that $KT(\mathbf{Z}^n)$ contains $F_n = K(H^2(\mathbf{Z}^n))$, and using equation $(**)$, we conclude that for $\varphi \in C_{n-1}$ and $\psi \in C_1$, the Toeplitz operator $T_{\varphi \otimes \psi}$ is Fredholm if and only if T_φ is invertible in T_{n-1} and ψ is invertible in C_1. (If $T_{\varphi \otimes \psi} + KT(\mathbf{Z}^n)$ is invertible, then φ and ψ are invertible, and if also $T_\varphi \otimes \psi$ is invertible, then T_φ is invertible.)

The isometric homomorphism

$$(\mathbf{Z}^{n-1})^+ \to T_n, \qquad x \mapsto V_{(x,0)},$$

induces a unique $*$-homomorphism $\alpha: T_{n-1} \to T_n$ such that $\alpha(V_x) = V_{(x,0)}$ for all $x \in (\mathbf{Z}^{n-1})^+$. Consequently $\alpha(T_{\varepsilon_x}) = T_{\varepsilon_{(x,o)}} (x \in \mathbf{Z}^{n-1})$, and therefore since the elements ε_x have closed linear span C_{n-1}, we get $\alpha(T_\varphi) = T_{\varphi \otimes 1}$ for all $\varphi \in C_{n-1}$. Hence if T_φ is invertible, so is $T_{\varphi \otimes 1}$.

Now let $m \in \mathbf{Z}^+$ and set $S = T_{1 \otimes \varepsilon_m}$. As S is an isometry $\ker(S) = 0$. From the equation

$$S^*(\varepsilon_{(x,y)}) = \begin{cases} \varepsilon_{(x,y-m} & \text{if } (x, y-m) \geqq 0 \\ 0 & \text{if } x = 0 \text{ and } y < m \end{cases}$$

we have $\ker(S^*) = \mathbf{C}\varepsilon_{(0,0)} \oplus \mathbf{C}\varepsilon_{(0,1)} \oplus \cdots \oplus \mathbf{C}\varepsilon_{(0,m-1)}$. Thus S is Fredholm of index $-m$. It is immediate from this that

$$\operatorname{index}(T_{1 \otimes \varepsilon_m)}) = -m$$

for all $m \in \mathbf{Z}$.

Recall that if m is the winding number of a non-vanishing $\psi \in C_1$ then $\psi \bar{\varepsilon}_m$ has a logarithm in C_1, so $\psi = \varepsilon_m e^{\psi'}$ for some $\psi' \in C_1$. Suppose that $\varphi \in C_{n-1}$ and that T_φ is invertible. Then the function

$$[0, 1] \to \mathbf{Z}, t \mapsto \operatorname{index}(T_{(\varphi \otimes \varepsilon_m e^{t\psi'})}),$$

is continuous and therefore its range is a singleton set. Thus

$$\operatorname{index}(T_{\varphi \otimes \psi}) = \operatorname{index}(T_{\varphi \otimes \varepsilon_m}).$$

However if $m \in \mathbf{Z}^+$ then $\varepsilon_{(0,m)} \in H^\infty(\mathbf{Z}^n)$, and so by Proposition 3.3 of [6] we have $T_{\varphi \otimes \varepsilon_m} = T_{(\varphi \otimes 1)(1 \otimes \varepsilon_m)} = T_{\varphi \otimes 1} T_{1 \otimes \varepsilon_m}$. Thus

$$\operatorname{index}(T_{\varphi \otimes \varepsilon_m}) = \operatorname{index}(T_{\varphi \otimes 1}) + \operatorname{index}(T_{1 \otimes \varepsilon_m}) = -m,$$

as $T_{\varphi \otimes 1}$ is invertible. We thus have

$$\operatorname{index}(T_{\varphi \otimes \varepsilon_m}) = -m$$

if $m \in \mathbf{Z}^+$, and it is immediate that this formula extends to all $m \in \mathbf{Z}$. Hence $\operatorname{index}(T_{\varphi \otimes \psi})$ is equal to $-m$, i.e. to minus the winding number of ψ about the origin. □

It follows from Theorem 3.1 that if $\varphi \in C_{n-1}$ and $\psi \in C_1$ and 0 belongs neither to the range of ψ nor to the closed convex hull of the range of φ then $T_{\varphi \otimes \psi}$ is Fredholm (use the fact that $\sigma(T_\varphi) \subseteq \operatorname{hull} \sigma(\varphi)$).

For the case $n = 2$ we can sharpen our result. Recall that if $\varphi \in C_1$ then T_φ is invertible if and only if T_φ is Fredholm of index zero (see [4] for example). If we use this and the Krein-Widom-Devinatz Theorem, we can reformulate Theorem 3.1 as:

Theorem 3.2 *If $\varphi, \psi \in C(\mathbf{T})$ then $T_{\varphi \otimes \psi}$ is Fredholm if and only if φ, ψ never vanish on $\mathbf{T}$ and φ has winding number zero about the origin. In this case the Fredholm index of $T_{\varphi \otimes \psi}$ is equal to minus the winding number of ψ about the origin.*

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