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Approximation and cohomology vanishing properties of low-dimensional compact sets in a Stein manifold

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1 Introduction

Let Λ^α denote the α-dimensional Hausdorff measure in $\mathbb{C}^N$, $N \geqq 2$, computed with respect to the Euclidean distance function. It is known that a compact set $K \subset \mathbb{C}^N$ such that $\Lambda^1(K) = 0$ is polynomially convex. Indeed its projections $\pi_j(K)$, $j = 1, \ldots, N$ into the coordinate complex axes are totally disconnected and hence $P(\pi_j(K)) = C(\pi_j(K))$; then it follows readily that also $P(K) = C(K)$, which implies the polynomial convexity of K. It is also known that a compact set $K \subset \mathbb{C}^N$ such that $\Lambda^2(K) = 0$ is a Stein compactum. Indeed its projections $\pi_j(K)$ have zero planar measure and hence, by the Hartogs–Rosenthal theorem (see [6]), $R(\pi_j(K)) = C(\pi_j(K))$; then also $R(K) = C(K)$, which implies that K is rationally convex and so it is a Stein compactum.

These results extend in a straightforward manner to the setting of a Stein manifold M of complex dimension $n \geqq 2$, for which Λ^α denotes the α-dimensional Hausdorff measure computed with respect to an arbitrary distance function compatible with the topology of M. Namely, a compact set $K \subset M$ such that $\Lambda^1(K) = 0$ is $\mathcal{O}(M)$-convex, whereas a compact set $K \subset M$ such that $\Lambda^2(K) = 0$ is a Stein compactum. This follows from the above mentioned results in $\mathbb{C}^N$ via the Remmert imbedding theorem of M into $\mathbb{C}^{2n+1}$.

That being stated, it seems natural to raise the following question: What conclusions of a parallel kind can be drawn with regard to a compact set $K \subset M$ such that $\Lambda^\alpha(K) = 0$ for some integer α with $1 \leqq \alpha \leqq 2n - 2$?

The present paper is intended to partially answer this question by proving the following two theorems:

Theorem 1 *A compact set $K \subset M$ such that $\Lambda^{2n-3}(K) = 0$ has the following approximation property: For every coherent analytic sheaf $\mathcal{S}$ on M the restriction map $H^{n-2}(M; \mathcal{S}) \to H^{n-2}(K; \mathcal{S})$ has dense image.*

Moreover a compact set $K \subset M$ such that $\Lambda^{2n-2}(K) = 0$ has a neighbourhood basis $\mathcal{U}$ of open sets such that $H^{n-1}(U; \mathcal{S}) = 0$ for every $U \in \mathcal{U}$ and every coherent analytic sheaf $\mathcal{S}$ on M.

Theorem 2 *Let q be an integer with $0 \leqq q \leqq n-2$ and set $\tilde{q} = n - \left[\frac{n}{q+1}\right]$. A compact set $K \subset M$ such that $\Lambda^{2q+1}(K) = 0$ is $\tilde{q}$-convex in M and a compact set $K \subset M$ such that $\Lambda^{2q+2}(K) = 0$ has a neighbourhood basis of $\tilde{q}$-complete open sets.*

Here "$\tilde{q}$-convex in M" means the following property: For every open neighbourhood ω of K one can find a C^∞ strongly $\tilde{q}$-plurisubharmonic proper function $u: M \to \mathbb{R}$ such that $K \subset M_0(u) = \{z \in M \mid u(z) < 0\} \Subset \omega$. Such terminology agrees with that of [8]. By the Andreotti–Grauert approximation theorem [1, Theorem 12], for every coherent analytic sheaf $\mathcal{S}$ on M the restriction map $H^{\tilde{q}}(M; \mathcal{S}) \to H^{\tilde{q}}(M_0(u); \mathcal{S})$ has dense image. Therefore an inductive limit consideration gives the following corollary of Theorem 2:

Corollary 1 *A compact set $K \subset M$ such that $\Lambda^{2q+1}(K) = 0$ has the following approximation property: For every coherent analytic sheaf $\mathcal{S}$ on M the restriction map $H^{\tilde{q}}(M; \mathcal{S}) \to H^{\tilde{q}}(K; \mathcal{S})$ has dense image ($0 \leqq q \leqq n-2$).*

Moreover a compact set $K \subset M$ such that $\Lambda^{2q+2}(K) = 0$ has a neighbourhood basis $\mathcal{U}$ of open sets such that $H^j(U; \mathcal{S}) = 0, j > \tilde{q}$, for every $U \in \mathcal{U}$ and every coherent analytic sheaf $\mathcal{S}$ on M ($0 \leqq q \leqq n-2$).

Some comments are in order.

(I.1) For $n = 2$ the second statement of Theorem 1 is equivalent to saying that K is a Stein compactum, provided $\Lambda^2(K) = 0$. It follows that for $n = 2$ the first statement of Theorem 1 is equivalent to saying that K is $\mathcal{O}(M)$-convex, provided $\Lambda^1(K) = 0$.[1] Hence for $n = 2$ Theorem 1 reduces to state the above mentioned known facts. The same is true of Theorem 2, since for $n = 2$ the only possibility is $q = 0$, hence $\tilde{q} = 0$ too, and it is known that convexity with respect to 0-plurisubharmonic functions is equivalent to $\mathcal{O}(M)$-convexity and 0-completeness is equivalent to being Stein.

(I.2) As M is Stein, $H^j(M; \mathcal{S}) = 0, j \geqq 1$, for every coherent analytic sheaf $\mathcal{S}$ on M; hence for $n \geqq 3$ the first statements of Theorem 1 and Corollary 1 are equivalent to saying that the separated spaces ${}^\sigma H^{n-2}(K; \mathcal{S})$, ${}^\sigma H^{\tilde{q}}(K; \mathcal{S})$ associated with $H^{n-2}(K; \mathcal{S})$, $H^{\tilde{q}}(K; \mathcal{S})$[2] are null provided $\Lambda^{2n-3}(K) = 0, q \geqq 1$ and $\Lambda^{2q+1}(K) = 0$, respectively.

(I.3) Theorem 2 and Corollary 1 are meaningful only for q such that $\tilde{q} \leqq n-2$, i.e. for $0 \leqq q \leqq \left[\frac{n-2}{2}\right]$. Indeed they do not give any informations when $\tilde{q} \geqq n-1$, since every co-connected compact subset of M is $(n-1)$-convex in M (see [8, Theorem 2]) and every compact subset of M has a neighbourhood basis of $(n-1)$-complete open sets.[3] Therefore for $n \geqq 3$ Theorem 1 provides informations

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[1] We recall that a compact set $K \subset M$ is $\mathcal{O}(M)$-convex if and only if is a Stein compactum and the restriction map $\Gamma(M; \mathcal{S}) \to \Gamma(K; \mathcal{S})$ has dense image for every coherent analytic sheaf $\mathcal{S}$ on M

[2] We recall that, given a topological vector space V, the separated space ${}^\sigma V$ associated with V is the quotient of V modulo the closure of the zero-element

[3] We recall that every complex manifold of dimension n with no compact connected components is $(n-1)$-complete (see [7])

independent of Corollary 1 with regard to any compact set $K \subset M$ with $\Lambda^{2q+1}(K) = 0$ or $\Lambda^{2q+2}(K) = 0$ for some integer q such that $\left[\frac{n-2}{2}\right] + 1 \leqq q \leqq n-2$. We shall see (Sect. V) that such informations are in general optimal for $q = n-2$, as far as approximation and cohomology vanishing properties are concerned; however we do not know how to decide whether for $n \geqq 4$ and $\left[\frac{n-2}{2}\right] \leqq q \leqq n-3$ it is possible or not to obtain any sharper informations.

We wish to mention two corollaries of Theorem 1.

Corollary 2 *A compact set $K \subset M$ such that $\Lambda^{2n-3}(K) = 0$ has the following approximation property: Every C^∞ $\bar{\partial}$-closed $(p, n-2)$-form defined on a neighbourhood of K can be approximated uniformly on K together with all derivatives of the coefficients by $C^\infty \bar{\partial}$-closed $(p, n-2)$-forms defined on the whole M $(0 \leqq p \leqq n)$.*

Moreover a compact set $K \subset M$ such that $\Lambda^{2n-2}(K) = 0$ verifies $H^{p,n-1}_{\bar{\partial}}(K) = 0$ $(0 \leqq p \leqq n)$.

Corollary 3 *Let $D \Subset M$ be an open domain and K a proper closed subset of bD in such a way that $bD \setminus K$ is C^1-smooth. Then:*

(a) *If $\Lambda^{2n-3}(K) = 0$ and bD is connected, every continuous* CR*-function f on $bD \setminus K$ has a continuous extension F to $\bar{D} \setminus K$ which is holomorphic on D;*

(b) *If $\Lambda^{2n-2}(K) = 0$ and f is a continuous* CR*-function on $bD \setminus K$ that satisfies the moment condition*

$$\int_{bD \setminus K} f\alpha = 0 \, ,$$

for every $C^\infty \bar{\partial}$-closed $(n, n-1)$-form α defined on a neighbourhood of $\bar{D}$, such that $(\operatorname{supp}\alpha) \cap K = \emptyset$, *then f has a continuous extension F to $\bar{D} \setminus K$ which is holomorphic on D.*

Corollary 2 follows straightforwardly from Theorem 1 by applying the Dolbeault isomorphisms, since it is known that these isomorphisms are not only algebraic, but topological as well.

Corollary 3 is in turn a consequence of Corollary 2, but not an immediate consequence: it depends on some recent work on removable singularities for the boundary values of holomorphic functions (see [9, Theorem 3 and Theorem 4]).

Corollary 3 improves and extends to the context of a Stein manifold the results of [10].

II Preliminary facts

On account of the Remmert imbedding theorem, there is no loss of generality in assuming that M be a closed complex submanifold of dimension n of some Euclidean space $\mathbb{C}^N$ $(2 \leqq n \leqq N)$.

Moreover we may assume that the Hausdorff measures on M are those computed with respect to the restriction to M of the Euclidean distance function of $\mathbb{C}^N$.

Throughout the paper we shall use the notation that, for each integer p with $1 \leqq p \leqq n$, $\mathscr{L}^p(M)$ means the set of all the holomorphic maps $l = (l_1, \ldots, l_p)$: $M \to \mathbb{C}^p$ having the following two properties:

(i) l is the restriction to M of a surjective linear map $L = (L_1, \ldots, L_p)$: $\mathbb{C}^N \to \mathbb{C}^p$;

(ii) l is non-degenerate, i.e. the analytic set $C(l)$ of critical points of l has dimension $\leqq n - 1$.

In the first place we have:

Lemma 1 *Let $K \subset M$ be a compact set such that $\Lambda^{2q+2}(K) = 0$. Then for every point $z^\circ \in M \backslash K$ it is possible to choose a holomorphic map $l \in \mathscr{L}^{q+1}(M)$ such that $l(z^\circ) \notin l(K)$ $(0 \leqq q \leqq n-2)$.*

Proof. Consider the Grassmann manifold $G_{N,N-q-1}(z^\circ)$ of $(N-q-1)$-dimensional complex affine subspaces of $\mathbb{C}^N$ passing through z°. By a result of Shiffman [11] almost every $\Pi \in G_{N,N-q-1}(z^\circ)$ does not meet K. On the other hand, it is also true that, for almost every $\Pi \in G_{N,N-q-1}(z^\circ)$, the intersection $M \cap \Pi$ is transverse in a neighbourhood of z°.

Therefore we can choose a $\Pi \in G_{N,N-q-1}(z^\circ)$ such that $K \cap \Pi$ is empty and $M \cap \Pi$ is transverse in a neighbourhod of z°. Let

$$\sum_{k=1}^{N} a_{hk}(z_k - z^\circ{}_k) = 0, \quad h = 1, \ldots, q+1$$

be independent linear equations that represent Π.

Then consider the surjective linear map $L : \mathbb{C}^N \to \mathbb{C}^{q+1}$ given by

$$L_h(z) = \sum_{k=1}^{N} a_{hk} z_k, \quad h = 1, \ldots, q+1 \ ,$$

for every $z \in \mathbb{C}^N$, and set

$$l = L|_M : M \to \mathbb{C}^{q+1} \ .$$

The analytic set $C(l)$ of critical points of l is the subset of M at which the holomorphic $(q+1)$-form $dl_1 \wedge \cdots \wedge dl_{q+1}$ vanishes. Since this form does not vanish on a neighbourhood of z° in M, it follows that $C(l)$ has dimension $\leqq n-1$, hence $l \in \mathscr{L}^{q+1}(M)$.

Moreover it is plain that $l(z^\circ) \notin l(K)$, and so the proof of the lemma is completed. q.e.d.

Next we can prove:

Lemma 2 *Let $K \subset M$ be a compact set such that $\Lambda^{2q+2}(K) = 0$. Then, given arbitrarily an open neighbourhood ω of K in M, there exist finitely many q-complete open subsets $U_1, \ldots, U_s$ of M such that $K \subset U_1 \cap \ldots \cap U_s \Subset \omega$ $(0 \leqq q \leqq n-2)$.*

Proof. Choose open neighbourhoods ω_1 and ω_2 of K in M such that $\omega_1 \Subset \omega$, $\omega_1 \subset \omega_2 \Subset M$ and ω_2 is Stein, and consider the compact set $\bar{\omega}_2 \backslash \omega_1$. By Lemma 1 we can, for every point $z \in \bar{\omega}_2 \backslash \omega_1$, find a holomorphic map $l \in \mathscr{L}^{q+1}(M)$ such that $l(z) \notin l(K)$. Then we can also find an open neighbourhood I_z of z in M and an open neigbourhood J_z of $l(K)$ in $\mathbb{C}^{q+1}$ so that $I_z \cap l^{-1}(J_z)$ is empty. It follows, since

$\bar{\omega}_2 \backslash \omega_1$ is compact, that there exist finitely many holomorphic maps $l_i \in \mathscr{L}^{q+1}(M)$ and open neighbourhoods J_i of $l_i(K)$ in $\mathbb{C}^{q+1}$, $i = 1, \ldots, s-1$, so that

$$\left(\bigcap_{i=1}^{s-1} l_i^{-1}(J_i) \right) \cap (\bar{\omega}_2 \backslash \omega_1) = \varnothing .$$

Each $l_i^{-1}(J_i)$ is q-complete. Indeed it is the intersection of M and an open subset of $\mathbb{C}^N$ which is biholomorphically equivalent to $J_i \times \mathbb{C}^{N-q-1}$ and so is q-complete.[4]

Then, after setting $U_i = l_i^{-1}(J_i)$, $i = 1, \ldots, s-1$ and $U_s = \omega_2$, we get the desired conclusion.

q.e.d.

Lemma 3 *Let $K \subset M$ be a compact set such that $\Lambda^{2q+1}(K) = 0$. Let $F \subset M$ be an $\mathcal{O}(M)$-compact set and $l: M \to \mathbb{C}^{q+1}$ a holomorphic map in $\mathscr{L}^{q+1}(M)$, and consider the compact set*

$$E = F \cap l^{-1}(l(K)) .$$

Then E is q-convex in M $(0 \leqq q \leqq n-2)$.

Proof. Let ω be an arbitrary open neighbourhood of E. Then it suffices to prove that there exists a C^∞ strongly q-plurisubharmonic proper function $u: M \to \mathbb{R}$ such that $E \subset M_0(u) = \{z \in M \mid u(z) < 0\} \Subset \omega$.

We can find a C^∞ strongly plurisubharmonic proper function $\varphi: M \to \mathbb{R}$ and an open neighbourhood J of $l(K)$ in $\mathbb{C}^{q+1}$ in such a way that

$$F \subset M_0(\varphi), \quad M_0(\varphi) \cap l^{-1}(J) \Subset \omega ,$$

where $M_0(\varphi) = \{z \in M \mid \varphi(z) < 0\}$. Moreover, by the assumption that $\Lambda^{2q+1}(K) = 0$, we also have $\Lambda^{2q+1}(l(K)) = 0$, so that $\mathbb{C}^{q+1} \backslash l(K)$ is connected; hence, according to [8, Theorem 2], $l(K)$ is q-convex in $\mathbb{C}^{q+1}$, which implies that we can find a C^∞ strongly q-plurisubharmonic proper function $\psi: \mathbb{C}^{q+1} \to \mathbb{R}$ such that

$$l(K) \subset \{w \in \mathbb{C}^{q+1} \mid \psi(w) < 0\} \Subset J .$$

Consider the functions $\exp(\varphi)$, $\exp(\psi \circ l): M \to \mathbb{R}$; then $\exp(\varphi) < 1$ on F, $\exp(\varphi) \geqq 1$ on $M \backslash M_0(\varphi)$ and $\exp(\psi \circ l) < 1$ on $l^{-1}(l(K))$, $\exp(\psi \circ l) > 1$ on $M \backslash l^{-1}(J)$. Therefore we can choose a positive integer m large enough so that $\exp(m\varphi) + \exp(m\psi \circ l) < 1$ on E and $\exp(m\varphi) + \exp(m\psi \circ l) > 1$ on $M \backslash (M_0(\varphi) \cap l^{-1}(J))$. Then set

$$u = \exp(m\varphi) + \exp(m\psi \circ l) - 1 : M \to \mathbb{R} .$$

It is plain that $E \subset M_0(u) \Subset \omega$. Moreover it is easily seen that u is strongly q-plurisubharmonic and proper, and so the proof is completed.

q.e.d.

Lemma 4 *Let $K \subset M$ be a compact set such that $\Lambda^{2q+1}(K) = 0$. Then, given arbitrarily an open neighbourhood ω of K in M, it is possible to find finitely many C^∞ strongly q-plurisubharmonic proper functions $u_1, \ldots, u_t$: $M \to \mathbb{R}$ such that $K \subset M_0(u_1) \cap \ldots \cap M_0(u_t) \Subset \omega$ $(0 \leqq q \leqq n-2)$.*

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[4] Indeed J_i is q-complete (see footnote 3)

Proof. Choose an open neighbourhood ω_1 of K in M with $\omega_1 \Subset \omega$ and a C^∞ strongly 0-plurisubharmonic proper function $\rho: M \to \mathbb{R}$ with $\bar{\omega}_1 \subset M_0(\rho)$, and consider the compact set $\overline{M_0(\rho)} \backslash \omega_1$. By Lemma 1 we can, for every point $z \in \overline{M_0(\rho)} \backslash \omega_1$, find a holomorphic map $l \in \mathscr{L}^{q+1}(M)$ such that $z \notin l^{-1}(l(K))$. Then we can also find an open neighbourhood I_z of z in M such $I_z \cap l^{-1}(l(K))$ is empty. It follows, since $\overline{M_0(\rho)} \backslash \omega_1$ is compact, that there exist finitely many holomorphic maps $l_i \in \mathscr{L}^{q+1}(M)$, $i = 1, \ldots, t-1$ so that

$$\left(\bigcap_{i=1}^{t-1} l_i^{-1}(l_i(K))\right) \cap (\overline{M_0(\rho)} \backslash \omega_1) = \varnothing \ .$$

Then, if $F \subset M$ is any $\mathcal{O}(M)$-convex compact set with $K \subset F$, and we set $E_i = F \cap l_i^{-1}(l_i(K))$, $i = 1, \ldots, t-1$, we also have

$$K \subset \bigcap_{i=1}^{t-1} E_i, \quad \left(\bigcap_{i=1}^{t-1} E_i\right) \cap (\overline{M_0(\rho)} \backslash \omega_1) = \varnothing \ .$$

Now, in view of Lemma 3, we can find C^∞ strongly q-plurisubharmonic proper functions $u_1, \ldots, u_{t-1}: M \to \mathbb{R}$ such that $E_i \subset M_0(u_i)$ and

$$\left(\bigcap_{i=1}^{t-1} M_0(u_i)\right) \cap (\overline{M_0(\rho)} \backslash \omega_1) = \varnothing \ .$$

Hence, setting $u_t = \rho$, we get the desired conclusion.
q.e.d.

For the proof of Theorem 1 we also need the following result:

Lemma 5 *Let $K \subset M$ be a compact set such that $\Lambda^{2n-3}(K) = 0$. Then for every holomorphic map $l \in \mathscr{L}^{n-1}(M)$ one has $\Lambda^{2n-1}(l^{-1}(l(K)) = 0$.*

Proof. Since l is non-degenerate, the set $C(l)$ of critical points of l verifies $\Lambda^{2n-1}(C(l)) = 0$; hence it suffices to show that $\Lambda^{2n-1}(l^{-1}(l(K)) \backslash C(l)) = 0$.

Let $z \in (l^{-1}(l(K)) \backslash C(l))$; then the differentials $dl_1, \ldots, dl_{n-1}$ are linearly independent at z, and so there exist an open neighbourhood U of z in M and a holomorphic function $l': U \to \mathbb{C}$ in such a way that $l_1, \ldots, l_{n-1}, l'$ are complex local coordinates of M valid on U. Let J be an open neighbourhood of $l(z)$ in $\mathbb{C}^{n-1}$ and J' an open neighbourhood of $l'(z)$ in $\mathbb{C}$ such that $J \times J'$ is contained in $(l, l')(U)$, and consider the open neighbourhood $U' = (l, l')^{-1}(J \times J')$ of z in M. Then

$$l(l^{-1}(K)) \cap U' \subset (l, l')^{-1}((l(K) \cap J) \times J') \ .$$

Now, since $\Lambda^{2n-3}(K) = 0$, one also has $\Lambda^{2n-3}(l(K)) = 0$, and hence $\Lambda^{2n-1}((l(K) \cap J) \times J') = 0$. It follows, as (l, l') is a diffeomorphism, that $\Lambda^{2n-1}(l^{-1}(l(K)) \cap U') = 0$. This shows that the $(2n-1)$-dimensional Hausdorff measure of $l^{-1}(l(K)) \backslash C(l)$ vanishes locally, from which the conclusion follows at once.
q.e.d.

III Proof of Theorem 1

As we have pointed out in (I.1), Theorem 1 is already known to be valid for $n = 2$. Therefore we shall consider the case $n \geqq 3$. We recall (I.2) that the first statement of the theorem is then equivalent to having

$$ {}^{\sigma}H^{n-2}(K;\mathscr{S}) = 0 \,, \tag{III.1} $$

for every coherent analytic sheaf $\mathscr{S}$ on M.

We first prove the second statement of the theorem. Thus suppose that $K \subset M$ is a compact set such that $\Lambda^{2n-2}(K) = 0$. By Lemma 2, K has a neighbourhood basis $\mathscr{U}$ of open sets which are intersections of finitely many $(n-2)$-complete open sets. Therefore it suffices to show that, if $U = U_1 \cap \ldots \cap U_s$, with each U_j being an $(n-2)$-complete open subset of M, then

$$ H^{n-1}(U;\mathscr{S}) = 0 \,, \tag{III.2} $$

for every coherent analytic sheaf $\mathscr{S}$ on M.

We can proceed by induction on the number of the U_j's, granted that, by the $(n-2)$-completeness of each U_j, (III.2) is true for $s = 1$. Thus let $s \geqq 2$, and assume inductively that (III.2) is true for $U' = U_2 \cap \ldots \cap U_s$, i.e. $H^{n-1}(U';\mathscr{S}) = 0$. Since $H^n(U_1 \cup U';\mathscr{S}) = 0$ (see footnote 3), the Mayer–Vietoris sequence

$$ \cdots \to H^{n-1}(U_1;\mathscr{S}) \oplus H^{n-1}(U';\mathscr{S}) \to H^{n-1}(U;\mathscr{S}) \to H^n(U_1 \cup U';\mathscr{S}) \to \cdots $$

implies immediately that (III.2) is valid.

Next we take up the proof of the first statement of Theorem 1, which requires more effort. Thus suppose that $K \subset M$ is a compact set such that $\Lambda^{2n-3}(K) = 0$.

In the first place we deal with the case that $\mathscr{S} = \mathscr{F}$ is a locally free analytic sheaf on M. Since M is Stein and $\mathscr{F}$ is locally free, it is known that the cohomology spaces with compact supports $H_c^q(M;\mathscr{F}) = 0$ for $q \neq n$ (see [2]). In particular, as $n \geqq 3$, $H_c^{n-2}(M;\mathscr{F})$ and $H_c^{n-1}(M;\mathscr{F})$ are null; hence the coboundary homomorphism

$$ \delta : H^{n-2}(K;\mathscr{F}) \to H_c^{n-1}(M\backslash K;\mathscr{F}) $$

is an algebraic isomorphism. On the other hand it is known that δ is a continuous map (see [3]); therefore the thesis will follow if we prove that ${}^{\sigma}H_c^{n-1}(M\backslash K;\mathscr{F}) = 0$. This in turn is equivalent to proving that ${}^{\sigma}H^1(M\backslash K;\check{\mathscr{F}} \otimes \Omega^n) = 0$, where $\check{\mathscr{F}} = \mathrm{Hom}_{\mathscr{O}}(\mathscr{F};\mathscr{O})$ is the dual of $\mathscr{F}$ and Ω^n is the sheaf of germs of holomorphic n-forms on M. As a matter of fact, by the refined version [2, VII.4.2] of the Serre duality theorem, there is a topological duality between ${}^{\sigma}H_c^{n-1}(M\backslash K;\mathscr{F})$ and ${}^{\sigma}\mathrm{Ext}^1(M\backslash K;\mathscr{F},\Omega^n)$, and since $\mathscr{F}$ is assumed to be locally free, one has $\mathrm{Ext}^*(M\backslash K;\mathscr{F},\Omega^n) \cong H^*(M\backslash K;\check{\mathscr{F}} \otimes \Omega^n)$. Indeed we shall prove that

$$ H^1(M\backslash K;\check{\mathscr{F}} \otimes \Omega^n) = 0 \,. \tag{III.3} $$

By Lemma 1 we have $K = \bigcap_l l^{-1}(l(K))$, where l ranges through $\mathscr{L}^{n-1}(M)$, and since M has a countable topology, we can find a sequence $\{l_i\}_{i=1}^{\infty}$ of maps $l_i \in \mathscr{L}^{n-1}(M)$ such that $K = \bigcap_{i=1}^{\infty} l_i^{-1}(l_i(K))$.

Let moreover $F \subset M$ be an $\mathscr{O}(M)$-convex compact set with $K \subset F$ and set, for every positive integer i,

$$ E_i = F \cap l_i^{-1}(l_i(K)), \quad U_i = M\backslash E_i \,. $$

Then $\mathscr{U} = \{U_i\}_{i=1}^{\infty}$ is an open covering of $M\backslash K$. Consider the nerve $N(\mathscr{U})$ of this covering and the Čech cohomology space $H^1(N(\mathscr{U});\check{\mathscr{F}} \otimes \Omega^n)$. We claim that

$$ H^1(N(\mathscr{U});\check{\mathscr{F}} \otimes \Omega^n) = 0 \,. \tag{III.4} $$

As a matter of fact, let f be a 1-cocycle of $N(\mathscr{U})$ with coefficients in $\check{\mathscr{F}} \otimes \Omega^n$, that is a function which associates to each ordered pair i, j of positive integers a section $f_{ij} \in \Gamma(U_i \cap U_j; \check{\mathscr{F}} \otimes \Omega^n)$, in such a way that $f_{ij} + f_{jk} = f_{ik}$ on $U_i \cap U_j \cap U_k$, for all positive integers i, j, k. We have $U_i \cap U_j = M \backslash (E_i \cup E_j)$ and, in view of Lemma 5,

$$\Lambda^{2n-1}(E_i \cup E_j) = 0 ;$$

hence $E_i \cup E_j$ is a co-connected compact subset of M. Then, since $\check{\mathscr{F}} \otimes \Omega^n$ is locally free, so that $H^1_c(M; \check{\mathscr{F}} \otimes \Omega^n) = 0$, the Hartogs theorem implies that f_{ij} extends to a section $\tilde{f}_{ij} \in \Gamma(M; \check{\mathscr{F}} \otimes \Omega^n)$. It follows that f is a coboundary, i.e. for each positive integer i there is a section $f_i \in \Gamma(U_i; \check{\mathscr{F}} \otimes \Omega^n)$ in such a way that $f_i - f_j = f_{ij}$ on $U_i \cap U_j$ for all positive integers i, j. Indeed it suffices to take $f_i = \tilde{f}_{i1}|_{U_i}$. Therefore (III.4) is valid.

Next we claim that

(III.5) For every positive integer i, $H^1(U_i; \check{\mathscr{F}} \otimes \Omega^n) = 0$.

As a matter of fact we have $U_i = M \backslash E_i$, and since, by Lemma 3, $0 = {}^\sigma H^{n-2}(E_i; \mathscr{F}) = H^{n-1}(E_i; \mathscr{F})$, the cohomology sequence with compact supports

$$\cdots \to H^{n-2}_c(M; \mathscr{F}) = 0 \to H^{n-2}(E_i; \mathscr{F}) \to H^{n-1}_c(U_i; \mathscr{F}) \to H^{n-1}_c(M; \mathscr{F}) =$$
$$= 0 \to H^{n-1}(E_i; \mathscr{F}) = 0 \to H^n_c(U_i; \mathscr{F}) \to H^n_c(M; \mathscr{F}) \to 0$$

implies that ${}^\sigma H^{n-1}_c(U_i; \mathscr{F}) = 0$ and that $H^n_c(U_i; \mathscr{F})$ is separated. It follows, on account of [2, VII.4.2], that (III.5) holds.

Now the desired conclusion that (III.3) is valid is a straightforward consequence of a special case of the Leray theorem applied to the open covering $\mathscr{U}$ of $M \backslash K$ (see [5, 12.8]).

Thus the proof of the first statement of Theorem 1 is completed for the case that $\mathscr{S} = \mathscr{F}$ is locally free.

Finally, assume that $\mathscr{S}$ is an arbitrary coherent analytic sheaf on M. Since M is Stein and K is compact, we can find a Stein open neighbourhood M' of K in M and a positive integer p such that an exact sequence of sheaves $\mathscr{O}^p \xrightarrow{\mu} \mathscr{S} \to 0$ is valid on M'. Let $\mathscr{R}$ be the kernel of μ; then $\mathscr{R}$ is a coherent analytic sheaf on M' and there is an exact cohomology sequence

$$\cdots \to H^{n-2}(K; \mathscr{O}^p) \xrightarrow{\mu_*} H^{n-2}(K; \mathscr{S}) \to H^{n-1}(K; \mathscr{R}) \to \cdots ,$$

and since, by the above, $H^{n-1}(K; \mathscr{R}) = 0$ and (III.1) is true for the sheaf $\mathscr{O}^p$, and since μ_* is a continuous map, it follows that (III.1) is true for the sheaf $\mathscr{S}$ as well.

Now the proof of Theorem 1 is completed.

IV Proof of Theorem 2

After having established Lemma 2 and Lemma 4 in Sect. II, Theorem 2 turns out to be a corollary of the following result of Diederich and Fornaess [4] on smoothing continuous q-plurisubharmonic functions: Let Ω be an n-dimensional complex manifold and $\rho:\Omega \to \mathbb{R}$ a function that is locally the supremum of finitely many C^∞ strongly q-plurisubharmonic functions. Then, given arbitrarily a continuous positive function $\eta:\Omega \to \mathbb{R}$, there exists a C^∞ strongly $\tilde{q}$-plurisubharmonic function $\sigma:\Omega \to \mathbb{R}$, $\tilde{q} = n - \left[\frac{n}{q+1}\right]$, such that $|\rho - \sigma| < \eta$ on Ω $(0 \leqq q \leqq n-1)$.

It should be observed that here we adopt the convention that "q-plurisubharmonic" means that the number of non-negative eigenvalues of the Levi form is $\geqq n-q$, whereas in [4] it means that this number is $\geqq n-q+1$ (likewise as in [1]). For this reason in [4] one finds $1 \leqq q \leqq n$ and $\tilde{q} = n - \left[\frac{n}{q}\right] + 1$.

That being stated, we take up the proof of Theorem 2.

We first prove the second statement of the theorem. Thus assume that $K \subset M$ is a compact set such that $\Lambda^{2q+2}(K) = 0$. By Lemma 2, K has a neighbourhood basis of open sets each of which is the intersection of finitely many q-complete open sets $U_1, \ldots, U_s$. Hence it suffices to show that $\Omega = U_1 \cap \ldots \cap U_s$ is $\tilde{q}$-complete. As a matter of fact, let $\rho_i: U_i \to \mathbb{R}$ be a C^∞ strongly q-plurisubharmonic proper function exhibiting the q-completeness of U_i, $i = 1, \ldots, s$, and consider on Ω the function $\rho = \sup\{\rho_1, \ldots, \rho_s\}$, which is a continuous proper function. Then there is a C^∞ strongly $\tilde{q}$-plurisubharmonic function $\sigma:\Omega \to \mathbb{R}$ with $\sigma > \rho - 1$. Hence σ is proper, which entails the $\tilde{q}$-completeness of Ω.

Next we prove the first statement of Theorem 2. Thus assume that $K \subset M$ is a compact set such that $\Lambda^{2q+1}(K) = 0$. By Lemma 4, given arbitrarily an open neighborhood ω of K, there is a continuous proper function $\rho: M \to \mathbb{R}$ that is globally the supremum of finitely many C^∞ strongly q-plurisubharmonic functions on M, such that $K \subset \{z \in M | \rho(z) < 0\} \Subset \omega$. Then if we take a small $\varepsilon > 0$, we still have $\{z \in M | \rho(z) < \varepsilon\} \Subset \omega$ and $K \subset \{z \in M | \rho(z) < -\varepsilon\}$. Let $\sigma: M \to \mathbb{R}$ be a C^∞ strongly $\tilde{q}$-plurisubharmonic function such that $|\rho - \sigma| < \varepsilon$ on M. Then σ is proper and $K \subset M_0(\sigma) \Subset \omega$. It follows that K is $\tilde{q}$-convex in M.

The proof of Theorem 2 is then completed.

V Concluding remarks

We claimed in (I.3) that Theorem 1 provides the best possible result on approximation and cohomology vanishing properties of an arbitrary compact set $K \subset M$ such that $\Lambda^{2n-3}(K) = 0$ or $\Lambda^{2n-2}(K) = 0$.

To prove this claim we consider the following two compact subsets of $\mathbb{C}^n$, $n \geqq 3$:

$$K_1 = \{z \in \mathbb{C}^n \,|\, |z_1|^2 + \cdots + |z_{n-2}|^2 = |z_{n-1}| = 1, z_n = 0\}\,,$$

$$K_2 = \{z \in \mathbb{C}^n \,|\, |z_1|^2 + \cdots + |z_{n-2}|^2 + |z_{n-1}|^2 = 1, z_n = 0\}\,.$$

Then $\Lambda^{2n-3}(K_1) = 0$ and $\Lambda^{2n-2}(K_2) = 0$, so that the two statements of Theorem 1 apply to K_1 and K_2, respectively, hence for every coherent analytic sheaf $\mathcal{S}$ on $\mathbb{C}^n$ the restriction map $H^{n-2}(\mathbb{C}^n; \mathcal{S}) \to H^{n-2}(K_1; \mathcal{S})$ has dense image and there is a neighbourhood basis $\mathcal{U}$ of K_2 of open sets such that $H^{n-1}(U; \mathcal{S}) = 0$ for every $U \in \mathcal{U}$, whence $H^{n-1}(K_2; \mathcal{S}) = 0$.

However it is not true that the restriction map $H^{n-3}(\mathbb{C}^n; \mathcal{S}) \to H^{n-3}(K_1; \mathcal{S})$ have dense image, and that $H^{n-2}(K_2; \mathcal{S}) = 0$. In fact, if the restriction map

$H^{n-3}(\mathbb{C}^n; \mathscr{S}) \to H^{n-3}(K_1; \mathscr{S})$ had dense image, the same would be true of the restriction map $H^{n-3}(\mathbb{C}^{n-1}; \mathscr{S}) \to H^{n-3}(K_1; \mathscr{S})$, since $K_1 \subset \mathbb{C}^{n-1}$ (the latter being imbedded in $\mathbb{C}^n$ as the hyperplane $z_n = 0$); and since K_1 has a neighbourhood basis of $(n-3)$-complete open sets, this would imply that $H^{2n-4}(K_1; \mathbb{C}) = 0$, whereas $H^{2n-4}(K_1; \mathbb{C}) \cong \mathbb{C}$. Similarly, if we had $H^{n-2}(K_2; \mathscr{S}) = 0$, then, since $K_2 \subset \mathbb{C}^{n-1}$, it would follow that $H^{2n-3}(K_2; \mathbb{C}) = 0$, whereas $H^{2n-3}(K_2; \mathbb{C}) \cong \mathbb{C}$.

Thus we have seen that there are topological obstructions to improving Theorem 1 with regard to an arbitrary K such that $\Lambda^{2n-3}(K) = 0$ or $\Lambda^{2n-2}(K) = 0$.

On the other hand, if we consider, for $n \geqq 4$, a compact set $K \subset M$ such that $\Lambda^{2q+1}(K) = 0$, or $\Lambda^{2q+2}(K) = 0$, for some q with $\left\lceil \frac{n-2}{2} \right\rceil \leqq q \leqq n-3$, such topological obstructions do not occur anymore, since $\check{H}^j(K; \mathbb{C}) = 0$ for $j \geqq 2q+1$, or $j \geqq 2q+2$, respectively. For this reason it seems rather difficult to exhibit any counterexamples to sharpening Theorem 1 for q in that range.

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