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Holomorphic actions on contractible domains without fixed points

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It is known that every continuous action of $\mathbb{Z}_{p^n}$ (with p prime), or of a compact connected abelian Lie group, on a contractible manifold has a fixed point (see for instance [B]). In particular, this holds for holomorphic actions on contractible complex manifolds. On the other hand, there are examples of continuous actions of $\mathbb{Z}_{pq}$ (with p, q relatively prime), and of any compact connected non-abelian Lie group, on an euclidean space of sufficiently high dimension without fixed points (see [CF, K, CM, HH]).

In this paper we shall give examples of *holomorphic* (actually, complex linear) actions of these groups on bounded contractible pseudoconvex domains without fixed points, proving

Theorem 1 *Let* $G = \mathbb{Z}_{pq}$, *with* p, q *relatively prime, or a compact connected non-abelian Lie group. Then there exists a bounded pseudoconvex taut contractible domain* $D \Subset \mathbb{C}^n$ *where* G *acts linearly without fixed points.*

This result is relevant, for instance, in iteration theory of holomorphic maps on taut manifolds. In [A] it was conjectured that given a holomorphic self-map $f \in \mathrm{Hol}(X, X)$ of a contractible taut manifold X, the sequence of iterates of f is not compactly divergent iff f has a fixed point in X. Theorem 1 provides a counter-example to this conjecture: the map $f \in \mathrm{Aut}(D)$ generating the $\mathbb{Z}_{pq}$-action is periodic – and thus its sequence of iterates is not compactly divergent – and fixed point free. So the results of [A], showing that the sequence of iterates is not compactly divergent iff the map has a *periodic* point, are in general the best possible.

We start the proof of Theorem 1 recalling the construction of the topological examples. Let G denote either $\mathbb{Z}_{pq}$ with $(p, q) = 1$ or a compact connected non-abelian Lie group. The main point in the construction of the topological examples is the

Proposition 2 *There exists an orthogonal representation* $\psi\colon G \to SO(m+1)$ *without fixed points in* S^m *for some* m *admitting a continuous equivariant map* $f\colon S^m \to S^m$ *of degree* 0.

The complete proof of this fact is in [CF] and [HH]; here we shall sketch the proof for $G = \mathbb{Z}_{pq}$.

First of all, let us introduce a topological construction. Let X and Y be two topological spaces. The *join* $X * Y$ is the topological space obtained taking the quotient of $X \times Y \times I$ (where $I = [0, 1]$) with respect to the equivalence relation $\sim$ generated by $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$, for all $x, x_1, x_2 \in X$ and $y_1, y_2, y \in Y$. Roughly speaking, $X * Y$ is obtained taking a copy of X, a copy of Y, and attaching strings connecting any point of X to any point of Y. For instance, $S^p * S^q \cong S^{p+q+1}$; and explicit homeomorphism $\Phi\colon S^p * S^q \to S^{p+q+1}$ is given by

$$\Phi([x, y, t]) = \left(\left(\cos \frac{\pi}{2} t\right)x, \left(\sin \frac{\pi}{2} t\right)y\right) \in \mathbb{R}^{(p+1)+(q+1)}, \tag{1}$$

where $[x, y, t] \in S^p * S^q$ denotes the class of $(x, y, t) \in S^p \times S^q \times I$.

If X is a G_1-space and Y a G_2-space, then $X * Y$ is naturally a $G_1 \times G_2$-space:

$$(g_1, g_2)\cdot[x, y, t] = [g_1 \cdot x, g_2 \cdot y, t] .$$

Analogously, if $G_1 = G_2 = G$, then $X * Y$ is also a G-space. In particular, if G_1 acts linearly on S^p (i.e., the action is the restriction of an orthogonal linear action on $\mathbb{R}^{p+1}$) and G_2 linearly on S^q, then $G_1 \times G_2$ acts linearly on S^{p+q+1}.

Take $G_1 = \mathbb{Z}_{p_1}$ and $G_2 = \mathbb{Z}_{p_2}$, with $p_1, p_2 \in \mathbb{N}$ relatively prime. Then G_j acts on $S^1 \subset \mathbb{C}$ by rotations: a generator ω_j of G_j acts by $\omega_j(z) = e^{2\pi i/p_j}z$. In this way we get a linear action of $G = G_1 \times G_2 = \mathbb{Z}_{p_1 p_2}$ on $S^1 * S^1 = S^3 \subset \mathbb{C}^2$, generated by

$$T(z_1, z_2) = (e^{2\pi i/p_1} z_1, e^{2\pi i/p_2} z_2) ;$$

this action has no fixed points on S^3.

Now we describe a map $f\colon S^3 \to S^3$ G-equivariant of degree 0. Let $m_1, m_2 \in \mathbb{Z}$ be such that $m_1 p_1 + m_2 p_2 = -1$. Then define f by

$$f([z_1, z_2, t]) = \begin{cases} [z_1^{m_1 p_1 + 1}, z_2, 3t] & \text{for } 0 \leqq t \leqq 1/3 , \\ [z_1, z_2, 2 - 3t] & \text{for } 1/3 \leqq t \leqq 2/3 , \\ [z_1, z_2^{m_2 p_2 + 1}, 3t - 2] & \text{for } 2/3 \leqq t \leqq 1 . \end{cases}$$

f is clearly continuous and G-equivariant, and it is not difficult to check that f has degree zero (see for instance [CF]). In particular, f is homotopic to a constant in S^3.

Now let G again be general, i.e., either $\mathbb{Z}_{pq}$ with $(p, q) = 1$ or compact connected non-abelian Lie. By Proposition 2, we can assume that G acts linearly on S^m without fixed points, and that there is a G-equivariant map $f\colon S^m \to S^m$ homotopic to a constant in S^m. Now, the *mapping cylinder* Y_0 of f is defined by

$$Y_0 = \{[x, f(x), t] \mid x \in S^m, t \in I\} \cup \{[x, x, 1] \mid x \in S^m\} \subset S^m * S^m = S^{2m+1} .$$

Roughly speaking, Y_0 is obtained by taking two copies of S^m in $S^m * S^m$, the *top* $\{[x, x, 0] \mid x \in S^m\}$ and the *bottom* $\{[x, x, 1] \mid x \in S^m\}$, and then attaching a string from each point in the top to its image via f in the bottom. Note that, under the identification (1) of $S^m * S^m$ with $S^{2m+1} \subset \mathbb{R}^{(m+1)+(m+1)}$, the top is the subset $\{(x, 0) \in S^{2m+1}\}$ and the bottom is $\{(0, x) \in S^{2m+1}\}$.

Since f is G-equivariant, the mapping cylinder Y_0 (as well as its top and its bottom) is invariant under the induced linear G-action on S^{2m+1}. Furthermore, Y_0 has two important topological properties:

(a) Y_0 can be retracted to its bottom: the homotopy $H\colon Y_0 \times I \to Y_0$ is given by

$$H([x, y, t], s) = [x, y, (1-t)s + t].$$

(b) The top can be contracted to a point in Y_0: if $H_1\colon S^m \times I \to S^m$ is the homotopy from f to a constant function, the homotopy we need is $\tilde{H}\colon S^m \times I \to Y_0$ given by

$$\tilde{H}([x, x, 0], s) = \begin{cases} [x, f(x), 2s] & \text{for } 0 \leqq s \leqq 1/2\,, \\ [x, H_1(x, 2s-1), 1] & \text{for } 1/2 \leqq s \leqq 1\,. \end{cases}$$

Using this we can construct a contractible space where G acts without fixed points. Let $\{Y_n\}_{n \in \mathbb{N}}$ be a sequence of disjoint copies of Y_0, and let Y be the space obtained by identifying the bottom of Y_n with the top of Y_{n+1}, for $n = 0, 1, \ldots$ Y has a natural structure of G-space, where G acts without fixed points. Furthermore, Y is contractible: let $\varphi\colon S^k \to Y$ be any continuous map. Then $\varphi(S^k)$ is contained in (the image of) the union of a finite number of Y_n's, in $Y_0 \cup \ldots \cup Y_{n_0}$, say. Then, by (a), $\varphi(S^k)$ can be retracted to the bottom of Y_{n_0} and thus, by (b), to a point in Y_{n_0+1}. From this it follows that all the homotopy groups of Y vanish, and so Y is contractible (see [M, Theorem 7.5.4], for instance).

Now, the trick is that we can, more or less, equivariantly imbed Y in $\mathbb{R}^{2(m+1)} \backslash \{0\}$. First of all, let $\tau\colon \mathbb{R}^{2(m+1)} \to \mathbb{R}^{2(m+1)} = \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ be given by $\tau(x_1, x_2) = (x_2, x_1)$, and let $Y_0' = \tau(Y_0) \subset S^{2m+1}$. Clearly, the top of Y_0' is the bottom of Y_0, and conversely; moreover, Y_0' is G-invariant too. Now define Y_n, $Z_n \subset \mathbb{R}^{2(m+1)} \backslash \{0\}$ for $n \in \mathbb{N}$ by

$$Y_n = \begin{cases} \dfrac{1}{n+1}\, Y_0 & \text{if } n \text{ is even}\,, \\[2ex] \dfrac{1}{n+1}\, Y_0' & \text{if } n \text{ is odd}\,; \end{cases}$$

$$Z_n = \begin{cases} \left\{ \left(0, \left[\dfrac{t}{n+1} + \dfrac{1-t}{n+2}\right] x\right) \,\middle|\, x \in S^{m+1}, t \in I \right\} & \text{if } n \text{ is even}\,, \\[2ex] \left\{ \left(\left[\dfrac{t}{n+1} + \dfrac{1-t}{n+2}\right] x, 0\right) \,\middle|\, x \in S^{m+1}, t \in I \right\} & \text{if } n \text{ is odd}\,, \end{cases} \tag{2}$$

and set

$$Y = \bigcup_{n=0}^{\infty} (Y_n \cup Z_n)\,.$$

Since Z_n connects linearly the bottom of Y_n to the top of Y_{n+1}, it is clear that Y is contractible, and G acts on Y without fixed points.

The main point now is that we can find an open contractible G-invariant neighbourhood Ω of Y (where G acts without fixed points) which can be (morally) retracted onto Y. If $G = \mathbb{Z}_{pq}$ there is no problem: the whole construction is symplicial, and so it suffices to take a regular neighbourhood of Y in $\mathbb{R}^{2(m+1)}\setminus\{0\}$. In the general case we need a slightly more refined construction (adapted from [CM]):

Lemma 3 *Let G be as usual. Then there exists a bounded contractible domain $\Omega \Subset \mathbb{R}^{2(m+1)}$ where G acts orthogonally without fixed points.*

Proof. Define $U_0 \subset S^m * S^m$ by

$$U_0 = \{[x, y, t] \mid x, y \in S^m, t \in I, \|y - f(x)\| < 1/16\} \cup \{[x, x, 1] \mid x \in S^m\} ,$$

where $f\colon S^m \to S^m$ is the G-equivariant map provided by Proposition 2. Clearly, U_0 is a G-invariant set containing Y_0; moreover, (a) and (b) hold for U_0 too. In fact, for $x_0 \in S^m$ set

$$C(x_0) = \{[x_0, y, t] \mid y \in S^m, \|y - f(x_0)\| < 1/16, t \in I\} ,$$

$$L(x_0) = \{[x_0, y, 1] \mid y \in S^m, \|y - f(x_0)\| < 1/16\} ,$$

$$\tilde{L}(x_0) = \{y \in S^m \mid \|y - f(x_0)\| < 1/16\} ,$$

$$\tilde{C}(x_0) = \{sy \in \mathbb{R}^{m+1} \mid y \in \tilde{L}(x_0), s \in I\} .$$

$\tilde{C}(x_0)$ is a convex cone with vertex at the origin homeomorphic to $C(x_0)$. A homeomorphism $\alpha_{x_0}\colon C(x_0) \to \tilde{C}(x_0)$ is given by

$$a_{x_0}([x_0, y, t]) = ty ,$$

and sends $L(x_0)$ homeomorphically onto $\tilde{L}(x_0)$.

Since $\tilde{C}(x_0)$ is convex, we can define a homotopy $\tilde{H}_{x_0}\colon \tilde{C}(x_0) \times I \to \tilde{C}(x_0)$ by setting

$$\tilde{H}_{x_0}(z, s) = z + s(\sqrt{1 + \langle z, f(x_0)\rangle^2 - \|z\|^2} - \langle z, f(x_0)\rangle) f(x_0) ,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^{m+1}$. $\tilde{H}_{x_0}$ is such that $\tilde{H}_{x_0}(z, 0) = z$ for all $z \in \tilde{C}(x_0)$, $\tilde{H}_{x_0}(y, s) = y$ for all $y \in \tilde{L}(x_0)$ and $s \in I$, $\tilde{H}_{x_0}(z, 1) \in \tilde{L}(x_0)$ for all $z \in \tilde{C}(x_0)$ and $\tilde{H}_{x_0}(0, 1) = f(x_0)$. Therefore the homotopy $H_{x_0}\colon C(x_0) \times I \to C(x_0)$ obtained by $\tilde{H}_{x_0}$ via α_{x_0} yields a continuous deformation of $C(x_0)$ onto $L(x_0)$ sending $[x_0, x_0, 0]$ to $[x_0, f(x_0), 1]$.

Since U_0 is the union of all $C(x_0)$ as x_0 varies in S^m (and this union is disjoint outside the bottom), we get a homotopy $H\colon U_0 \times I \to U_0$ that can be used to prove (a) and (b) exactly as we did for Y_0.

Now let $U'_0 = \tau(U_0)$, and define $U_n \subset \mathbb{R}^{2(m+1)}\setminus\{0\}$ for $n \in \mathbb{N}$ by

$$U_n = \begin{cases} \dfrac{1}{n+1} U_0 & \text{if } n \text{ is even} , \\ \dfrac{1}{n+1} U'_0 & \text{if } n \text{ is odd} ; \end{cases}$$

Then $K = \bigcup_n (U_n \cup Z_n)$ – where Z_n in defined as in (2) – is still a contractible subset of $\mathbb{R}^{2(m+1)}\setminus\{0\}$ where G acts without fixed points.

Our goal is to build a G-invariant neighbourhood Ω of K in $\mathbb{R}^{2(m+1)}\setminus\{0\}$ that can be continuously deformed onto K. Set

$$W_0^{-1} = \left\{(x, y) \in \mathbb{R}^{2(m+1)}\setminus\{(0,0)\} \,\middle|\, \frac{7}{8} < \|x\| < \frac{9}{8}, \|y\| < 1/16\right\},$$

$$W_0^1 = \left\{\left(x\cos\left(\frac{\pi}{2}t\right), y\sin\left(\frac{\pi}{2}t\right)\right) \,\middle|\, \frac{7}{8} < \|x\| < \frac{9}{8}, \left\|\frac{y}{\|y\|} - f\left(\frac{x}{\|x\|}\right)\right\| < 1/16\right\},$$

$$W_0^2 = \left\{(x, y) \in \mathbb{R}^{2(m+1)}\setminus\{(0,0)\} \mid \|x\| < 1/16, \frac{7}{8} < \|y\| < \frac{9}{8}\right\},$$

and for all $n \in \mathbb{N}$

$$W_n^1 = \begin{cases} \dfrac{1}{n+1} W_0^1 & \text{if } n \text{ is even}, \\ \dfrac{1}{n+1} \tau(W_0^1) & \text{if } n \text{ is odd}; \end{cases}$$

$$W_n^2 = \begin{cases} \dfrac{1}{n+1} W_0^2 & \text{if } n \text{ is even}, \\ \dfrac{1}{n+1} \tau(W_0^2) & \text{if } n \text{ is odd}; \end{cases}$$

Finally, set

$$\Omega = W_0^{-1} \cup \bigcup_{n=0}^{\infty} (W_n^1 \cup W_n^2).$$

Clearly, Ω contains K and it is G-invariant; we must show that it is open and that it can be continuously deformed onto K.

To prove that Ω is open, it suffices to show that W_0^1 contains an open neighbourhood of each of its points with $t \neq 0, 1$. Take $z_0 = \left(x_0\cos\left(\frac{\pi}{2}t_0\right), y_0\sin\left(\frac{\pi}{2}t_0\right)\right) \in W_0^1$ with $t_0 \neq 0, 1$ (x_0, y_0 and t_0 are not uniquely determined, but any choice will do). Then there is $\varepsilon_0 > 0$ such that

$$\left\|\frac{y_0}{\|y_0\|} - f\left(\frac{x_0}{\|x_0\|}\right)\right\| < \varepsilon_0 < 1/16;$$

choose $\delta_0 > 0$ so that $\varepsilon_0 + 2\delta_0 < 1/16$. Now, there is $\delta_1 > 0$ such that

$$\|y' - y_0\| < \delta_1 \Rightarrow \left\|\frac{y'}{\|y'\|} - \frac{y_0}{\|y_0\|}\right\| < \delta_0,$$

$$\|x' - x_0\| < \delta_1 \Rightarrow \left\|f\left(\frac{x'}{\|x'\|}\right) - f\left(\frac{x_0}{\|x_0\|}\right)\right\| < \delta_0 \quad \text{and} \quad \frac{7}{8} < \|x'\| < \frac{9}{8};$$

let $\delta = \min\left\{\delta_1 \sin\left(\frac{\pi}{2} t_0\right), \delta_1 \cos\left(\frac{\pi}{2} t_0\right)\right\}$. Then it is easy to check that $\{z \in \mathbb{R}^{2(m+1)} \mid \|z - z_0\|_\infty < \delta\}$ is contained in W_0^1, where $\|\cdot\|_\infty$ is the sup norm of $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, as claimed.

Finally, we have to deform Ω onto K. First of all, we can continuously deform $W_0^{-1} \cup \bigcup_n W_n^2$ into $\bigcup_n (W_n^1 \cup Z_n)$ with maps of the kind $((x, y), s) \mapsto (x, sy)$ (actually, the set we get in this way is slightly larger than $\bigcup_n (W_n^1 \cup Z_n)$, but it can immediately be continuously deformed into the latter). Now, these maps send each $W_n^1 \cap W_n^2$ into itself: therefore we can patch them together so to get a continuous deformation of Ω into $\bigcup_n (W_n^1 \cup Z_n)$ which is the identity near points of W_n^1 with $1/3 \leqq t \leqq 2/3$.

By definition, it is clear that we can radially retract each W_n^1 onto its U_n. This yields the required continuous deformation of $\bigcup_n (W_n^1 \cup Z_n)$ onto K, which is the identity near the center of each Z_n.

Summing up, we have shown how to continuously deform Ω onto K; since K is contractible, Ω is too. □

Note that, for $G = \mathbb{Z}_{pq}$ we have obtained a domain in $\mathbb{R}^8$.

There is a theorem (see [MZ] and [S]) saying that $\Omega \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^{2m+3}$, and so one gets a topological action (trivial on the second factor) of G on an euclidean space without fixed points. But we are interested in holomorphic actions, and so we proceed in another way.

We consider $\mathbb{R}^n$ (where from now on $n = 2(m + 1)$) imbedded in $\mathbb{C}^n$ as usual, and let $X = \Omega + i\mathbb{R}^n \subset \mathbb{C}^n$ be the tube over Ω. If we extend the action of G to $\mathbb{C}^n$ by complex linearity, X is G-invariant, G has no fixed points in X and Ω is a G-invariant totally real submanifold of X. Adapting an argument of [HW], we can find a G-invariant Stein neighbourhood $X_0 \subset X$ of Ω:

Lemma 4 *Let X be a complex manifold and G a compact (not necessarily connected) Lie group acting holomorphically on X. Let $\varphi_0\colon X \to [0, +\infty)$ be a strictly plurisubharmonic G-invariant smooth function. Then $\Omega = \varphi_0^{-1}(0)$ admits a fundamental system of G-invariant Stein neighbourhoods in X.*

Proof. Let $\{C_\mu\}$ be a covering of X by G-invariant open sets with $C_\mu \Subset C_{\mu+1}$. Set $U_0 = C_2$ and

$$U_\mu = C_{2\mu+2} \setminus \overline{C_{2\mu-1}} \quad \text{for } \mu \geqq 1 .$$

$\{U_\mu\}$ is a locally finite covering of X by G-invariant open sets. Let $\{\rho_\mu\}$ be a G-invariant partition of unity subordinate to this covering. Then it is easy to find a sequence $\{r_\mu\} \subset \mathbb{R}^+$ of positive real numbers such that the function $\varepsilon\colon X \to \mathbb{R}$ given by

$$\varepsilon(z) = \sum_\mu r_\mu \rho_\mu(z)$$

vanishes at infinity and so that $\varphi_0 - \varepsilon$ is still strictly plurisubharmonic.

Let U be a given neighbourhood of Ω in X. Then we can choose the sequence $\{r_\mu\}$ so small that $\varphi_0 > \varepsilon$ on $X \setminus U$, so that

$$\Omega_\varepsilon = \{z \in X \mid \varphi_0(z) < \varepsilon(z)\}$$

is contained in U. Since Ω_ε is a G-invariant neighbourhood of Ω, it suffices to show that Ω_ε is Stein.

Let $\psi\colon \Omega_\varepsilon \to \mathbb{R}$ be given by

$$\psi = \frac{1}{\varepsilon - \varphi_0}\,.$$

An easy computation shows that ψ is strictly plurisubharmonic on Ω_ε. For $r > 0$, the set

$$D_r = \{z \in \Omega_\varepsilon \,|\, \psi(z) < r\}$$

is contained in $\{z \in X \,|\, \varepsilon(z) > r^{-1}\}$, which is relatively compact in X. Hence $D_r \Subset \Omega_\varepsilon$, ψ is a strictly plurisubharmonic (G-invariant) exhaustion of Ω_ε and, by the solution of Levi's problem, Ω_ε is Stein. □

Let $z_j = x_j + iy_j$ be the coordinates in $\mathbb{C}^n$, and define $\varphi_0\colon X \to \mathbb{R}^+$ by

$$\varphi_0(z) = \sum_{j=1}^{n} y_j^2\,.$$

Then $\Omega = \varphi_0^{-1}(0)$, and φ_0 is a G-invariant (G acts orthogonally!) strictly plurisubharmonic function. So Lemma 4 yields a G-invariant Stein neighbourhood $X_0 \subset X$ of Ω.

Since X_0 is a neighbourhood of Ω, we can find a G-invariant continuous function $h\colon \Omega \to \mathbb{R}^+$ such that

$$x_1 = \{z = x + iy \in X \,|\, \varphi_0(z) < h(x)\} \subset X_0\,.$$

Now we want a G-invariant smooth function $h_0\colon \Omega \to \mathbb{R}^+$ such that:

(a) $h_0 \leqq h$;
(b) $h_0(x) \to 0$ as x goes to the boundary of Ω;
(c) $\rho = \varphi_0 - h_0 \circ \pi$ is strictly plurisubharmonic, where $\pi\colon X \to \Omega$ is the projection $\pi(x + iy) = x$.

Then

$$D = \{z \in X \,|\, \rho(z) < 0\} \subset X_1$$

will be a G-invariant contractible (π is a retraction of deformation of D onto Ω) pseudoconvex bounded domain in $\mathbb{C}^n$, where G acts without fixed points. Furthermore, by the maximum principle D will also be taut, and so it will be the domain whose existence is stated in Theorem 1.

Since Ω is finite dimensional and paracompact, we can find a locally finite open covering $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ of Ω such that $U_\alpha \Subset \Omega$ for all $\alpha \in A$, and there is $N > 0$ such that any point of Ω is contained in the closure of at most N elements of $\mathscr{U}$; N is something like $2n + 1$. Now for every $\alpha \in A$ choose $\psi_\alpha \in C^\infty(\Omega)$ such that

(i) $\operatorname{supp}(\psi_\alpha) \subset \overline{U_\alpha}$;
(ii) $0 \leqq \psi_\alpha \leqq c_\alpha$, where

$$c_\alpha = \frac{1}{N} \min\left\{ \min_{x \in \overline{U_\alpha}} h(x),\ \min_{x \in \overline{U_\alpha}} d(x, \partial\Omega) \right\};$$

here $d(x, \partial\Omega)$ is the euclidean distance, and thus it is G-invariant;
(iii) $\|\operatorname{Hess}(\psi_\alpha)_x\| < 1/N$ for all $x \in \Omega$.

Furthermore we can clearly assume

$$\forall x \in \Omega \qquad \sum_{\alpha \in A} \psi_\alpha(x) > 0 .$$

Then set

$$h_0(x) = \int_G \sum_{\alpha \in A} \psi_\alpha(g \cdot x) d\mu(g) ,$$

where μ is the Haar measure of G. It is clear that h_0 is well-defined, and thus G-invariant and smooth, because $\mathscr{U}$ is locally finite. Now take $x \in \Omega$, and for $g \in G$ let $r(x, g)$ be the number of closure of elements of $\mathscr{U}$ containing $g \cdot x$; clearly, $r(x, g) \leqq N$. Then

$$\begin{aligned} h_0(x) &= \int_G \sum_{\alpha \in A} \psi_\alpha(g \cdot x)\, d\mu(g) \leqq \int_G \frac{1}{N} r(g, x) h(g \cdot x)\, d\mu(g) \\ &\leqq \int_G h(x)\, d\mu(g) = h(x) , \end{aligned}$$

and (a) is proved (we have used the G-invariance of h). Analogously one proves

$$h_0(x) \leqq d(x, \partial\Omega) ,$$

and so (b) follows.

We are left to show (c). It is easy to check that

$$\mathrm{Levi}(\varphi_0)(v) = \frac{1}{2} \|v\|^2 ,$$

and that

$$\mathrm{Levi}(h_0 \circ \pi)_z(v) = \frac{1}{4} \langle \mathrm{Hess}(h_0)_x(v), v \rangle = \frac{1}{4} \int_G \sum_{\alpha \in A} \langle \mathrm{Hess}(\psi_\alpha \circ g)_x(v), v \rangle\, d\mu(g) ,$$

where $\langle\, , \rangle$ is the standard hermitian product on $\mathbb{C}^n$, and the real matrix $\mathrm{Hess}(h_0)_x$ acts on $\mathbb{C}^n$ by complex linearity.

Then for all $v \in \mathbb{C}^n$ with $\|v\| = 1$ we have

$$\begin{aligned} \mathrm{Levi}(\varphi_0 - h_0 \circ \pi)_z(v) &= \frac{1}{2} - \frac{1}{4} \int_G \sum_{\alpha \in A} \langle \mathrm{Hess}(\psi_\alpha \circ g)_x(v), v \rangle\, d\mu(g) \\ &= \frac{1}{2} - \frac{1}{4} \int_G \sum_{\alpha \in A} \langle {}^t g \cdot \mathrm{Hess}(\psi_\alpha)_{g \cdot x} g(v), v \rangle\, d\mu(g) \\ &= \frac{1}{2} - \frac{1}{4} \int_G \sum_{\alpha \in A} \langle \mathrm{Hess}(\psi_\alpha)_{g \cdot x} g(v), g(v) \rangle\, d\mu(g) \\ &\geqq \frac{1}{2} - \frac{1}{4} \int_G \sum_{\alpha \in A} \|\mathrm{Hess}(\psi_\alpha)_{g \cdot x}\| \, \|g(v)\|^2\, d\mu(g) \\ &\geqq \frac{1}{2} - \frac{1}{4} \int_G \frac{r(g, x)}{N}\, d\mu(g) \\ &\geqq \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0 , \end{aligned}$$

where we have used the fact that the action of G is orthogonal, and we are done.

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