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Complete space-like submanifolds in a de Sitter space with parallel mean curvature vector

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1 Introduction

Let $M_p^{n+p}(c)$ be an $n+p$-dimensional connected semi-Riemannian manifold of index p and of constant curvature c, which is called as indefinite space form of index p. If $c > 0$, we call it as a de Sitter space of index p. It is seen that a complete space-like hypersurface of a Minkowski space R_1^{n+p} possesses a remarkable Bernstein property in the maximal case by Calabi [2] and Cheng and Yau [3]. As a generalization of the Bernstein type problem, a complete space-like maximal submanifold M^n of $M_p^{n+p}(c)$ was characterized by Ishihara [5] under a certain conditions. An entire space-like hypersurface with constant mean curvature of a Minkowski space is investigated by Goddard [4] and Treibergs [10]. Akutagawa [1] and Ramanathan [9] investigated the complete space-like hypersurfaces in a de Sitter space. They obtained independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the following are satisfied

$$H^2 \leqq c, \qquad \text{when } n = 2\,; \tag{1.1}$$

$$n^2 H^2 < 4(n-1)c, \qquad \text{when } n \geqq 3\,. \tag{1.2}$$

In this paper, we consider general submanifolds in an indefinite space form, we obtain that a complete space-like submanifold in a de Sitter space with parallel mean curvature vector is totally umbilical if the conditions (1.1) or (1.2) are satisfied. Conditions (1.1) and (1.2) are best possible.

2 Local formulas and lemmas

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional semi-Riemannian manifold of constant curvature c whose index is p. Let M^n be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. As the semi-Riemannian metric of $M_p^{n+p}(c)$ induced the

Riemannian metric of M^n, the immersion is called space-like. We choose a local field of semi-Riemannian orthonormal frames $e_1, \ldots, e_{n+p}$ in $M_p^{n+p}(c)$ such that at each point of M^n, $e_1, \ldots, e_n$ span the tangent space of M^n and forms an orthonormal frame there. We make us of the following convention on the range of indices: $1 \leqq A, B, C, \ldots \leqq n+p$; $1 \leqq i, j, k, \ldots \leqq n$; $n+1 \leqq \alpha, \beta, \gamma, \ldots \leqq n+p$; and we shall agree that repeated indices under summation sign are summed over respective ranges. Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $M_p^{n+p}(c)$ is given by $ds^2_{M_p^{n+p}} = \sum \omega_i^2 - \sum \omega_\alpha^2 = \sum \varepsilon_A \omega_A^2$, where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$. Then the structure equations of $M_p^{n+p}(c)$ are given by

$$d\omega_A = -\sum \varepsilon_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0\,, \tag{2.1}$$

$$d\omega_{AB} = -\sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \tfrac{1}{2}\sum K_{ABCD}\omega_C \wedge \omega_D\,, \tag{2.2}$$

$$K_{ABCD} = c\varepsilon_A \varepsilon_B(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}) \tag{2.3}$$

We restricted these forms to M^n, then

$$\omega_\alpha = 0, \qquad \text{for } \alpha = n+1, \ldots, n+p, \tag{2.4}$$

and the Riemannian metric of M^n is written as $ds^2 = \sum \omega_i^2$. We put $\omega_{\alpha i} = \sum h_{ij}^\alpha \omega_j$. From Cartan's Lemma we have $h_{ij}^\alpha = h_{ji}^\alpha$, where h_{ij}^α are the components of the second fundamental form of M^n. Let

$$h = \frac{1}{n}\sum\sum_i (h_{ii}^\alpha) e_\alpha \tag{2.5}$$

$$H^2 = \frac{1}{n^2}\sum\left(\sum_i h_{ii}^\alpha\right)^2 \tag{2.6}$$

where H is called mean curvature. From (2.1), we obtain the structure equations of M^n

$$d\omega_i = -\sum \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0\,, \tag{2.7}$$

$$d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{kj} - \tfrac{1}{2}\sum R_{ijkl}\,\omega_k \wedge \omega_l\,, \tag{2.8}$$

and the Gaussian formula

$$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) - \sum (h_{il}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{jl}^\alpha) \tag{2.9}$$

The components of the Ricci curvature tensor Ric are given by

$$R_{jk} = c(n-1)\delta_{jk} - \sum h_{ii}^\alpha h_{jk}^\alpha + \sum h_{ik}^\alpha h_{ji}^\alpha \tag{2.10}$$

We have also the structure equations of the normal bundle of M^n

$$d\omega_\alpha = -\sum \omega_{\alpha\beta} \wedge \omega_\beta, \qquad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0\,, \tag{2.11}$$

$$d\omega_{\alpha\beta} = -\sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \tfrac{1}{2}\sum R_{\alpha\beta ij}\omega_i \wedge \omega_j\,, \tag{2.12}$$

$$R_{\alpha\beta ij} = -\sum (h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta) \tag{2.13}$$

For indefinite Riemannian manifolds, refer to O'Neill [8].

Let h_{ijk}^α denote the covariant derivatives of h_{ij}^α so that

$$\sum h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha - \sum h_{kj}^\alpha \omega_{kj} - \sum h_{jk}^\alpha \omega_{ki} - \sum h_{ij}^\beta \omega_{\beta\alpha}\,. \tag{2.14}$$

Similarly, let h^α_{ijkl} denote the covariant derivative of h^α_{ijk} so that

$$\sum h^\alpha_{ijkl}\,\omega_l = dh^\alpha_{ijk} - \sum h^\alpha_{ljk}\,\omega_{li} - \sum h^\alpha_{ilk}\,\omega_{lj} - \sum h^\alpha_{ijl}\,\omega_{lk} - \sum h^\beta_{ijk}\,\omega_{\beta\alpha} \tag{2.15}$$

Then we obtain

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \qquad \text{for any } \alpha\,, \tag{2.16}$$

and the Ricci formula

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum h^\alpha_{im} R_{mjkl} - \sum h^\alpha_{jm} R_{mikl} - \sum h^\beta_{ij} R_{\alpha\beta kl}\,. \tag{2.17}$$

The Laplacian Δh^α_{ij} of the second fundamental form h is defined by $\Delta h^\alpha_{ij} = \sum h^\alpha_{ijkk}$. From (2.17) we have

$$\Delta h^\alpha_{ij} = \sum h^\alpha_{kkij} - \sum h^\alpha_{km} R_{mijk} - \sum h^\alpha_{mi} R_{mkjk} - \sum h^\beta_{ki} R_{\alpha\beta jk}\,. \tag{2.18}$$

Lemma 2.1 (see Omori [7] and Yau [11]). *Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2-function bounded from above on M^n, then for any $\varepsilon > 0$, there exists a point p in M^n such that*

$$\begin{aligned} \sup F - \varepsilon &< F(p)\,, \\ |\operatorname{grad} F| &< \varepsilon\,, \\ \Delta F &< \varepsilon\,. \end{aligned} \tag{2.19}$$

Lemma 2.2 (see Okumura [6]), *Let $a_1, \ldots, a_n$ be real numbers satisfies $\sum a_j = 0$ and $\sum a_j^2 = K^2\,(K > 0)$, then we have*

$$|\textstyle\sum a_j^3| \leqq (n-2)[n(n-1)]^{-1/2} K^3\,.$$

3 Theorem and its proof

Theorem. *Let M^n be an n-dimensional complete space-like submanifold in $M^{n+p}_p(c)$ with parallel mean curvature vector. If*

$$H^2 \leqq c, \qquad \text{when } n = 2\,, \tag{3.1}$$

$$n^2 H^2 < 4(n-1)c, \qquad \text{when } n \geqq 3\,. \tag{3.2}$$

then, M^n is totally umbilical.

Proof. From (2.18) and Gaussian formula, we have

$$\begin{aligned} \Delta h^\alpha_{ij} = {} & \sum h^\alpha_{kkij} + nch^\alpha_{ij} - c\sum h^\alpha_{kk}\delta_{ij} + \sum h^\alpha_{km} h^\beta_{mk} h^\beta_{ij} - 2\sum h^\alpha_{mk}\, h^\beta_{mj} h^\beta_{ik} \\ & + \sum h^\alpha_{mi} h^\beta_{mk} h^\beta_{kj} - \sum h^\alpha_{mi} h^\beta_{mj} h^\beta_{kk} + \sum h^\alpha_{jm} h^\beta_{mk} h^\beta_{ki}\,. \end{aligned} \tag{3.3}$$

Because the mean curvature vector is parallel, we have that mean curvature H is constant.

If $H = 0$, M^n is maximal. From the theorem 1.1 in [5], we know that M^n is totally geodesic.

If $H \neq 0$, we can choose e_{n+1} in such a way that its direction coincides with that of mean curvature vector h. Then

$$\omega_{\beta, n+1} = 0, \quad H = \text{constant}\,, \tag{3.4}$$

$$H^\alpha H^{n+1} = H^{n+1} H^\alpha\,, \tag{3.5}$$

$$\operatorname{tr} H^{n+1} = nH, \qquad \operatorname{tr} H^\alpha = 0, \quad \alpha \neq n+1\,, \tag{3.6}$$

where H^α denote the matrix (h_{ij}^α).

Putting

$$\mu_{ij} = h_{ij}^{n+1} - H\delta_{ij}, \qquad \tau_{ij}^\alpha = h_{ij}^\alpha, \quad \alpha \neq n+1 , \tag{3.7}$$

we have

$$\|\mu\|^2 = \operatorname{tr}(\mu)^2 = \sum \mu_{ij}^2 = \operatorname{tr}(H^{n+1})^2 - nH^2 , \tag{3.8}$$

$$\|\tau\|^2 = \sum_{\beta \neq n+1} \operatorname{tr}(\tau^\beta)^2 = \sum_{\beta \neq n+1} \|\tau^\beta\|^2 = \sum_{\beta \neq n+1} (\tau_{ij}^\beta)^2 = \sum_{\beta \neq n+1} (h_{ij}^\beta)^2 , \tag{3.9}$$

$$\operatorname{tr}\mu = 0, \qquad \operatorname{tr}(\tau^\beta) = 0, \quad \beta \neq n+1 , \tag{3.10}$$

$$S = \|\mu\|^2 + \|\tau\|^2 + nH^2 , \tag{3.11}$$

where S is the square of length of the second fundamental form. Hence, it may be seen that $\|\tau\|^2$ as well $\|\mu\|^2$ are independent of the choice of the frame field and are functions globally defined on M^n.

A submanifold M^n is said to be pseudo-umbilical if it is umbilical with respect to the direction of the mean curvature vector h, i.e., $h_{ij}^{n+1} = H\delta_{ij}$. From (3.7) $\sim$ (3.11) one can easily see that M^n is pseudo-umbilical if and only if $\|\mu\|^2 = 0$, M^n is totally umbilical if and only if it is pseudo-umbilical and $\|\tau\|^2 = 0$.

$$\begin{aligned}\Delta h_{ij}^{n+1} &= nch_{ij}^{n+1} - ncH\,\delta_{ij} + \sum h_{km}^{n+1} h_{mk}^\beta h_{ij}^\beta - 2\sum h_{mk}^{n+1} h_{mj}^\beta h_{ik}^\beta \\ &\quad + \sum h_{mi}^{n+1} h_{mk}^\beta h_{kj}^\beta - nH\sum h_{mi}^{n+1} h_{mj}^{n+1} + \sum h_{jm}^{n+1} h_{mk}^\beta h_{ki}^\beta ,\end{aligned} \tag{3.12}$$

$$\begin{aligned}\Delta h_{ij}^\alpha &= nch_{ij}^\alpha + \sum h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta - 2\sum h_{mk}^\alpha h_{mj}^\beta h_{ik}^\beta + \sum h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta \\ &\quad - nH\sum h_{mi}^\alpha h_{mj}^{n+1} + \sum h_{jm}^\alpha h_{mk}^\beta h_{ki}^\beta, \quad a \neq n+1 .\end{aligned} \tag{3.13}$$

$$\begin{aligned}\tfrac{1}{2}\Delta \|\mu\|^2 &= \sum (h_{ijk}^{n+1})^2 + nc\sum (h_{ij}^{n+1})^2 - n^2cH^2 + \sum h_{mk}^{n+1} h_{mk}^\beta h_{ij}^\beta h_{ij}^{n+1} \\ &\quad - 2\sum h_{mk}^{n+1} h_{mj}^\beta h_{ik}^\beta h_{ij}^{n+1} + \sum h_{mi}^{n+1} h_{mk}^\beta h_{ki}^\beta h_{ij}^{n+1} - nH\sum h_{mi}^{n+1} h_{mj}^{n+1} h_{ij}^{n+1} \\ &\quad + \sum h_{jm}^{n+1} h_{mk}^\beta h_{ki}^\beta h_{ij}^{n+1} \\ &= \sum (h_{ijk}^{n+1})^2 + nc\sum (h_{ij}^{n+1})^2 - n^2cH^2 - nH\operatorname{tr}(H^{n+1})^3 \\ &\quad + \sum_{\beta \neq n+1} \operatorname{tr}(H^{n+1}H^\beta)^2 + [\operatorname{tr}(H^{n+1})^2]^2 .\end{aligned} \tag{3.14}$$

Here we use $H^{n+1}H^\beta = H^\beta H^{n+1}$.

$$\begin{aligned}\tfrac{1}{2}\Delta \|\tau\|^2 &= \sum_{\alpha \neq n+1} (\tau_{ijk}^\alpha)^2 + nc\,\|\tau\|^2 + \sum_{\alpha \neq n+1} h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta h_{ij}^\alpha \\ &\quad - 2\sum_{\alpha \neq n+1} h_{mk}^\alpha h_{mj}^\beta h_{ik}^\beta h_{ij}^\alpha + \sum_{\alpha \neq n+1} h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta h_{ij}^\alpha \\ &\quad - nH\sum_{\alpha \neq n+1} h_{mi}^\alpha h_{ij}^\alpha h_{mj}^{n+1} + \sum_{\alpha \neq n+1} h_{jm}^\alpha h_{mk}^\beta h_{ki}^\beta h_{ij}^\alpha .\end{aligned} \tag{3.15}$$

On the other hand,

$$\operatorname{tr}(H^{n+1})^3 = \operatorname{tr}\mu^3 + 3H[\operatorname{tr}(H^{n+1})^2 - nH^2] + nH^3 . \tag{3.16}$$

(3.14) and (3.16) imply

$$\begin{aligned}\tfrac{1}{2}\Delta \|\mu\|^2 &\geqq (\|\mu\|^2 + nH^2)^2 - nH[\operatorname{tr}(\mu)^3 + 3H\|\mu\|^2 + nH^3] + nc\|\mu\|^2 \\ &= \|\mu\|^2[\|\mu\|^2 + nc - nH^2] - nH\operatorname{tr}(\mu)^3 .\end{aligned} \tag{3.17}$$

Since $\operatorname{tr}\mu = 0$, we can apply Lemma 2.2 to the eigenvalues of μ and hence

$$|\operatorname{tr}(\mu)^3| \leqq (n-2)[n(n-1)]^{-1/2} \|\mu\|^3 . \tag{3.18}$$

Hence,

$$\begin{aligned}\tfrac{1}{2}\Delta \|\mu\|^2 &\geqq \|\mu\|^2[\|\mu\|^2 + nc - nH^2] - n|H|(n-2)[n(n-1)]^{-1/2} \|\mu\|^3 \\ &= \|\mu\|^2\{\|\mu\|^2 - n|H|(n-2)[n(n-1)]^{-1/2} \|\mu\| + nc - nH^2\} . \end{aligned} \tag{3.19}$$

From (2.10), we know that the Ricci curvature of M^n is bounded from below. Putting $F = -(\|\mu\|^2 + a)^{-1/2}$, since M^n is space-like and F is bounded, we can apply Lemma 2.1 to the function F. For given any positive number $\varepsilon > 0$, there exists a point p at which F satisfies the properties (2.19) in Lemma 2.1. Consequently, the following relationship.

$$\tfrac{1}{2}F^4(p)\Delta \|\mu(p)\|^2 < 3\varepsilon^2 - F(p)\varepsilon . \tag{3.20}$$

can be derived by the simple and direct calculation. For a convergent $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$ $(m \to \infty)$, there exists a point sequence $\{p_m\}$ such that the sequence $\{F(p_m)\}$ converges to F_0 because $\{F(p_m)\}$ is a bounded sequence, by taking a subsequence, if necessary. From the difinition of the supremum and (2.19), we have $F_0 = \sup F$ and hence the definition of F gives rise to $\lim \|\mu(p_m)\|^2 = \sup \|\mu\|^2$.

On the other hand, it follows from (3.20) that we have

$$\tfrac{1}{2}F^4(p_m)\Delta \|\mu(p_m)\|^2 < 3\varepsilon_m^2 - F(p_m)\varepsilon_m . \tag{3.21}$$

The right hand side of (3.21) converges to 0 because F is bounded. Accordingly, for any positive number $\varepsilon > 0$ $(\varepsilon < 2)$ there is a sufficiently large integer m for which we have

$$F(p_m)^4 \Delta \|\mu(p_m)\|^2 < \varepsilon . \tag{3.22}$$

This relationship and (3.19) yield

$$\begin{aligned}(2-\varepsilon)\|\mu(p_m)\|^4 - 2(n-2)[n(n-1)]^{-1/2} n|H| \|\mu(p_m)\|^3 \\ + 2(nc - nH^2 - \varepsilon a)\|\mu(p_m)\|^2 - \varepsilon a^2 < 0 .\end{aligned}$$

Hence, $\{\|\mu(p_m)\|\}$ is bounded. Thus the supremum of F satisfies $F_0 = \sup F > 0$. According to (3.22), we have

$$\limsup_{m\to\infty} \Delta \|\mu(p)\|^2 \leqq 0 . \tag{3.23}$$

(3.19) implies

$$\sup\|\mu\|^2[\sup\|\mu\|^2 - (n-2)[n(n-1)]^{-1/2} n|H| \sup\|\mu\| + (nc - nH^2)] \leqq 0 . \tag{3.24}$$

1), when $n = 2$ and $H^2 \leqq c$,

$$\sup\|\mu\|^2[\sup\|\mu\|^2 + nc - nH^2] \leqq 0 .$$

Hence $\sup\|\mu\|^2 = 0$, i.e., $\|\mu\|^2 \equiv 0$.

2), when $n \geqq 3$ and $n^2H^2 < 4(n-1)c$, we have also

$$\sup\|\mu\|^2 = 0, \text{ hence } \|\mu\|^2 \equiv 0 .$$

That is, M^n is pseudo-umbilical. Hence $h_{ij}^{n+1} = H\delta_{ij}$.

From (3.15), we get

$$\begin{aligned}\tfrac{1}{2}\Delta \|\tau\|^2 &= \sum_{\alpha \neq n+1} (\tau^\alpha_{ijk})^2 + nc\|\tau\|^2 + \sum_{\alpha \neq n+1} h^\alpha_{km} h^\beta_{mk} h^\beta_{ij} h^\alpha_{ij} \\ &\quad - 2 \sum_{\alpha \neq n+1} h^\alpha_{mk} h^\beta_{mj} h^\beta_{ik} h^\alpha_{ij} + \sum_{\alpha \neq n+1} h^\alpha_{mi} h^\beta_{mk} h^\beta_{kj} h^\alpha_{ij} \\ &\quad - nH \sum_{\alpha \neq n+1} h^\alpha_{mi} h^{n+1}_{mj} h^\alpha_{ij} + \sum_{\alpha \neq n+1} h^\alpha_{jm} h^\beta_{mk} h^\beta_{ki} h^\alpha_{ij} \\ &= \sum_{\alpha \neq n+1} (\tau^\alpha_{ijk})^2 + nc\|\tau\|^2 + \sum_{\alpha,\beta \neq n+1} [\mathrm{tr}(H_\alpha H_\beta)]^2 \\ &\quad - nH\|\tau\|^2 - 2 \sum_{\beta,\alpha \neq n+1} h^\alpha_{mk} h^\beta_{mj} h^\beta_{ik} h^\alpha_{ij} \\ &\quad + \sum_{\alpha,\beta \neq n+1} h^\alpha_{mi} h^\beta_{mk} h^\beta_{kj} h^\alpha_{ij} + \sum_{\alpha,\beta \neq n+1} h^\alpha_{jm} h^\beta_{mk} h^\beta_{ki} h^\alpha_{ij} . \end{aligned} \tag{3.25}$$

Here we make use of $h^{n+1}_{ij} = H\delta_{ij}$.

We put $S_{\alpha\beta} = \sum h^\alpha_{ij} h^\beta_{ij}$ for α, $\beta \neq n+1$. Then $(S_{\alpha\beta})$ is a $(p-1) \times (p-1)$ symmetrix matrix.. It can be assumed to be diagonal for a suitable choice of $e_{n+2}, \ldots, e_{n+p}$. Set $S_\alpha = S_{\alpha\alpha}$ and we have $\|\tau\|^2 = \sum S_\alpha$. In general, for a matrix $A = (a_{ij})$, we put $N(A) = \mathrm{tr}(A^t A)$. Now we have from (3.25)

$$\begin{aligned}\tfrac{1}{2}\Delta \|\tau\|^2 &= \sum_{\alpha \neq n+1} (\tau^\alpha_{ijk})^2 + (nc - nH^2)\|\tau\|^2 + \sum_{\alpha \neq n+1} S^2_\alpha \\ &\quad + \sum_{\alpha,\beta \neq n+1} N(H^\alpha H^\beta - H^\beta H^\alpha) .\end{aligned}$$

Obviously, $N(H^\alpha H^\beta - H^\beta H^\alpha) \geqq 0$. Let

$$(p-1)\sigma_1 = \sum_{\alpha \neq n+1} S_\alpha = \|\tau\|^2, \; [(p-1)(p-2)\sigma_2]/2 = \sum_{\substack{\alpha<\beta \\ \alpha,\beta \neq n+1}} S_\alpha S_\beta$$

Thus we have

$$\sum_{\alpha \neq n+1} S^2_\alpha = (p-1)\sigma_1^2 + (p-1)(p-2)(\sigma_1^2 - \sigma_2) ,$$

$$(p-1)^2(p-2)(\sigma_1^2 - \sigma_2) = \sum_{\substack{\alpha<\beta, \\ \alpha,\beta \neq n+1}} (S_\alpha - S_\beta)^2 .$$

Hence, $\sum_{\alpha \neq n+1} S^2_\alpha = [1/(p-1)]\|\tau\|^4 + [1/(p-1)] \sum_{\substack{\alpha<\beta \\ \alpha,\beta \neq n+1}} (S_\alpha - S_\beta)^2$. Thus we obtain

$$\tfrac{1}{2}\Delta \|\tau\|^2 \geqq (nc - nH^2)\|\tau\|^2 + [1/(p-1)]\|\tau\|^4 .$$

We make use of the similar methods of proof of $\|\mu\|^2$ for $\|\tau\|^2$. We have $\|\tau\|^2 \equiv 0$. Hence, M^n is totally umbilical.

Remark. When $n = 2$, condition (3.1) is best possible from [1] and [9]. When $n \geqq 3$, condition (3.2) is also best possible from following example.

Example. We consider Riemannian product $H^1(c_1) \times S^{n-1}(c_2)$, where $n \geqq 3$, $c_1 = (2-n)c$ and $c_2 = [(n-2)/(n-1)]c$. Because $H^1(c_1) \times S^{n-1}(c_2) \subset S^{n+1}_1(c)$, then we can get $n^2H^2 = 4(n-1)c$.

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