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A characterization of $\mathbb{P}_n$ by vector bundles

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1 Introduction

In this short note we want to give a characterization of the complex projective space via vector bundles which had been conjectured by Mukai [Kat].

Theorem. *Let X be a compact complex manifold of dimension n, E an ample vector bundle on X of rank $n+1$ satisfying*

$$c_1(E)=c_1(X).$$

Then $X \simeq \mathbb{P}_n$ and $E \simeq \mathcal{O}_{\mathbb{P}_n}(1)^{n+1}$.

Here $c_1(X)$ means the first Chern class of X i.e.: $c_1(X)$ is the anti-canonical class of X.

The theorem being "clear" for $n \leqq 2$, Mukai gave a proof in case $n=3$.

For the general proof given here it is essential to examine carefully extremal rational curves (in the sense of Mori) on X and on the projectivized bundle $\mathbb{P}(E)$.

2 Proof of the theorem

We begin with the easy

Lemma 1. *Let E be an ample vector bundle of rank $n+1$ on P_n. Assume $c_1(E)=c_1(\mathbb{P}_n)$. Then $E \simeq \mathcal{O}_{\mathbb{P}_n}(1)^{n+1}$.*

Proof. Let $l \subset \mathbb{P}_n$ be a line. Then the condition on the Chern class and the ampleness of E imply

$$E|l \simeq \mathcal{O}_{\mathbb{P}_1}(1)^{n+1}$$

So the vector bundle

$$F=E \otimes \mathcal{O}_{\mathbb{P}_n}(-1)$$

is trivial on any line. Hence F is trivial [OSS, p. 51] and our claim follows.

Now let X denote a compact manifold of dimension n and E an ample $(n+1)$-bundle on X with $c_1(E)=c_1(X)$.

Then the anti-canonical bundle K_X^{-1} is ample, i.e. X is Fano. Our strategy is to look at the compact manifold

$$\mathbb{P}(E) \xrightarrow{\pi} X.$$

($\mathbb{P}$ is always taken in Grothendieck's sense).

$\mathbb{P}(E)$ is a $2n$-dimensional manifold with anti-canonical bundle

$$K_{\mathbb{P}(E)}^{-1} = \mathcal{O}_{\mathbb{P}(E)}(n+1).$$

This is an easy consequence of $c_1(E)=c_1(X)$.

E being ample, $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample and hence $\mathbb{P}(E)$ is a Fano manifold.

Lemma 2. $\mathrm{Pic}(X)=\mathbb{Z}$

The proof of Lemma 2 relies on Mori theory. We refer for this to [Mo] and [KMM]. Some of the facts coming up in the proof are also important for our later considerations.

Proof. Since K_X is not nef, there is an extremal ray R on X, which is represented by an extremal rational curve C_0 satisfying

$$(*) \qquad 0<(K_X^{-1}\cdot C_0)\leqq n+1$$

([Mo, 1.4]).

Since

$$c_1(E)=c_1(X)$$

and since clearly

$$(c_1(E)\cdot C_0)\geqq n+1,$$

we have

$$(**) \qquad (K_X^{-1}\cdot C_0)=n+1.$$

So in the notation of [Wi] R has length $n+1$. By (**) and [Wi, 2.4.1] we conclude $\mathrm{Pic}(X)=\mathbb{Z}$.

On $\mathbb{P}(E)$, besides the extremal ray R_1 defining the projection π we have a second extremal ray R_2 since $K_{\mathbb{P}(E)}^{-1}$ is ample 0 and $b_2(\mathbb{P}(E))\geqq 2$ (see [Mo, 1.4]). R_2 defines a surjective morphism $\psi:\mathbb{P}(E)\to Z$ to a normal projective variety Z. ψ has connected fibers and the following property:

(+) for any irreducible curve $C\subset X$, $\dim\psi(C)=0$ holds if and only if its class $[C]$ belongs to R_2 (see [KMM, Io]).

Lemma 3. *If* $\dim Z<2n$, *then* $X\simeq\mathbb{P}_n$ *(and* $\mathbb{P}(E)\simeq\mathbb{P}_n\times\mathbb{P}_n$, $Z\simeq\mathbb{P}_n$*).*

Proof. Let F_s be a fiber of ψ. We first claim:

(1) $\pi|F_s$ is finite.

Assume to the contrary that π contracts a curve in F_s.

Because of (+), all curves on F_s are homologous (up to positive multiples). We conclude that π contracts all curves in F_s, hence $\dim\pi(F_s)=0$.

So $F_s\subset\pi^{-1}(x)\simeq\mathbb{P}_n$ for some $x\in X$.

Consequently $\psi|\pi^{-1}(x)$ has some positive-dimensional fiber. This is only possible if $\psi(\pi^{-1}(x))$ is a point. Hence $F_s=\pi^{-1}(x)$. But the extremal rays R_1 and R_2 are different, contradiction!

So $\pi|F_s$ is finite for all $s\in Z$. In particular $\dim F_s \leqq n$.

Take s general so that F_s is smooth. F_s is a Fano manifold since we have by the adjunction formula

$$K_{F_s}^{-1} \cong \mathcal{O}_{\mathbb{P}(E)}(n+1)|F_s .$$

This formula also shows that F_s has index $\geqq (n+1)$. Recall that the index of a Fano manifold X is the biggest $r\in\mathbb{N}$ such that there is some $L\in\operatorname{Pic}(X)$ with $L^r=K_X$.

Since the index of a Fano manifold is always bounded by $\dim+1$, we conclude that $\operatorname{index}(F_s)=n+1$ and the Kobayashi-Ochiai theorem [KO] says that $F_s\simeq\mathbb{P}_n$.

By (1) we obtain a finite surjective map $\mathbb{P}_n\to X$. Then $X\simeq\mathbb{P}_n$ by a theorem of Lazarsfeld [La].

What remains to treat is the case where $\dim Z=2n$, i.e. ψ is a modification. Of course this case must be excluded.

We will use the following generalization of a theorem of Ionescu [Io] communicated to me by J. Wisniewski:

Lemma 4 (Wisniewski). *Let X be a projective manifold with extremal ray R. Let*

$$l(R)=\min\{K_X^{-1}\cdot C \mid C \text{ a rational curve in } R\}$$

be the length of R. Let A = union of all curves in R.

Assume that the contraction of R has a non-trivial fiber of dimension $\leqq d$. Then:

$$\dim A \geqq \dim X + l(R) - d - 1 .$$

Remark. Ionescu's result is $\dim A \geqq 1/2\ (\dim X + 1(R) - 1)$, not involving d; Wisniewski's proof is to look carefully to Ionescu's method.

Lemma 5. *ψ is not a modification.*

Proof. We apply Lemma 4 to the extremal ray R_2 on the projective $2n$ – fold $\mathbb{P}(E)$. In order to compute $l(R_2)$, take an extremal curve l belonging to R_2.

Since $K_{\mathbb{P}(E)}^{-1}=\mathcal{O}_{\mathbb{P}(E)}(n+1)$, we have

$$(K_{\mathbb{P}(E)}^{-1}\cdot l)=m(n+1), \qquad m\in\mathbb{N}.$$

On the other hand

$$(K_{\mathbb{P}(E)}^{-1}\cdot l)\leqq \dim\mathbb{P}(E)+1=2n+1 ,$$

l being extremal. Hence $m=1$ and $l(R_2)=n+1$.

By the same arguments as in Lemma 3, every fiber of ψ has dimension $\leqq n$. So we can apply Lemma 4 with $d=n$ to obtain:

$$\dim A \geqq \dim\mathbb{P}(E), \quad \text{i.e.} \quad A=\mathbb{P}(E),$$

and ψ cannot be a modification.

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Note added in proof

Meanwhile Ye-Zhang and T. Fujita also proved the theorem by different methods.