Math. Z. 207, 305–318 (1991)

Mathematische Zeitschrift

Calabi-Yau manifolds and a conjecture of Kobayashi

Thomas Peternell

Mathematisches Institut der Universität Bayreuth, Postfach 101251, W-8580 Bayreuth, Federal Republic of Germany

Received May 7, 1990

Introduction

A Calabi-Yau manifold is by definition a projective manifold with trivial canonical class but without any holomorphic 1-forms. One of the basic problems for those class of Ricciflat manifolds is the existence of rational curves. In dimension 2 the Calabi-Yau manifolds are just the K3-surfaces and the existence of rational curves is well-known. In dimension 3 Wilson [W] recently showed that any Calabi-Yau 3-fold with $b_2(X) > 19$ contains rational curves and he also shows existence of rational curves if there are certain special divisors on X. In this paper we show (Sect. 1).

Theorem. *Let X be a Calabi-Yau 3-fold and assume the existence of an (non-zero) effective non-ample divisor on X. Then X contains a rational curve.*

Of course such a divisor can only exist if $b_2(X) \geqq 2$.

The proof is based on some results of Wilson's and on a careful analysis of surfaces $S \subset X$ which are not ample divisors on X. In many cases we find rational curves inside S (e.g. when S is even not nef) but not always.

In Sect. 2 we consider hyperbolic 3-folds X. Hyperbolicity means that there is not non-constant holomorphic $f: \mathbb{C} \to X$. We prove that – following a conjecture of Kobayashi – X has ample canonical class except possibly for Calabi-Yau 3-folds without rational curves.

§ 0 Preliminaries

We always denote by X a smooth complex projective manifold. ω_X will always be the canonical sheaf (bundle), $\kappa(X)$ denotes the Kodaira dimension, and $h^q(X, \mathscr{F})$ is the dimension of $H^q(X, \mathscr{F})$ for a suitable sheaf $\mathscr{F}$ on X.

(0.1) Div (X) is by definition the group of Cartier-divisors modulo linear equivalence. A $\mathbb{Q}$-divisor on X is an element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, a $\mathbb{R}$-divisor an element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. $N^1_{\mathbb{R}}(X)$ will be the vector space of all real divisors modulo linear equivalence, analogously we define $N^1_{\mathbb{Q}}(X)$. $N^1_{\mathbb{R}}(X)$ is a finite-dimensional $\mathbb{R}$-vector space of dimension $\rho(X)$, the Picard number of X. Often we will not distinguish between a divisor D and its class in $N^1_{\mathbb{R}}(X)$. We denote $K \subset N^1_{\mathbb{R}}(X)$ the cone generated by the ample divisors. K is called the ample cone.

We let $N_1(X) := \{1\text{-cycles on } X\}/\approx$ where $Z \approx Z'$ iff $DZ = DZ'$ for all $D \in \mathrm{Div}(X)$. $\overline{NE}(X) \subset N_1(X)$ will always denote the closed convex cone generated by the classes of irreducible curves.

(0.2) Lemma *Let X be a smooth projective 3-fold. Let $D \in \mathrm{Div}(X)$ be nef and H ample. If $D^2 \neq 0$ then $D^2 H > 0$.*

Proof. By (0.3) we have $D^2 H \geqq 0$ (Kleiman). So assume $D^2 \neq 0$ but $D^2 H = 0$. Let $V = \{E \in N_{\mathbb{R}}(X) \mid ED^2 = 0\}$. Then codim $V = 1$. Let $Z_- = \{E \mid ED^2 < 0\}$. Since $D^2 H = 0$, $V \cap K \neq \{0\}$ and hence $Z_- \cap K \neq \emptyset$. Thus we find an ample H' with $D^2 H' < 0$. This contradicts the nefness of D by Kleiman's result.

(0.3) Let us fix two notations from the theory of algebraic surfaces (cf. [BPV]).

(a) An elliptic surface is a smooth projective surface S together with a surjective morphism $p: S \to C$ to a compact Riemann surface C such that the general fiber is a smooth elliptic curve.

(b) A hyperelliptic surface S is an elliptic surface $p: S \to C'$ such that p is holomorphically locally trivial and C is an elliptic curve.

All what we need of the theory of surfaces can be found in [BPV].

(0.4) We need some facts concerning non-normal surfaces. These can be found in [Mo, 3.36]. Let S be a projective non-normal Gorenstein surface. Let $E \subset S$ be the non-normal locus, with structure given by the conductor ideal. Let f: $\tilde{S} \to S$ be the normalization of S, $\tilde{E}$ the analytic preimage of E.

Then there are exact sequences.

$$0 \to \omega_{\tilde{S}} \to f^*(\omega_S) \to f^*(\omega_S) \otimes \mathcal{O}_{\tilde{E}} \to 0$$
$$0 \to \mathcal{O}_S \to f_*(\mathcal{O}_{\tilde{S}}) \to \omega_S^{-1} \otimes \omega_E \to 0.$$

(0.5) Definition. A Calabi-Yau manifold is a projective manifold with trivial canonical bundle ω_X and $H^1(X, \mathcal{O}_X) = 0$.

(0.6) Remark. Let X be an arbitrary projective manifold with $\omega_X \simeq \mathcal{O}_X$. Then a theorem of Beauville [Be] states the existence of a finite étale covering of the form

$$T \times \Pi V_j \times \Pi X_i$$

with T a torus, X_i simply connected sympletic manifolds (of even dimension, in particular) and V_j simply connected manifold of dimension $\geqq 3$ satisfying $H^q(V_j, \mathcal{O}_{V_j}) = 0$, $0 < q < \dim V_j$. So the V_j are Calabi-Yau.

(0.7) Proposition (Wilson). *Let X be a Calabi-Yau 3-fold. Let $D \in \mathrm{Div}(X)$, $D^3 = 0$, D nef, $D \cdot c_2(X) \neq 0$ or $h^0(\mathcal{O}_X(mD)) \geqq 2$ for some $m \in \mathbb{N}$ and $D^2 H > 0$ for some ample H.*

Then there exists $n \in \mathbb{N}$ such that $\mathcal{O}_X(nD)$ is generated by global sections and the associated morphism $\phi: X \to Y$ is an elliptic fiber space over a normal projective surface Y.

Proof. [W, 3.2]. The assumption $D \cdot c_2(X) \neq 0$ is only used to conclude $h^0(\mathcal{O}_X(mD)) \geqq 2$ for some m.

(0.8) Proposition. *Let X be a Calabi-Yau 3-fold, $D \in \mathrm{Div}(X)$, D nef, $D^3=0$, $Dc_2(X) \neq 0$, and $D^2 H > 0$ for some ample H. Then X contains a rational curve.*

Proof. Let $\phi: X \to S$ be the associated elliptic fiber space. As Wilson remarked, S is rational. Let us give an easy argument. After eventually lifting back ϕ to a desingularisation of S, we may assume S smooth. From $H^1(X, \mathcal{O}_X)=0$ we deduce by a Leray spectral sequence argument:

$$H^1(S, \mathcal{O}_S)=0.$$

Hence it is sufficient to know $\kappa(S) = -\infty$ in order to conclude rationality.

By Iitaka's $(C_{3,1})$: $\kappa(S)+\kappa(X_y)$ for the general smooth fiber X_y, we conclude:

$$\kappa(S) \leqq 0.$$

$H^2(X, \mathcal{O}_X)=0$ implies $H^2(S, \mathcal{O}_S)=0$ (think of holomorphic 2-forms). Thus the Kodaira classification of surfaces says: S is hyperelliptic or an Enriques surface. $H^1(S, \mathcal{O}_S)=0$ excludes the hyperelliptic surfaces.

S being an Enriques surface, S carries – whether minimal or not – an "elliptic pencil" $S \to \mathbb{P}_1$ [BPV, p. 274]. Thus we obtain a map $\psi: X \to \mathbb{P}_1$. Let Y be its general smooth fiber. We see easily $\kappa(Y)=0$. By $\omega_Y \simeq \omega_X | Y$, we deduce $H^2(\mathcal{O}_Y) = H^0(\omega_Y) \neq 0$. So the minimal model Y_m is K3 or a torus. Since Y_m admits an elliptic fibration over an elliptic curve, this is not possible and finally S is rational.

Now take a rational curve $C \subset S$. Let $X_C = \phi^{-1}(C)$. The general fiber of $\phi | X_C$ is a smooth elliptic curve (observe that we may assume that any fiber over $y \in S \setminus \mathrm{Sing}(Y)$ is (after reduction) elliptic because otherwise the fiber would contain rational curves).

Let f: $\tilde{C} \simeq \mathbb{P}_1 \to C$ be the normalization, $\tilde{X}_C = X_C \times_C \tilde{C}$ and ρ: $\hat{X}_C \to \tilde{X}_C$ a minimal desingularisation. Since we may assume X_C without rational curves, $\hat{\phi}$: $\hat{X}_C \to \tilde{C}$ is relatively minimal (in the sense of [BPV]).

Clearly $\kappa(\hat{X}_C) \leqq 1$. If $\kappa(\hat{X}_C) = -\infty$, X contains rational curves. Next suppose $\kappa(\hat{X}_C)=0$. Clearly $\hat{X}_C$ is minimal. If $\hat{X}_C$ has a fiber whose reduction is not smooth elliptic, then X_C contains a rational curve. If the reductions of all fibers are smooth elliptic, then $c_2(\hat{X}_C)=0$ ([BPV, p. 97]). Hence $\hat{X}_C$ is a torus or hyperelliptic. Both is clearly not possible.

Last assume $\kappa(\hat{X}_C)=1$. Again red $\hat{\phi}^{-1}(y)$ is smooth elliptic for all y, otherwise we are done. Now again $c_2(\hat{X}_C)=0$, hence $\chi(\mathcal{O}_{\hat{X}_C})=0$ by Riemann-Roch. Moreover $R^1 \hat{\phi}_*(\mathcal{O}_{\hat{X}_C})$ is topologically trivial ([BPV, p. 162]), hence trivial ($\tilde{C} \simeq \mathbb{P}_1$). The Leray spectral sequence gives $H^1(\mathcal{O}_{\hat{X}_C}) \simeq \mathbb{C}$, so $H^2(\mathcal{O}_{\hat{X}_C})=0$.

On the other hand by [BPV, p. 162]:

$$\omega_{\hat{X}_C} \simeq p^*(\mathcal{O}_{\mathbb{P}_1}(a)) \text{ with some } a > 0.$$

Hence $H^0(\mathcal{O}_{\hat{X}_C}) = H^2(\mathcal{O}_{\hat{X}_C}) \neq 0$. This ends the proof.

The following is well known (cp. [W]):

(0.9) Proposition. *Let X be a Calabi-Yau, 3-fold $D \in \mathrm{Div}(X)$. If D is big ($D^3 > 0$) and nef but not ample, there exists a rational curve on X.*

§ 1 Non-ample divisors in Calabi-Yau 3-folds

In this section we prove the main result of the paper:

(1.1) Theorem. *Let X be a Calabi-Yau 3-fold, $S \subset X$ an irreducible hypersurface. Assume that S is not an ample divisor. Then X contains a rational curve.*

Proof. (0) We begin with some general observations. Let $f: \tilde{S} \to S$ be the normalization, $\pi: \hat{S} \to \tilde{S}$ a minimal desingularization. First observe that $\hat{S}$ may supposed to be minimal. In fact, if $\hat{S}$ is not minimal, pick up a (-1)-curve $C \subset \hat{S}$. Since π is minimal, dim $\pi(C)=1$, hence $f\pi(C)$ is a rational curve in S and we are done.

Also we may assume $\kappa(\hat{S}) \geqq 0$. Moreover, if $\hat{S}$ is an elliptic surface $p: \hat{S} \to C$, the only singular fibers can be multiples of smooth elliptic curves, otherwise we obtain a rational curve in X. In fact, a singular fiber different from a multiple of a smooth elliptic curve consists of rational curves ([BPV, p. 150/151]). Since $\dim f\pi(p^{-1}(x))=1$ we are done in this case. If $S^3>0$ and S is nef then X contains rational curves by (0.9). We distinguish two cases:

(1) S is not nef

(2) S is nef.

(1.2) Lemma. *If S is not nef, X contains a rational curve.*

Proof. S not being nef there is a curve C with $S \cdot C<0$, in particular $C \subset S$. So the normal bundle $\mathcal{N}_{S|X}$ is not nef. By adjunction formula

$$\omega_S \simeq \mathcal{N}_{S|X},$$

so ω_S is not nef.

If the reflexive sheaf $\omega_{\tilde{S}}$ is not nef, choose a curve $C \subset \tilde{S}$ with $(c_1(\omega_{\tilde{S}}) \cdot C)<0$.

Since $\pi_*(\omega_{\hat{S}}^{\mu}) \subset \omega_{\tilde{S}}^{\mu}$ for all $\mu \in \mathbb{N}$ and since $\kappa(\hat{S}) \geqq 0$ (so $\omega_{\hat{S}}^{\mu}$ is generated by global sections for $\mu \gg 0$), $\omega_{\tilde{S}}^{\mu}$ is generated by global sections outside a finite set (Sing(S)) for suitable big μ. This clearly contradicts $(c_1(\omega_{\tilde{X}}) \cdot C)<0$.

Hence $\omega_{\tilde{S}}$ must be nef. This already excludes the case "S normal".

In the case "S non-normal" we still have some informations on the curves $C \subset \tilde{S}$ with

$$(*) \qquad (c_1(f^*(\omega_S)) \cdot C)<0.$$

Namely $C \subset \tilde{E}$, $\tilde{E}$ the preimage of the non-normal locus E of S. In fact, if $C \not\subset \tilde{E}$, we would obtain by

$$\omega_{\tilde{S}} \simeq I_E \otimes f^*(\omega_S) \quad (0.4):$$
$$(c_1(\omega_{\tilde{S}}) \cdot C)<0,$$

a contradiction. So (*) holds. (*) implies that there are only finitely many curves $C_1, \ldots, C_s$ with

$$(c_1(\omega_S) \cdot C_i)<0,$$

hence

$$(S \cdot C_i)<0.$$

Now let $t_0 = \inf\{t \in \mathbb{R}_+ \mid S + tH \text{ ample}\}$ and put $D_0 = S + t_0 H$.

Clearly $t_0 \in \mathbb{Q}_+$, in fact

$$t_0 = \max_{1 \leqq i \leqq s} \{t \mid (S + tH \cdot C_i) = 0\}.$$

If $D_0^3 > 0$ we conclude by (0.9).

If $D_0^3 = 0$ observe that $S\, c_2(X) \geqq 0$ [M, Y], hence $D_0\, c_2(X) > 0$. Since $S \cdot C \geqq 0$ for all but a finite number of curves, $S^2 H \geqq 0$ for any ample H, and so $D_0^2 H > 0$. Now apply (0.8) to get a rational curve.

(1.3) Lemma. *If S is nef and $S^2 \neq 0$ (but $S^3 = 0$), X contains a rational curve or the normalization of S is a hyperelliptic surface (in particular smooth).*

Proof. Assume that X does not contain a rational curve. Assume that $H^2(S, \mathcal{O}_S) = H^0(\omega_S) \neq 0$. Then the exact sequence

$$(S) \qquad 0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

yields a cohomology sequence

$$\begin{array}{ccccccc} H^2(\mathcal{O}_X) & \to & H^2(\mathcal{O}_S) & \to & H^3(\mathcal{O}_X(-S)) & \to & H^3(\mathcal{O}_X) \to 0 \\ \| & & & & \|\wr & & \|\wr \\ 0 & & & & H^0(\mathcal{O}_X S)) & & \mathbb{C} \end{array}$$

thus $h^0(\mathcal{O}_X(S)) \geqq 2$. Since S is by assumption nef, X contains a rational curve by (0.8) as soon as we know $S^2 H > 0$. But $S^2 H > 0 \Leftrightarrow S^2 \neq 0$. So we must have $H^2(\mathcal{O}_S) = 0$, moreover $S \cdot c_2(X) = 0$ because otherwise we can again apply (0.8). By Riemann-Roch: $\chi(\mathcal{O}_X(S)) = \chi(\mathcal{O}_X(-S)) = 0$. Sequence (S) yields $\chi(\mathcal{O}_S) = 0$ since $\chi(\mathcal{O}_X) = 0$. The vanishing of $H^2(\mathcal{O}_X)$ implies therefore

$$h^1(\mathcal{O}_S) = 1.$$

We want to get some more informations from $S^3 = 0$, $S^2 \neq 0$. $S^3 = 0$ just says $c_1(N_{S|X})^2 = 0$, hence $c_1(\omega_S)^2 = 0$. $S^2 \neq 0$ implies (self-intersection formula) $c_1(N_{S|X}) \neq 0$ in $H^2(S, \mathbb{R})$, so $c_1(\omega_S) \neq 0$.

(1) Assume S to be non-singular. Then $c_1(\omega_S)^2 = 0$, $c_1(\omega_S) \neq 0$ gives $\kappa(S) = 1$. So $p: S \to C$ is an elliptic surface. By $h^1(\mathcal{O}_C) \leqq h^1(\mathcal{O}_S) = 1$, we obtain $g(C) \leqq 1$. By [BPV, p. 162]:

$$\deg(R^1 p_*(\mathcal{O}_X)^*) = \chi(\mathcal{O}_S) = 0.$$

So the canonical bundle formula [BPV, p. 161] gives

$$\omega_S \simeq p^*(\omega_C \otimes \mathcal{L}) \otimes \mathcal{O}_S\left(\sum_{i=1}^{S} (m_i - 1) F_i\right)$$

where $m_i F_i$ are the multiple fibers and $\mathcal{L} = R^1 p_*(\mathcal{O}_X)^*$ is topologically trivial.

If $g(C) = 1$ the formula implies the non-existence of any singular fiber because otherwise $h^0(\omega_S) > 0$. But then $\kappa(S) = 0$! If $g(C) = 0$ our formula and $h^0(\omega_S) = 0$ imply $s = 1$ and $m_1 = 2$ or $s = 0$. But if $s = 1$, $m = 2$ we have $\kappa(S) = -\infty$ and if $s = 0$: $\kappa(S) = 0$. So (1) does not occur if X contains no rational curve.

(2) Now let S be singular but normal. In this case if $c_1(\omega_{\hat{S}})^2 < c_1(\omega_S)^2 = 0$, the minimality of $\hat{S}$ yields $\kappa(\hat{S}) = -\infty$, which gives rational curves. Hence $c_1(\omega_{\hat{S}})^2 = c_1(\omega_S)^2 = 0$, which just means $\omega_{\hat{S}} \simeq \pi^*(\omega_S)$, i.e. S has only rational double points. So $\kappa(\hat{S}) = 0$. In particular $\hat{S}$ contains a rational curve, consequently $\hat{S}$ cannot be a torus or hyperelliptic. By [BPV, p. 67] we easily see: $\hat{S}$ is not Enriques, hence $\hat{S}$ must be a K3 surface. Then $\omega_S \simeq \mathcal{O}_S$, contradiction.

(3) Finally let S be non-normal.

(3.1) $$\kappa(\hat{S}) = 0.$$

Since $\pi_*(\omega_{\tilde{S}}) \subset \omega_{\hat{S}}$ and $f_*(\omega_{\tilde{S}}) \subset \omega_S$ we have

$$H^0(\omega_{\hat{S}}) = 0.$$

So $\hat{S}$ is Enriques or hyperelliptic. The Enriques case is excluded as before. If $\hat{S}$ is hyperelliptic, we must have $\hat{S} = \tilde{S}$ and we are done by our assumption.

(3.2) $$\kappa(\hat{S}) = 1.$$

Again we have for the elliptic surface $p: \hat{S} \to C$:

$$H^0(\omega_{\hat{S}}) = 0.$$

Since p has at most multiple fibers (as singular fibers) we have $c_2(\hat{S}) = 0$ ([BPV, p. 97]), hence $\chi(\mathcal{O}_{\hat{S}}) = 0$ and $h^1(\mathcal{O}_{\hat{S}}) = 0$. Now the contradiction is the same as in (1).

(3.3) $$\kappa(\hat{S}) = 2.$$

Again $H^2(\mathcal{O}_{\hat{S}}) = 0$ and by the positivity of $\chi(\mathcal{O}_{\hat{S}})$: $H^1(\mathcal{O}_{\hat{S}}) = 0$. Since we also have $H^q(\mathcal{O}_{\tilde{S}}) = 0$, $q = 1, 2$, $\tilde{S}$ has only rational singularities and since the only smooth rational curves in $\hat{S}$ are (-2)-curves, $\tilde{S}$ even has at most rational double points, in particular $\omega_{\tilde{S}}$ is locally free. It follows: $c_1(\omega_{\tilde{S}})^2 = c_1(\omega_{\hat{S}})^2 > 0$. Using $\omega_{\tilde{S}} \simeq I_{\tilde{E}} \otimes f^*(\omega_S)$ and $c_1(\omega_S)^2 = 0$:

$$0 < \tilde{E}^2 - 2(c_1 f^*(\omega_S) \cdot \tilde{E}).$$

Since ω_S is nef, we obtain $\tilde{E}^2 > 0$. If we take squares in $f^*(\omega_S) \simeq \omega_{\tilde{S}} \otimes \mathcal{O}_{\tilde{S}}(\tilde{E})$, we obtain $0 = c_1(\omega_{\tilde{S}})^2 + \tilde{E}^2 + 2(c_1(\omega_{\tilde{S}}) \cdot \tilde{E})$. Since $\omega_{\tilde{S}}$ is nef and $\tilde{E}^2 > 0$, $c_1(\omega_{\tilde{S}})^2 > 0$, this is impossible!

(1.4) Lemma. *Assume S to be nef, $S^2 \neq 0$ and that the normalization of S is hyperelliptic. Then X contains a rational curve.*

Proof. By (0.4) we have

(*) $$\omega_{\tilde{S}} \simeq I_{\tilde{E}} \otimes f^*(\omega_S)$$

where $\tilde{E}$ is the analytic preimage of the non-normal locus E of S – equipped with the structure given by the conductor ideal. Since $c_1(\omega_S)^2 = 0$, we have $c_1(f^*(\omega_S))^2 = 0$. Since $f^*(\omega_S)$ is nef, we can describe the structure of $f^*(\omega_S) \cdot \tilde{S}$ being hyperelliptic, $\tilde{S}$ is an analytic fiber bundle $\tilde{p}: \tilde{S} \to \tilde{C}$ over an elliptic curve

$\tilde{C}$. On the other hand we have an elliptic fibration $\tilde{q}$: $\tilde{S} \to \mathbb{P}_1$. We denote by F_i fibers of $\tilde{p}$, by G_i fibers of $\tilde{q}$. Then we have:

(a) $$f^*(\omega_S) \equiv \Sigma n_i F_i, \qquad n_i > 0$$

or

(b) $$f^*(\omega_S) \equiv \Sigma m_j G_j, \qquad m_j > 0.$$

This follows from $b_2(\tilde{S}) = 2$. From (*) we conclude:

(a) $\tilde{E}$ consists of fiber of $\tilde{p}$

(b) $\tilde{E}$ consists of fibers of $\tilde{q}$.

We want to prove first:
(**) f^*: $H^2(S, \mathbb{Z}) \to H^2(\tilde{S}, \mathbb{Z})$ is an isomorphism. First assume (a). $\tilde{p}$ clearly induces a continous map p: $S \to C$ to some curve C. Moreover there is a diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{f} & S \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{C} & \xrightarrow{g} & C \end{array}$$

Now it is easy to check that C carries a complex structure such that both g and p are holomorphic; g is a modification. Since $h^1(\mathcal{O}_S) = 1$, we must have (by $p_*(\mathcal{O}_S) = \mathcal{O}_C$)) $h^1(\mathcal{O}_C) \leqq 1$, hence C is smooth elliptic and g is biholomorphic. So the picture is as follows: there are points $x_1, \ldots, x_s \in C$ such that $f|\tilde{p}^{-1}(x_i)$ is étale onto $f(p^{-1}(x_i))$ (all fibers of $\tilde{p}$, p are smooth elliptic: for the fibers of $\tilde{p}$ this is clear; for the fibers of p it is true since otherwise $R^1 p_*(\mathcal{O}_S)$ would have torsion, consequently $h^1(\mathcal{O}_S) > 1$ by the Leray spectral sequence).

Now our claim is an immediate consequence of the exact sequence ([BK, 3.A.7]):

$$\ldots \longrightarrow H^q(S, \mathbb{Z}) \longrightarrow H^q(\tilde{S}, \mathbb{Z}) \oplus H^q(E, \mathbb{Z}) \longrightarrow H^q(\tilde{E}, \mathbb{Z})) \longrightarrow \ldots$$

Now assume (b). Similarly as in (a) we have a diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{f} & S \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{P}_1 & \xrightarrow{h} & 0 \end{array}$$

with a possibly singular rational curve D. The only thing to do is to prove smoothness of D; then we can argue as in (a). Taking direct images of the exact sequence (0.4) we obtain:

$$0 \to q_*(\omega_S^{-1} \otimes \omega_E) \to R^1 q_*(\mathcal{O}_S) \to R^1 q_*(f_*(\mathcal{O}_{\tilde{S}})) \to R^1 q_*(\omega_S^{-1} \otimes \omega_E) \to 0.$$

Assume D to be singular, so $h^1(\mathcal{O}_D)=1$. By Leray's spectral sequence and $h^2(\mathcal{O}_S)=0$:

$$(+) \qquad H^0(R^1 q_*(\mathcal{O}_S))=0.$$

Since $q_*(\omega_S^{-1}\otimes\omega_E)$ is concentrated on points, we get $q_*(\omega_S^{-1}\otimes\omega_E)=0$.
Taking cohomology of (0.4) and using $h^1(\mathcal{O}_S)=1$, $h^2(\mathcal{O}_S)=0$, we deduce:

$$H^1(\omega_S^{-1}\otimes\omega_E)=0.$$

By Leray's spectral sequence again, it follows

$$R^1 q_*(\omega_S^{-1}\otimes\omega_E)=0.$$

So $(+)$ gives

$$R^1 q_*(\mathcal{O}_S)\simeq R^1 q_*(f_*(\mathcal{O}_{\tilde{S}})).$$

Now use the so-called Serre spectral sequence

$$E_2^{p,q}=R^p q_*(R^q f_*(\mathcal{O}_{\tilde{S}}))$$

converging to $R^{p+q}(q\circ f)_*(\mathcal{O}_{\tilde{S}})$ to deduce

$$R^1 q_*(f_*(\mathcal{O}_{\tilde{S}}))\simeq R^1(q\circ f)_*(\mathcal{O}_{\tilde{S}}).$$

Since $q\circ f=h\circ\tilde{q}$ we can again use this spectral sequence to compute

$$R^1(q\circ f)_*(\mathcal{O}_{\tilde{S}})\simeq h_*(\mathcal{O}_{\mathbb{P}_1}).$$

In summary:

$$R^1 q_*(\mathcal{O}_S)\simeq h_*(\mathcal{O}_{\mathbb{P}_1}),$$

contradicting $(+)$. So (**) is proved completely.

Since $H^2(S,\mathcal{O})=0$, (**) gives us a nef line bundle $L_S\not\equiv 0$ on S which is not a multiple of ω_S. In fact there exists a nef line bundle $\tilde{L}$ on $\tilde{S}$ which is not a multiple of $f^*(\omega_S)$ and $\tilde{L}$ is by (**)-up to numerical equivalence – of the form $f^*(L_S)$. Now consider the restriction map

$$r\colon\ H^2(X,\mathbb{Q})\to H^2(S,\mathbb{Q}).$$

Since $c_1(\omega_S)\in H^2(S,\mathbb{Q})$ is non-zero and not ample (nor negative), r must be onto.

Hence there is a line bundle L on X with $L|S\simeq L_S^m$ for some $m\in\mathbb{N}$ (use $H^q(X,\mathcal{O})=0$ for $q=1,2$). We may assume $m=1$.

First assume that L is nef.

By (0.9) we may assume $L^3=0$. $L+S$ is nef and moreover $L+S$ is big: it is sufficient to see $LS^2>0$, which follows from

$$\begin{aligned} LS^2&=(c_1(f^*(L_S)\cdot c_1(f^*(\omega_S)))\\ &=(c_1\,\tilde{p}^*(F_1)\cdot c_1\,\tilde{q}^*(F_2))>0\end{aligned}$$

(F_i ample line bundles on $\tilde{C}$ resp. $\mathbb{P}_1$). If $L+S$ is not ample, we get again a rational curve. If $L+S$ is ample, we deduce

$$(L+S\cdot c_2(X))>0,$$

hence $L\cdot c_2(X)>0$.

Now $L^2 H>0$, so X contains a rational curve by (0.8). So we may assume that L is not nef.

(1) Suppose first $\rho(X)\geqq 3$.

(α) Assume furthermore

$$\{D\,|\,S\equiv 0\}\subset\{D^3=0\}.$$

By our assumption, there is $D\in\mathrm{Div}(X)$ such that $D|S\equiv 0$, $D^3>0$. Choose $\mu\gg 0$ such that $(L+\mu D)^3>0$. Put $\tilde{L}=L+\mu D$. Then $\tilde{L}|S\equiv L|S$ and $\tilde{L}^3>0$. I claim that there is $H\in\mathrm{Div}(X)\otimes\mathbb{Q}$ ample with the following property: if t_0 is chosen such that $\tilde{L}+t_0 H\in\partial K$ then either $(\tilde{L}+t_0 H)^3>0$ or $\tilde{L}+tS\in K$ for every $t\gg 0$.

In fact, assume $\tilde{L}+tS\notin K$ for $t>0$. For $H\in K$ let $t_0(H)$ be the unique number with

$$\tilde{L}+t_0(H)\cdot H\in\partial K.$$

If $H_\nu\to S$ in $N^1(X)$, then $\lim t_0(H_\nu)=\infty$ (otherwise we find $c>0$ such that $\tilde{L}+cS\in\partial K$). Now choose $H\in(\mathrm{Div}(X)\otimes\mathbb{Q})\cap K$ near to S such that

$$t_0^2(H)>t_0(H)$$

and

$$\tilde{L}H^2>\tilde{L}^2 H.$$

The second inequality can be achieved since

$$\tilde{L}S^2=LS^2>0, \qquad \tilde{L}^2 S=L^2 S=0.$$

Hence: $(\tilde{L}+t_0 H)^3>0$.

But then it is easy to get $D\in\mathrm{Div}(X)$, not nef, with $D^2 H>0$, $DH^2>0$. Hence there is a rational curve by [W].

(β) Now let $\{D\,|\,S\equiv 0\}\subset\{D^3=0\}$.

Let $V\subset N^1(X)$ be the linear space generated by S and $\{D\,|\,S\equiv 0\}$. Since $\rho(S)=2$, V is hypersurface. Let $r\colon N^1(X)\to N^1(S)$ be the restriction. Then $r(\partial K)$ is a cone in $N^1(S)\simeq\mathbb{R}^2$ containing interior points. In fact, otherwise $r(\partial K)=\mathbb{R}_+[S]$, hence for any nef non-ample D we would have $D|S\equiv aS$. In other words: $\partial K\subset V$, which is clearly impossible. So for every $D\in\partial K$ we find D' arbitrary near to D such that $D'\in\partial K$ and $D'|S\neq aS$, i.e. $D'\notin V$ and also D'' with $D''\notin\partial K$ and $D''\in V$.

In particular: $V\not\subset\partial K$ (otherwise we would locally have $V=\partial K$).

Let $D_0\in\{D^3=0\}$. If D_0 is "general", i.e. a smooth point of $\{D^3=0\}$, then $D_0^2\neq 0$. Then any neighborhood of D_0 contains points D with $D^3>0$ as well as $D^3<0$ (consider the cubic polynominal $q(t)=(D_0+tH)^3$ where H is ample with $D_0^2 H\neq 0$, then q changes sign at $t=0$). We conclude – by our above remarks – that in every neighborhood U of S there are $D\in U$, D not nef, with $D^3>0$.

In fact there are $\tilde{D}\in U\cap V$, $\tilde{D}\notin\partial K$, and since $S^2\neq 0$ we find $D'\in U$ with $D'^3>0$, $D'^2 H>0$, $D'H^2>0$. So by [W] X contains rational curves.

(2) Now assume $\rho(X)=2$.

In this case $\partial(K \cup -K) = L_1 \cup L_2$, L_i lines in $N^1(X)$. Since we may assume $\partial K \subset \{D^3 = 0\}$, $p(D) = D^3$ vanishes on L_i and it follows easily $p = p_1 p_2 p_3$, p_i linear, with $L_i = \{p_i = 0\}$, $i = 1, 2$.

If $p_3 \neq p_i$, $i = 1, 2$, choose a Cartier divisor $D \in L_3$. Let H be ample and let $D_0 = D + t_0 H \in \partial K$. Then one may assume $t_0 \in \mathbb{Q}$ and using D_0 it is quite easy to construct a rational curve.

So let $p_3 = p_2$. Then: $S \in L_2$, and $D^2 = 0$ for all $D \in L_1$. Now take $D \in L_1$. By the existence of L we know $D|S$ to be ample. So $D^2 S > 0$ and $D^2 \neq 0$, contradiction.

This finishes the proof of (1.4).

(1.5) Lemma. *If S is nef and $S^2 = 0$, X contains a rational curve.*

Proof. Assume again that X has no rational curve. Since $S^2 = 0$, the self-intersection formula gives $c_1(\omega_S) = 0$ in $H^2(S, \mathbb{R})$. We first assume

$$H^2(\mathcal{O}_S) = H^0(\omega_S) \neq 0. \tag{1}$$

By $c_1(\omega_S) = 0$ we conclude $\omega_S \simeq \mathcal{O}_S$. Since $f_* \pi_*(\omega_{\hat{S}}^{\mu}) \subset \omega_S^{\mu}$: $\kappa(\hat{S}) = 0$. Hence we can find μ_0 such that $\omega_{\hat{S}}^{\mu_0} \simeq \mathcal{O}_S$.

If π or f are not isomorphisms we would obtain a non-zero section of $\omega_S^{\mu_0}$ with zeroes. So S must be smooth. ω_S being trivial, S cannot be hyperelliptic. S cannot be Enriques since then X has a rational curve.

Hence S is $K3$ or a torus.

Now $h^0(\omega_S) = h^2(\mathcal{O}_S) = 1$. Consequently

$$h^0(\mathcal{O}_X(S)) = 2.$$

Observe that we are not allowed to apply (0.8) since $S^2 H = 0$ for all ample H!

Instead we consider the meromorphic map

$$\psi\colon X \dashrightarrow \mathbb{P}_1 = \mathbb{P}(H^0(\mathcal{O}_X(S))).$$

We claim that ψ is everywhere defined. So let B be the base locus of ψ and take $x_0 \in B$. Then for all $S' \in |S|$: $x_0 \in S'$. Choose $S_1, S_2 \in |S|$ irreducible. Then $x_0 \in S_1 \cap S_2$, and hence $S_1 \cap S_2$ contains a curve. So $S_1 S_2$ is an effective 1-cycle on S_1 and non-zero, so $S_1 S_2 \neq 0$, and $S^2 = S_1 S_2 \neq 0$, contradiction. Thus $B \neq \varnothing$ and ψ a morphism. We also have

$$\mathcal{O}_X(S) \simeq \psi^*(\mathcal{O}_{\mathbb{P}_1}(1)).$$

(1.5.1) Any fiber of ψ is smooth. If S is $K3$ any fiber is $K3$ and we have $R^q \psi_*(\mathbb{Z}) = 0$ for $q = 1, 3$ and $R^2 \psi_*(\mathbb{Z}) \simeq \mathbb{Z}^{22}$. By Leray spectral sequence we deduce

$$H^3(X, \mathbb{Z}) = 0.$$

But $b_3(X)>0$ since $H^3(X, \mathcal{O})\simeq\mathbb{C}$ (then use Hodge decomposition). If S is a torus, any fiber of ψ is a torus. Then $R^1\psi_*(\mathbb{Z})\simeq\mathbb{Z}^4$. For the Leray spectral sequence (E_r^{pq}) associated to ψ and the sheaf $\mathbb{Z}$ we have

$$E_2^{0,1}=H^0(R^1\psi_*(\mathbb{Z}))=\mathbb{Z}^4,$$
$$E_2^{2,0}=H^2(\mathbb{P}_1, \mathbb{Z})=\mathbb{Z},$$

hence $E_3^{0,1}$ contains a $\mathbb{Z}^3$, so by $E_3^{0,1}\subset H^1(X,\mathbb{Z})=0$, we obtain a contradiction.

(1.5.2) Now let us consider the case where ψ has some singular fibers X_s, $s\in\mathbb{P}_1$.

If $X_s=kS'$ with S' irreducible reduced then $kS'\sim S$, so S' has the same properties as S and we obtain the contradiction by substituting simply S by S' in our previous considerations or – if $h^2(\mathcal{O}_{S'})=0$ – go to (2). Now assume that X_s has several components S_i, $1\leqq i\leqq r$, $r\geqq 2$. Let k_i be the multiplicity of S_i'. If some S_i' is not nef we are done. Thus we may assume all S_i' to be nef.

Since $(\sum k_i S_i')^2 H=0$ for any ample H, there is either some i_0 with $S_{i_0}'^2 H>0$ or $S_i'\cap S_j'=\emptyset$ for $i\neq j$. In the first case $S_{i_0}'^3=0$ because otherwise S_{i_0}' is big and nef and X has rational curves. So we can apply our results from (1) to S_{i_0}', and finish. In the second case ψ has some disconnected fibers X_s and therefore $h^0(\mathcal{O}_{X_s})\geqq 2$, moreover $\phi_*(\mathcal{O}_X)$ has rank $\geqq 2$ at some points. So $\psi_*(\mathcal{O}_X)$ has torsion and thus $h^0(\psi_*(\mathcal{O}_X)=h^0(\mathcal{O}_X))\geqq 2$, contradiction.

(2) $$H^2(\mathcal{O}_S)=H^0(\omega_S)=0.$$

The arguments at the beginning of the proof apply here, too. So S is smooth and $\kappa(S)=0$. By our assumption S cannot be $K3$ or a torus. If S is Enriques, X contains a rational curve. So assume S to be hyperelliptic. Fix an ample divisor H and let $g(t)=(S+tH)^3$. Since $S^2=0$, g has a double zero at 0. Let t_0 be the third zero. Necessarily t_0 is rational. Put

$$D_0=-(S+t_0H).$$

First assume that D_0 is nef. Then $D_0^3=0$ but $D_0^2\neq 0$, in fact $D_0^2H>0$. Assume $D_0\, c_2(X)=0$. So

$$S+t_0H\cdot c_2(X)=0.$$

Since $S\cdot c_2(X)\geqq 0$ and $H\cdot c_2(X)>0$, we get $t_0=0$, which is impossible. Hence $D_0\, c_2(X)>0$. Now apply (0.8) to obtain a rational curve.

In case D_0 not nef there is an uniquely determined t_1 such that $-(S+t_1H)$ is nef but not ample, $t_1\neq t_0$.

Let $D_1=-(S+t_1H)$. So $D_1^3>0$, D_1 is nef but not ample.

By [CP] there is an irreducible curve C or an irreducible surface $Y\subset X$ with $D_1\cdot C=0'$ or $D_1^2\cdot Y=0$. If $D_1\cdot C=0$, t_1 is rational and we are done by (0.9). If $D_1^2\cdot Y=0$ and Y is not nef we are done by the first part of the proof. If Y is nef we conclude from

$$(S+t_1H)^2\,Y=0, \qquad S^2=0$$

that t_1 is rational and finish as before.

The existence of a rational curve in case $D^3>0$ for some nef D can also easily be deduced from the results of Wilson [W].

Combining (1.2)–(1.5), Theorem (1.1) is proved completely.

(1.6) Corollary. *Let X be a Calabi-Yau 3-fold, f: $X \to Y$ a surjective non-finite map to a projective variety of positive dimension. Then X contains a rational curve.*

§ 2 A conjecture of Kobayashi

In this section we deal with hyperbolic manifolds. Hyperbolicity is defined (originally) via the Kobayashi pseudo-metric but for our purposes the following equivalent statement for compact manifolds known as Brody's theorem is more convenient:

(2.1) A compact complex manifold X is hyperbolic iff any holomorphic map f: $\mathbb{C} \to X$ is constant.

As general reference for hyperbolicity we use the survey article of Kobayashi [K] and Lang's introductory book [L].

We want to study the following

(2.2) Conjecture of Kobayashi. *Let X be a projective manifold. If X is hyperbolic, the canonical bundle ω_X is ample.*

For Riemann surfaces the proof is 19[th] century, for surfaces it follows essentially from Enriques-Kodaira classification. Hence $\dim X = 3$ is the first interesting case. The results of Sect. 2 put us into position to almost solve Kobayashi's conjecture in dimension 3. The result is this.

(2.3) Theorem. *A 3-dimensional projective hyperbolic manifold X has ample canonical bundle ω_X with the following possible exception that X is a Calabi-Yau manifold with any effective divisor being ample and $\rho(X) \leqq 19$.*

The rest of this section is more or less the proof of (2.3). It will be given in several steps some of which hold in any dimension.

(2.4) Proposition. *Let X be a n-dimensional hyperbolic projective manifold. Then ω_X is nef.*

Proof. If ω_X is not nef then X contains a rational curve by Mori's fundamental theory [Mo], [KMM].

(2.5) Corollary. *Let X be a smooth projective hyperbolic 3-fold. Then $\kappa(X) \geqq 0$. Moreover $h^0(\omega_X) > 0$ or $q(X) = h^1(\mathcal{O}_X) > 0$.*

Proof. ω_X being nef by (2.4) this is just a theorem of Miyaoka [M]: what we need is (by Riemann-Roch): $c_1(X)\, c_2(X) \geqq 0$.

(2.6) Proof of Theorem (2.3) in case $\kappa(X) = 0$. So assume $\kappa(X) = 0$ in (2.3).

1. Case: $q(X) > 0$.
Then by [V, U], X is up to bimeromorphic equivalence

a) an abelian variety
b) a fiber space τ: $X \to C$ over a smooth curve C with general fiber an abelian surface or a $K3$ surface.
c) a fiber space τ: $X \to Y$ over a smooth surface Y with general fiber an elliptic curve.

Obviously in cases b) and c) X cannot be hyperbolic. In case a) we find a diagram

$$\begin{array}{ccc} & Z & \\ {\scriptstyle\tau}\swarrow & & \searrow{\scriptstyle\pi} \\ A & & X \end{array}$$

where A is an abelian variety, π is a modification and τ a sequence of blow-up's with smooth centers. Now for general $x \in Z$ we find $f\colon \mathbb{C} \to Z$ holomorphic and non-constant such that $f(0)=x$ – this is an easy exercise. Thus X cannot be hyperbolic.

2. Case: $q(X)=0$.
By (2.5) we know $h^0(\omega_X)>0$ (in fact $h^0(\omega_X)=1$). So we can view ω_X as an effective divisor $D=\sum_{i=1}^{s} k_i D_i$ with $s \geqq 0$, $k_i>0$ and D_i prime divisors.

Suppose first $s>0$. Then by [W2]: $\kappa(D_i)\leqq 0$ where $\hat{D}_i$ is a desingularisation of D_i. But then D_i contains rational curves and we are done. If $s=0$, ω_X is trivial. By Beauville's theorem already mentioned in (1.2), the universal cover $\tilde{X}$ of X is of the form

$$\tilde{X}=\mathbb{C}^k \times \prod_{j=1}^{r} V_j \times \prod_{i=1}^{t} X_i$$

where X_i are (even-dimensional) sympletic manifolds and V_j simply-connected Calabi-Yau manifolds. If $k>0$, either $\tilde{X}=\mathbb{C}^3$ of $t=1$ and $\tilde{X}=\mathbb{C}\times X_1$ and X is not hyperbolic. If $k=0$, we must have $r=1$, $t=0$. Hence X is Calabi-Yau! Now apply (1.14) to conclude.

(2.7) Proposition. *A projective 3-fold X with $0<\kappa(X)<3$ is never hyperbolic.*

Proof. Because of the existence of the Iitaka fibration, X has up to bimeromorphic equivalence the structure of a fiber space $X \to S$ whose general fiber X_s satisfies $\kappa(X_s)=0$. Since X_s is a curve or a surface we obtain rational or elliptic curves in X_s and hence X cannot be hyperbolic.

(2.8) Proposition. *Let X be a projective 3-fold of general type ($\kappa(X)=\dim X$). If X is hyperbolic, ω_X is ample.*

Proof. By (2.4) ω_X is nef. Hence ω_X^m is generated by global sections for some $m \gg 0$ (see e.g. [KMM]). So we obtain a modification $\phi\colon X \to X'$ to a normal projective variety X' with the following properties:

a) X' is $\mathbb{Q}$-Gorenstein, the reflexive sheaf $\mathcal{O}_{X'}(mK_{X'})$ is locally free ($K_{X'}$ denote a canonical divisor)
b) $\omega_X^m=\phi^*(\mathcal{O}_{X'}(mK_{X'}))$
c) X' has only canonical singularities. (see [R])

By passing to the canonical cover of an affine neighborhood of a singularitiy of X' (see [R, KMM]) we may assume that X' is Gorenstein.

Since canonical singularities are rational ([KMM]), we can apply [R, 2.14] to obtain (a lot of) rational curves in the exceptional locus of a suitable desingularisation of X' and hence also in the exceptional locus of ϕ. Alternatively, apply directly [R, 2.6].

The proof of (2.3) is now complete.

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Note added in proof

Recently Y. Kawamata has proved (2.8) in any dimension (Y. Kawamata: Moderate degenerations of algebraic surfaces, to appear in Proceedings of the Algebraic Geometry Conference Bayreuth 1990)