Strong positivity in $C(\bar{\Omega})$ for elliptic systems*

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The book on maximum principles by Protter and Weinberger contains a maximum principle for systems of essentially positive elliptic equations. These systems are weakly coupled, that is: no coupling in the derivatives. Recently the problem has been revisited by several authors, e.g. [21] and [28]. Nagel uses semigroup theory for operator matrices and finds as an application a positivity result for the elliptic system. De Figueiredo and Mitidieri use the maximum principle for one equation. In this note we will give a direct proof by using an extension of the Krein-Rutman Theorem. The underlying space will be $(C(\bar{\Omega}))^k$. In our approach it is sufficient to have operators with continuous coefficients. The three conditions we use can be described by: (i) essentially positive coupling matrix; (ii) full coupling; (iii) existence of a positive supersolution. We will show the existence of a unique first eigenfunction. Furthermore we will investigate the necessity of the three basic conditions. A partial result will be shown for some systems that are not far from essentially positive. (Essentially positive is also known as cooperative.) For the last result we need pointwise estimates for Green functions. Recent results for such estimates are listed in an appendix. Implications for the parabolic system will be given.

1 Main result

The domain Ω is a bounded, open and connected subset of $\mathbb{R}^n$ that satisfies a uniform exterior cone condition. We consider for $\mathbf{u}, \mathbf{f}: \bar{\Omega} \to \mathbb{R}^k$ the following system of differential equations:

$$(1.1) \qquad L\mathbf{u} = H\mathbf{u} + \mathbf{f} \quad \text{in} \quad \Omega$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on} \quad \partial\Omega;$$

where:

L is a diagonal matrix of strictly elliptic second order operators, that is:

$$L=\begin{pmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_k \end{pmatrix}, \quad \text{with} \quad L_\mu = -\sum_{ij}^n a_{ij}^\mu(\cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i^n b_i^\mu(\cdot) \frac{\partial}{\partial x_i} + c^\mu(\cdot),$$

and for some $\lambda > 0$: $\sum_{ij}^n a_{ij}^\mu(x)\xi_i\xi_j \geq \lambda |\xi|^2$ for all $x, \xi \in \mathbb{R}^n$; H is $k \times k$ matrix of functions on $\bar{\Omega}$ with zero diagonal elements.

We assume that the coefficients of L and H are in $C(\bar{\Omega})$; hence the elliptic operators L_μ are uniformly elliptic.

The three basic conditions in order to obtain a strong positivity result are the following.

Condition 1 H is essentially positive (cooperative).

A matrix H is called essentially positive if $H_{\mu\nu}(x) \geq 0$ for all $\mu \neq \nu$ and $x \in \bar{\Omega}$.

Condition 2 H is fully coupled.

That is, the index set $\{1, 2, \ldots, k\}$ cannot be split up in two disjoint nonempty sets α and β such that $H_{\mu\nu}(x) \equiv 0$ in Ω for $\mu \in \alpha$, $\nu \in \beta$.

For an essentially positive matrix it means $\tilde{H}$, with $\tilde{H}_{\mu\nu} = \max\{H_{\mu\nu}(x); x \in \bar{\Omega}\}$, is irreducible. See [19].

Condition 3 There is a positive strict supersolution of (1.1) *with* $\mathbf{f} = \mathbf{0}$.

That is: there is $\boldsymbol{\phi} \in (W_{\text{loc}}^{2;n}(\Omega) \cap C(\bar{\Omega}))^k$ such that $\boldsymbol{\phi} \geq 0$, $(L-H)\boldsymbol{\phi} \geq 0$ and either $\boldsymbol{\phi} \neq 0$ on $\partial\Omega$ or $(L-H)\boldsymbol{\phi} \neq 0$ in Ω.

Some notations Let $\mathbf{u}$ be a (vector)function in $(C(\bar{\Omega}))^k$. By $\mathbf{u} \gg 0$ we mean that $\mathbf{u}_\nu(x) > 0$ for all $x \in \Omega$ (Ω is open) and all components $\nu \in \{1, 2, \ldots, k\}$.

For φ in an ordered vector space we write $\varphi > 0$ if $0 \neq \varphi \geq 0$.

An operator A is called strictly positive if $A\varphi > 0$ for $\varphi > 0$.

Remark 1.1 If all three conditions hold the supersolution $\boldsymbol{\phi}$ satisfies $\boldsymbol{\phi} \gg 0$.

Indeed, if $(L-H)\boldsymbol{\phi} \geq 0$ and $\boldsymbol{\phi} \geq 0$, then $L\boldsymbol{\phi} \geq H\boldsymbol{\phi} \geq 0$, hence $L_\nu\boldsymbol{\phi}_\nu \geq 0$, hence $\boldsymbol{\phi}_\nu \equiv 0$ or $\boldsymbol{\phi}_\nu \gg 0$ by the scalar minimum principle ([23, Theorem 9.6]). Now if $\boldsymbol{\phi}_\mu \equiv 0$ for $\mu \in \alpha$ and $\boldsymbol{\phi}_\nu \gg 0$ for $\nu \in \beta$ and if both sets are nonempty, then $\sum_{\nu=1}^k H_{\mu\nu}\boldsymbol{\phi}_\nu = L_\mu\boldsymbol{\phi}_\mu = 0$ for $\mu \in \alpha$. Hence $H_{\mu\nu} = 0$ for $\nu \in \beta$ and H is not fully coupled, a contradiction.

Theorem 1.1 *Let Conditions* 1, 2 *and* 3 *be satisfied and* $\mathbf{f} \in (L^p(\bar{\Omega}))^k$ *with* $p \geq n$. *Then the following holds.*

(i) *There is a unique* $\mathbf{u} \in (W_{\text{loc}}^{2;p}(\Omega) \cap C_0(\bar{\Omega}))^k$ *that satisfies* (1.1).

(ii) *If* $\mathbf{f} \geq 0$ *then* $\mathbf{u} \geq 0$; *if* $\mathbf{f} > 0$ *then* $\mathbf{u} \gg 0$.

(iii) *There is a unique positive eigenfunction* $\boldsymbol{\psi} \in (W_{\text{loc}}^{2;n}(\Omega) \cap C_0(\bar{\Omega}))^k$; $\boldsymbol{\psi} \gg 0$ *and* $(L-H)\boldsymbol{\psi} = \lambda\boldsymbol{\psi}$ *for some* $\lambda > 0$. *(Unique after normalizing).*

Remark 1.2 If the Conditions 1 and 3 hold but the system is not fully coupled one still obtains (i) when $\boldsymbol{\phi}$ in Condition 3 is a supersolution that is strict in

every component (or in every fully coupled subset of components). Now the second part of (ii) only holds componentwise. There is still a first eigenfunction $\psi > 0$, but not necessarily $\psi \gg 0$. The eigenfunction is unique if and only if the natural ordering of the fully coupled subsets is complete.

Remark 1.3 In our proof we use Theorem 9.30 of [23] in order to solve $(L_\mu + \beta)\mathbf{u}_\mu = \mathbf{f}_\mu$ in Ω, $\mathbf{u}_\mu = 0$ on $\partial\Omega$. Assuming b_i^μ, $c^\mu(\cdot) \in L^\infty(\Omega)$, instead of continuity, will be sufficient. In that theorem it is also assumed that the domain satisfies an exterior cone condition. That is the only part of our proof where the regularity of the domain plays a role.

On the other hand, if one assumes Lipschitz-continuity of the coefficients of L_μ one can solve the elliptic equation above if all boundary points are regular for the Laplacian. Hence the exterior cone condition in Theorem 1.1 can be replaced by this regularity. For a survey concerning sufficient conditions on the regularity of the boundary, see [23, p. 139].

Remark 1.4 Assuming $f \in (L^p(\bar{\Omega}))^k$ with $p > n$, it is possible to have the coefficients of H in $L^q(\bar{\Omega})$ with $q \geq p\,n/(p-n)$.

Remark 1.5 The related eigenvalue problem:

$$L\mathbf{u} = \lambda H \mathbf{u} \quad \text{in} \quad \Omega$$

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega$$

has been studied by several authors. See e.g. [32, 24, 9, 17].

Proof of Theorem 1.1 We will solve the system in two steps. In the first step we show for β large enough the following:

$$\begin{array}{ll} L\mathbf{u} = H\mathbf{u} + \mathbf{f} & \text{in} \quad \Omega \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega \end{array} \Leftrightarrow \mathbf{u} = (L + \beta I)_0^{-1}(H + \beta I)\mathbf{u} + (L + \beta I)_0^{-1}\mathbf{f},$$

where $(L + \beta I)_0^{-1}$ is a diagonal matrix containing the inverse of $L_i + \beta$ with zero Dirichlet boundary condition on the i-th element of the diagonal.

With $A = (L + \beta I)_0^{-1}(H + \beta I)$ the second step will be:

$$(I - A)\mathbf{u} = (L + \beta I)_0^{-1}\mathbf{f}$$

$$\Leftrightarrow$$

$$\mathbf{u} = (I - A)^{-1}(L + \beta I)_0^{-1}\mathbf{f} = \sum_{\nu = 0}^{\infty} A^\nu (L + \beta I)_0^{-1}\mathbf{f}.$$

Step 1 Set $\beta = 1 \vee \max\{-c^\mu(x); x \in \bar{\Omega}, 1 \leq \mu \leq k\}$. Then the inverse of $L_\mu + \beta$ with Dirichlet boundary conditions is well defined by [23, Theorem 9.30] as an operator on $L^p(\Omega) \supset C(\bar{\Omega})$ and $(L_\mu + \beta)_0^{-1}\mathbf{f}_\mu \in W_{loc}^{2,p}(\Omega) \cap C_0(\bar{\Omega})$ for all $p \in [n, \infty)$. The maximum principle in [23, Theorem 9.6] shows that if $\mathbf{f}_\mu > 0$ then $(L_\mu + \beta)_0^{-1}\mathbf{f}_\mu \gg 0$. The restriction to $C(\bar{\Omega})$ or $C_0(\bar{\Omega})$ we will also denote by $(L_\mu + \beta)_0^{-1}$. Now $(L + \beta I)_0^{-1} \in \mathscr{L}((C(\bar{\Omega}))^k; (C_0(\bar{\Omega}))^k)$ is well defined and strictly positive.

This operator $(L_\mu + \beta)_0^{-1}$ is compact as well. Indeed, let S denote the unit ball in $C(\bar\Omega)$ and let $v \in S$. Since Ω is bounded, $|(L_\mu + \beta)_0^{-1} v| \leq (L_\mu + \beta)_0^{-1} 1$ and $\lim_{x \to \partial\Omega} ((L_\mu + \beta)_0^{-1} 1)(x) = 0$ and since $W_{loc}^{2,p}(\Omega) \subset C^1(\Omega)$ the set $(L_\mu + \beta)_0^{-1} S$ is bounded and equicontinuous, and hence relatively compact by Arzela-Ascoli.

Lemma 1.2 $(L + \beta I)_0^{-1} \in \mathscr{L}((C(\bar\Omega))^k; (C_0(\bar\Omega))^k)$ is *strictly positive and compact*. *Moreover, if* $\mathbf{f} \in (L^p(\Omega))^k$ *with* $\mathbf{f}_\mu > 0$, *then* $((L + \beta I)_0^{-1} \mathbf{f})_\mu \gg 0$.

Next we will show some properties of $A = (L + \beta I)_0^{-1}(H + \beta I) \in \mathscr{L}((C_0(\bar\Omega))^k)$.

Since the restriction to $C_0(\bar\Omega)$ of every component operator is compact too, $(L + \beta I)_0^{-1} \in \mathscr{L}((C_0(\bar\Omega))^k)$ is compact. Hence A is strictly positive and compact as the product of a bounded, strictly positive and a compact, strictly positive operator. The strong maximum principle [23, Theorem 9.6] also shows that $(L_\mu + \beta)_0^{-1} \in \mathscr{L}(C_0(\bar\Omega))$ is irreducible, that is: $C_0(\bar\Omega)$ and $\{0\}$ are the only closed lattice ideals in $C_0(\bar\Omega)$ which are invariant under $(L_\mu + \beta)_0^{-1}$. The closed lattice ideals in $C_0(\bar\Omega)$ are sets $\{f \in C_0(\bar\Omega); f(x) = 0$ for $x \in K\}$ with K a closed set in $\bar\Omega$. See the first example on p. 157 of [33]. Since every component of $(L + \beta I)_0^{-1}$ is irreducible and since $H + \beta I \geq I$, the only possible lattice ideals, that are invariant under A, are subsets of $(C_0(\bar\Omega))^k$ with every component equal to $C_0(\bar\Omega)$ or $\{0\}$. From the fact that H is fully coupled it follows that they can only be the trivial ones $(C_0(\bar\Omega))^k$ or $\{\mathbf{0}\}$. Indeed, let I be a non-zero closed A-invariant ideal, then there is $\mathbf{0} < \mathbf{f} \in I$ with $\mathbf{f}_v \gg 0$. If $H_{\mu v} \neq 0$ then $(A \mathbf{f})_\mu \gg \mathbf{0}$. Repeating this argument k times we find, since H is fully coupled, that $(A^k \mathbf{f})_v \gg \mathbf{0}$ for every component v. Using the second part of Lemma 1.2 we can summarize:

Lemma 1.3 $A = (L + \beta I)_0^{-1}(H + \beta I) \in \mathscr{L}((C_0(\bar\Omega))^k)$ *is positive, irreducible and compact. Moreover, if* $\mathbf{0} < \mathbf{f} \in (L^p(\Omega))^k$ *then* $\mathbf{0} \ll A^{k+1} \mathbf{f}$.

Step 2 The operator $\sum_{v=0}^{\infty} A^v$ is well defined (and positive and bounded) if $r(A) < 1$.

By a theorem of de Pagter [29] one finds that the spectral radius $r(A)$ is positive. Since A is compact and positive, the Krein-Rutman Theorem [27] shows that $r(A)$ is an eigenvalue with a positive eigenfunction. Similarly, since the dual operator A' too is compact (Schauder) and positive, $r(A)$ is an eigenvalue of A' with a positive eigenfunction $\Psi \in ((C_0(\bar\Omega))^k)'$. Let $\langle \cdot, \cdot \rangle$ denote the pairing between $(C_0(\bar\Omega))^k$ and its dual. The operator A' is defined by $\langle A'\Phi, \mathbf{u} \rangle = \langle \Phi, A\mathbf{u} \rangle$. Since A is irreducible it follows that Ψ is strictly positive. Indeed, let $\mathbf{u} \in (C_0(\bar\Omega))^k$ satisfy $\mathbf{u} > 0$. Then there is m such that $\langle \Psi, A^m \mathbf{u} \rangle > 0$ and it follows that $\langle \Psi, \mathbf{u} \rangle = (r(A))^{-m} \langle \Psi, A^m \mathbf{u} \rangle > 0$. See [33, Proposition III.8.3]. In the first appendix we state some of the results on Banach lattices that are used here.

By Condition 3 there is $\phi \in (C(\bar\Omega))^k$ such that $(L + \beta I)\phi \geq (H + \beta I)\phi > 0$. By the strong maximum principle [23, Theorem 9.6] and the fact that ϕ is a strictly positive strict supersolution, one finds that $\phi > (L + \beta I)_0^{-1}(H + \beta I)\phi = A\phi > 0$. Since Ψ is strictly positive this results in $\langle \Psi, \phi \rangle > \langle \Psi, A\phi \rangle = \langle A'\Psi, \phi \rangle = r(A)\langle \Psi, \phi \rangle > 0$. This shows that $r(A) < 1$. With Lemma 1.3 we find that $A^{k+1}(L + \beta I)_0^{-1} \mathbf{f} \gg 0$ if $\mathbf{f} > 0$. We may conclude:

Lemma 1.4 $T = \sum_{v=0}^{\infty} A^v (L + \beta I)_0^{-1} \in \mathscr{L}((L^p(\Omega))^k; (W_{loc}^{2,p}(\Omega) \cap C_0(\bar\Omega))^k)$ *is well defined. The restriction of* $T \in \mathscr{L}((C_0(\bar\Omega))^k)$ *is positive, irreducible and compact. Moreover, if* $\mathbf{0} < \mathbf{f} \in (L^p(\Omega))^k$, *then* $\mathbf{0} \ll T\mathbf{f}$.

A solution of (1.1) is defined by $u = Tf$. The positivity shows uniqueness.

It follows from the results stated in the first appendix, among them the Krein-Rutman Theorem [27], that T has a unique positive eigenfunction ψ with eigenvalue $r(T) > 0$. Then ψ is an eigenfunction of $L - H$ with eigenvalue $r(T)^{-1}$. Moreover, every eigenfunction of $L - H$ in $(C_0(\bar{\Omega}))^k$ is also an eigenfunction of T. This shows the uniqueness of the positive eigenfunction. $\square$

Bandle [5] remarked that a similar proof holds for other boundary conditions as well. We might even have different boundary conditions for different components. Consider Neumann boundary condition $B_\nu u_\nu = -\dfrac{d}{d\,n}\, u_\nu$, where n is the outward normal. We have (with sufficient regularity) when β is large enough, for

$$(L_\nu + \beta)\, u_\nu = f_\nu \quad \text{in} \quad \Omega,$$

$$B_\nu u_\nu = 0 \quad \text{on} \quad \partial\Omega,$$

a compact and strongly positive solution operator $(L_\nu + \beta)_{b_\nu}^{-1} \in \mathcal{L}(C(\bar{\Omega}); C(\bar{\Omega}))$.

Corollary 1.5 *Let Conditions* 1, 2 *and* 3 *be satisfied. Let* $p \geq n$.
If $u \in (W_{\text{loc}}^{2,p}(\Omega) \cap C(\bar{\Omega}))^k$ *satisfies*

(1.2)
$$Lu = Hu + f \quad \text{in} \quad \Omega$$

$$u = \psi \quad \text{on} \quad \partial\Omega;$$

with $0 \leq f \in (L^p(\Omega))^k$ *and* $0 \leq \psi \in (C(\partial\Omega))^k$, *then* $0 \ll u$ *or* $0 = u$.

Proof. Let $\beta > 0$ be large enough. By [23, Theorem 9.18/9.30] there is a solution $w \in (W_{\text{loc}}^{2,p}(\Omega) \cap C(\bar{\Omega}))^k$ of

$$(L + \beta I)\, w = 0 \quad \text{in} \quad \Omega$$

$$w = \psi \quad \text{on} \quad \partial\Omega.$$

The function w is nonnegative and satisfies $w_\nu \gg 0$ if $\psi_\nu > 0$. Then $v = u - w$ is the nonnegative solution of

$$Lv = Hv + f + (H + \beta I)\, w \quad \text{in} \quad \Omega$$

$$v = 0 \quad \text{on} \quad \partial\Omega.$$

By Theorem 1.1 one finds that $v \gg 0$ if $(f + (H + \beta I)\, w) > 0$. Since $u = v + w \geq v$ the result follows. $\square$

The maximum principle for one equation does not use any regularity for the domain. To have a similar result for systems we have to modify Condition 3.

Corollary 1.6 *Let* Ω *be an open, bounded and connected subset of* $\mathbb{R}^n$. *Suppose Conditions* 1, 2 *and* 3 *are satisfied. Moreover suppose* ϕ *in Condition* 3 *satisfies* $\phi \geq \alpha 1$ *on* $\bar{\Omega}$ *for some* $\alpha > 0$. *Then* $u \in (W_{\text{loc}}^{2,p}(\Omega) \cap C(\bar{\Omega}))^k$, *with* $p \geq n$, *such that*

$$(L - H)u \geq 0 \quad \text{in} \quad \Omega$$

$$u \geq 0 \quad \text{on} \quad \partial\Omega$$

satisfies either $u = 0$ *or* $u \gg 0$.

Proof. Suppose that $\mathbf{u}_v(x_0) < 0$ for some $x_0 \in \Omega_1$. Let $c > 0$ be the smallest number such that $\mathbf{w} = \mathbf{u} + c\,\boldsymbol{\phi} \geqq \mathbf{0}$. Since $\mathbf{w}_v(x) > 0$ for x in a neighbourhood of $\partial\Omega$, there is $\Omega_1 \subset \Omega$ with smooth boundary, with $\mathbf{w}_v(x) > 0$ on $\Omega \setminus \Omega_1$ and such that Condition 2 still holds on Ω_1. Theorem 1.1 on Ω_1 shows $\mathbf{w} \gg \mathbf{0}$, a contradiction. If $\mathbf{u} \geqq \mathbf{0}$ then for every smooth domain $\Omega_1 \subset \Omega$, with $\Omega \setminus \Omega_1$ small enough, Theorem 1.1 shows $\mathbf{u} \gg \mathbf{0}$ or $\mathbf{u} = \mathbf{0}$ in Ω_1 and hence in Ω. $\square$

2 On conditions similar to Condition 3

In order to have the literal version of the strong maximum principle, that is the following implication holds for all $M \in \mathbb{R}^+$ and $\mathbf{f} \leqq \mathbf{0}$ with $\mathbf{1} = (1, \ldots, 1)^T$:

$$\begin{aligned}
L\,\mathbf{u} &= H\,\mathbf{u} + \mathbf{f} \quad \text{in } \Omega \\
\mathbf{u} &\leqq M\,\mathbf{1} \qquad \text{on } \partial\Omega
\end{aligned} \quad \Rightarrow \quad \mathbf{u} \ll M\,\mathbf{1} \quad \text{or} \quad \mathbf{u} = M\,\mathbf{1},$$

we would need Condition 3 with $\boldsymbol{\phi} = \mathbf{1}$.

In other words

$$(2.1) \qquad\qquad -c^\mu + \sum_{v=0}^{k} H_{\mu v} \leqq 0 \qquad \text{for all } \mu,$$

which is the genuine restriction of Protter and Weinberger [31, (7) p. 190, 192].

De Figueiredo and Mitidieri use in [21, 22] a supersolution $\boldsymbol{\phi}$ for which every component lies above $c\,d(x, \partial\Omega)^\vartheta$ $(c > 0, \vartheta < 1)$.

Consider the related parabolic system:

$$(2.2) \qquad\qquad \left(I\,\frac{\partial}{\partial t} + L\right)\mathbf{u} = H\,\mathbf{u} + \mathbf{f} \quad \text{in } E = \Omega \times (0, T)$$

$$\mathbf{u}(t) = \boldsymbol{\varphi}(t) \qquad \text{on } \partial\Omega \quad \text{for } t \in (0, T)$$

$$\mathbf{u}(0) = \mathbf{u}_0.$$

From semigroup-theory (see [14, Proposition 7.1]) one knows that a C_0-semigroup is positive if (and only if) the resolvent operator $(\lambda I + L - H)^{-1}$ is positive for all λ large enough. For Condition 3 this corresponds with the existence of $\boldsymbol{\phi} > \mathbf{0}$ such that $(\lambda I + L - H)\boldsymbol{\phi} > \mathbf{0}$ for all λ large. Since $\boldsymbol{\phi} = \mathbf{1}$ will do, Conditions 1 and 2 are sufficient for a similar theorem in the parabolic case.

Let $\mathbf{f}, \boldsymbol{\varphi}, \mathbf{u}_0 \geqq \mathbf{0}$. One obtains if $\mathbf{u}_m(t_1) \neq 0$ that $\mathbf{u}(t) \gg \mathbf{0}$ for all $t > t_1$.

For a strong maximum principle for one equation Walter in [35] uses $\boldsymbol{\phi}$ that is strictly positive but not necessarily a strict supersolution. A similar result holds for systems.

Condition 3 a. There is a supersolution $\boldsymbol{\phi} \in (W^{2;n}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega}))^k$ *of* (1.1) *with* $\boldsymbol{\phi} > \mathbf{0}$ *and* $(L - H)\boldsymbol{\phi} \geqq \mathbf{0}$.

Assuming $\boldsymbol{\phi} \gg \mathbf{0}$ instead of $\boldsymbol{\phi} > \mathbf{0}$ is not really a restriction.

Corollary 2.1 *Let Conditions* 1, 2 *and* 3a *be satisfied. Let* $p \geq n$.
(i) *Then there is a unique positive eigenfunction* $\psi \in (W_{loc}^{2,p}(\Omega) \cap C_0(\bar{\Omega}))^k$ *with*
$(L-H)\psi = \lambda\psi$ *for some* $\lambda \geq 0$. *(Unique after normalizing).*
(ii) *Let* $\mathbf{u} \in (W_{loc}^{2,p}(\Omega) \cap C(\bar{\Omega}))^k$ *with* $\mathbf{u} \geq \mathbf{0}$ *on* $\partial\Omega$ *satisfy* $(L-H)\mathbf{u} \geq \mathbf{0}$. *If* $\mathbf{u}$ *satisfies*
$\mathbf{u} \geq -M\psi$ *for some* $M > 0$, *then one of the following holds.*
(a) $\mathbf{u} = \mathbf{0}$ *or* (b) $\mathbf{u} \gg \mathbf{0}$ *or* (c) $\mathbf{u} = \alpha\psi$ *with* $\alpha < 0$.

Remark 2.1 If Ω has a uniform interior ball condition it follows from the strong maximum principle that $\psi_v(x) \geq \gamma d(x, \partial\Omega)$ for every component v (some $\gamma > 0$). See [35, Lemma 1]. Hence, if $\mathbf{u}$ is Lipschitz continuous and $\mathbf{u} \geq \mathbf{0}$ on $\partial\Omega$, then $\mathbf{u} \geq -M\psi$ for some M. Then statement (ii) becomes similar to the one in [35].

Proof. After replacing $(L-H)$ by $(L+I-H)$ Theorem 1.1 shows part (i).
(ii a/b) Condition 3a is weaker only if $(L-H)\phi = \mathbf{0}$ in Ω and $\phi = \mathbf{0}$ on $\partial\Omega$. Hence, if these two equalities do not hold, we can apply Theorem 1.1. Indeed, let β be large enough. Since $\mathbf{u} \geq \mathbf{0}$ on $\partial\Omega$ and $(L+\beta I)\mathbf{u} \geq (H+\beta I)\mathbf{u}$ Theorem 9.6 of [23] shows that $\mathbf{u} \geq (L+\beta I)_0^{-1}(H+\beta I)\mathbf{u}$. Then $\mathbf{w} = (L+\beta I)_0^{-1}(H+\beta I)\mathbf{u}$ satisfies

(2.2)
$$Lw = Hw + f \quad \text{in} \quad \Omega$$

$$w = 0 \quad \text{on} \quad \partial\Omega.$$

The function $\mathbf{f} = (H+\beta I)(I - (L+\beta I)_0^{-1}(H+\beta I))\mathbf{u}$ satisfies $\mathbf{0} \leq \mathbf{f} \in (C(\bar{\Omega}))^k$. From Theorem 1.1 it follows that $\mathbf{w} = \mathbf{0}$ or $\mathbf{w} \gg \mathbf{0}$ and hence $\mathbf{u} = \mathbf{0}$ or $\mathbf{u} \geq \mathbf{w} \gg \mathbf{0}$.
(ii c) If $(L-H)\phi = \mathbf{0}$ in Ω and $\phi = \mathbf{0}$ on $\partial\Omega$ it follows from Theorem 1.1 that ϕ is (a multiple of) the unique positive eigenfunction and the eigenvalue $\lambda = 0$. Notice that $\mathbf{v} = M\psi + \mathbf{u} \geq \mathbf{0}$ and $(L-H)\mathbf{v} = \mathbf{f} \geq \mathbf{0}$. If $\mathbf{v} > \mathbf{0}$ on $\partial\Omega$ or $\mathbf{f} > \mathbf{0}$ then Condition 3 is satisfied, with ϕ replaced by $(L+\beta I)_0^{-1}(H+\beta I)\mathbf{v}$, and the first eigenvalue is positive, a contradiction. Hence $\mathbf{v} = \mathbf{0}$ on $\partial\Omega$ and $(L-H)\mathbf{v} = \mathbf{0}$, which shows that $\mathbf{v}$, and hence $\mathbf{u}$, is a multiple of ψ. $\square$

We will end this section with a special case of (1.1).

Corollary 2.2 *Suppose that the Conditions* 1 *and* 2 *are satisfied and suppose* H *is a constant matrix. Moreover let the operators* L_μ *have a common eigenfunction* $v \gg 0$ ($v = 0$ *on* $\partial\Omega$) *with* $L_\mu v = \lambda_\mu v$ *for all* μ.
Let $\Lambda = (\lambda_\mu)$. *Then the following two statements are equivalent.*
(i) *There is a vector* $\rho \in \mathbb{R}^k$ *with* $\rho > 0$ *and* $(\Lambda - H)\rho > 0$.
(ii) *If* $\mathbf{u}$ *satisfies* (1.1) *with* $\mathbf{f} \geq \mathbf{0}$, *then* $\mathbf{u} \geq \mathbf{0}$; *if* $\mathbf{f} > \mathbf{0}$ *then* $\mathbf{u} \gg \mathbf{0}$.

Proof of Corollary 2.2 (i) $\Rightarrow$ (ii): Condition 3 is satisfied with $\phi = v\rho$.
(ii) $\Rightarrow$ (i): If $(\Lambda - H)\rho = \mathbf{0}$ then it follows from (ii) and $(L-H)\rho v = v(\Lambda - H)\rho = \mathbf{0}$ that $\rho v = \mathbf{0}$, and hence $\rho = \mathbf{0}$. So $\Lambda - H$ is nonsingular and there exists a vector ρ with $(\Lambda - H)\rho > 0$. Again by (ii) and $(L-H)\rho v = v(\Lambda - H)\rho > 0$ it follows that $\rho v > 0$ and hence $\rho > 0$. $\square$

Remark 2.2 Suppose Condition 1 holds. Then the following four statements are equivalent for constant H (see [7, pp. 134-138]):
(iii) $\Lambda - H$ is a nonsingular M-matrix;
(iv) $\Lambda - H$ is semipositive in matrix sense; there is $\rho \in \mathbb{R}^k$ with $\rho \gg 0$ and $(\Lambda - H)\rho \gg 0$ ([7, I_{27}]);
(v) all the leading principal minors of $\Lambda - H$ are positive ([7, E_{17}]);

(vi) $\Lambda - H$ is inverse-positive in matrix sense: $(\Lambda - H)^{-1}$ exists and each element of $(\Lambda - H)^{-1}$ is positive $([7, N_{38}])$.

De Figueiredo and Mitidieri show in [21] by a different proof that, for cooperative systems with constant coefficients and $L_\mu = \Delta$, (v) is a necessary and sufficient condition in order to have a maximum principle. Another proof of this result by using (vi) instead of (v) is given by Clément and Egberts in [15].

Remark 2.3 Suppose Conditions 1 and 2 hold and H is constant. Then the statements (i) in Corollary 2.2 and (iv) are equivalent. One direction is trivial, the other is shown as follows.

Let ρ be the vector in (i), and set $A = (\Lambda + \beta I)^{-1}(H + \beta I)$, with β positive and large enough for $\Lambda + \beta I$ to have strictly positive diagonal elements. Then A is a positive operator and $\rho > A\rho > A^2 \rho > \ldots > A^k \rho > 0$. Since the system is fully coupled one finds $\rho \gg A^k \rho$. Then $\kappa = \sum_{m=0}^{k-1} A^m \rho$ satisfies $\kappa \gg 0$ and

$$(\Lambda - H)\kappa = (\Lambda + \beta I)(\rho - A^k \rho) \gg 0.$$

3 Near the first eigenvalue

In this section we will fix $p > n$. Condition 3 will be satisfied if one replaces L by $L - c$ for some c sufficiently negative (take $\phi = 1$ and $-c\mathbf{1} > |(L-H)\phi|$). Then T^c: $(L^p(\Omega))^k \to (C_0(\bar\Omega))^k$ defined, for $\beta > 0$, by

$$T^c = (L - cI - H)_0^{-1} = \sum_{v=0}^{\infty} ((L + (\beta - c)I)_0^{-1}(H + \beta I))^v (L + (\beta - c)I)_0^{-1}$$

is compact and strictly positive. $T^c((L^p(\Omega))^k)$ with norm $\|(L - cI - H)u\|_p$ is a Banach space. Since $L - \lambda_0 I - H = (I - (c - \lambda_0)T^c)(L - cI - H)$ we find from Theorem 1.1:

Proposition 3.1 *Suppose Conditions 1 and 2 are satisfied. Then there exists $\lambda_0 \in \mathbb{R}$ such that*
(i) *$L - \lambda_0 I - H$: $T^c((L^p(\Omega))^k) \to (L^p(\Omega))^k$ is a Fredholm operator of index 0, and $\mathcal{N}(L - \lambda_0 I - H) = \text{span}\{\mathbf{u}_0\}$ with $\mathbf{u}_0 \gg 0$;*
(ii) *for $\lambda < \lambda_0$ the operator $T^\lambda = (L - \lambda I - H)_0^{-1}$: $(L^p(\Omega))^k \to (C_0(\bar\Omega))^k$ exists and is compact. Moreover, if $0 < \mathbf{u} \in (L^p(\Omega))^k$ then $0 \ll T^\lambda \mathbf{u}$.*

Remark 3.1 Barta proved in [6] that the lowest eigenvalue λ_0 of

$$-\Delta u = \lambda u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

satisfies, if $w \in C^2(\bar\Omega)$ and $w(x) > 0$ for all $x \in \Omega$,

$$\lambda_0 \geqq \inf \left\{ \frac{-\Delta w(x)}{w(x)}; \, x \in \Omega \right\}.$$

The result has been extended to nonselfadjoint scalar problems in [30]. A similar result holds for systems that satisfy Conditions 1 and 2. For a function $\mathbf{w} \in (C(\bar{\Omega}) \cap C^2(\Omega))^k$ that satisfies $\mathbf{w} \gg 0$, we find:

$$\lambda_0 \geq \inf \left\{ \frac{((L-H)\mathbf{w})_\nu(x)}{\mathbf{w}_\nu(x)} ; \ x \in \Omega, \ \nu \in \{1, \ldots, k\} \right\}.$$

Denote this infimum by λ_w. Since $\mathbf{w} \gg 0$ it follows that $(L - \lambda_w I - H)\mathbf{w} \geq 0$. If the last inequality is not an equality or if $\mathbf{w} \neq 0$ on $\partial\Omega$, we find that Condition 3 is satisfied for $L - \lambda_w I - H$. Hence by Theorem 1.1 $\lambda_0 - \lambda_w > 0$. In the case of two equalities $\mathbf{w}$ is a multiple of the first eigenfunction and $\lambda_0 = \lambda_w$. From the last fact we also find:

$$\lambda_0 = \sup \{\lambda_w; \ 0 \ll \mathbf{w} \in (C(\bar{\Omega}) \cap C^2(\bar{\Omega}))^k\}.$$

For $\lambda - \lambda_0 > 0$ but small enough there is an anti-maximum principle as in [12] at least when the boundary $\partial\Omega$ is $C^{1,1}$. The proof is similar to the one of Clément and Peletier. Instead of assuming this regularity, we prefer to assume the consequence of this regularity that is used in the proof, namely a compact imbedding of appropriate Banach spaces. As a result one will still have an anti-maximum principle for a smaller class of right hand sides when the boundary is less smooth.

   Following arguments from [2, 3] we define, using the eigenfunction $\mathbf{u}_0$ from Proposition 3.1, the ordered Banach space (even a Banach lattice)

$$\mathscr{B} = \{\mathbf{u} \in (C_0(\bar{\Omega}))^k; \ |\mathbf{u}(\cdot)| \leq \kappa \, \mathbf{u}_0(\cdot) \ \text{ for some } \kappa > 0\}$$

with norm

$$\|\mathbf{u}\|_w = \sup \left\{ \left| \frac{\mathbf{u}_\nu(x)}{(\mathbf{u}_0)_\nu(x)} \right| ; \ x \in \Omega, \ \nu \in \{1, \ldots, k\} \right\}.$$

Condition 4 There exists a Banach space $\mathscr{B}_1$ with $\mathscr{B} \hookrightarrow \mathscr{B}_1 \hookrightarrow (L^p(\Omega))^k$, such that $T^c : \mathscr{B}_1 \to \mathscr{B}$ is compact.

   If Ω has a $C^{1,1}$-boundary it follows from the construction of T^c and [23, Theorem 9.15] that $T^c \mathbf{u} \in (W^{2,p}(\Omega) \cap C_0(\bar{\Omega}))^k$ for $\mathbf{u} \in (L^p(\Omega))^k$. By the Rellich-Kondrachov Theorem ([1, Theorem 6.2]) $W^{2,p}(\Omega)$ is compactly imbedded in $C^1(\bar{\Omega})$. The strong maximum principle shows that $\mathbf{u}_0 \gg 0$ and $\mathbf{u}_0(x) \geq \alpha \, d(x, \partial\Omega)\mathbf{1}$ (some $\alpha > 0$) for $x \in \Omega$. Then $v \in (C^1(\bar{\Omega}) \cap C_0(\bar{\Omega}))^k$ satisfies $|v| \leq c \, \mathbf{u}_0$. Hence we may take $\mathscr{B}_1 = (L^p(\Omega))^k$.

Theorem 3.2 *Suppose Conditions 1 and 2 are satisfied and let λ_0 be as in Proposition 3.1. Take $0 < \mathbf{f} \in (L^p(\Omega))^k$ and let $\mathbf{u} \in (W^{2,p}_{\mathrm{loc}}(\Omega) \cap C_0(\bar{\Omega}))^k$ be a solution of $(L - \lambda I - H)\mathbf{u} = \mathbf{f}$.*
(i) If $\lambda_0 < \lambda$, then $0 \nleq \mathbf{u}$, that is, for some component ν and some $x \in \Omega$: $\mathbf{u}_\nu(x) < 0$.
(ii) Suppose Condition 4 also holds and that $\mathbf{f} \in \mathscr{B}_1$. Then there is $\delta > 0$, which depends on $\mathbf{f}$, such that, whenever $\lambda_0 < \lambda < \lambda_0 + \delta$, it follows that $\mathbf{u} \ll 0$.

Remark 3.2 If $\lambda = \lambda_0$ and $0 < \mathbf{f}$, then there is no solution in $\mathscr{B}$. If $\mathbf{u} \in \mathscr{B}$ is a solution, then for κ large enough $\mathbf{u} + \kappa \mathbf{u}_0$ is positive and satisfies $(L - \lambda_0 I - H)(\mathbf{u} + \kappa \mathbf{u}_0) = \mathbf{f} > 0$. Hence the conditions of Theorem 1.1 for $L - \lambda_0 I - H$ are satisfied and its first eigenvalue is positive, a contradiction.

Proof. (i) Suppose $\lambda_0 < \lambda$ and $0 \leq \mathbf{u}$ ($\mathbf{u} \neq \mathbf{0}$ since $\mathbf{f} \neq \mathbf{0}$), then Condition 3 is satisfied for $L - \lambda I - H$ and $\lambda < \lambda_0$ by Theorem 1.1, a contradiction.

(ii) The proof of the second part uses the ideas of [12].

(iia) Define the Banach space

$$\mathscr{R} = \mathrm{T}^c(\mathscr{B}_1),$$

with the norm $\|(L - cI - H)\mathbf{u}\|_1$ and set $\mathscr{S} = (L - \lambda_0 I - H)\mathscr{R}$. First we show that

(3.1) $$\mathscr{B}_1 = \mathrm{span}\{\mathbf{u}_0\} \oplus \mathscr{S}.$$

Since $(L - \lambda_0 I - H) = (I - (\lambda_0 - c)\,\mathrm{T}^c)(L - cI - H) \in \mathscr{L}(\mathscr{R}; \mathscr{B}_1)$ is a Fredholm operator of index zero, it is sufficient to show that $\mathbf{u}_0 \notin \mathscr{S}$. Let $T \in \mathscr{L}((C_0(\bar{\Omega}))^k)$ denote the restriction of T^c. T is compact, strictly positive and irreducible. Similar as in step 2 of the proof of Theorem 1.1, the dual operator T' is also compact and strictly positive, and $r(T') = r(T) = (\lambda_0 - c)^{-1} > 0$. By the Krein-Rutman Theorem there exists a strictly positive eigenvector Ψ_0 of T' with $T'\Psi_0 = (\lambda_0 - c)^{-1}\Psi_0$. Hence $\langle \Psi_0, T\mathbf{u}_0 \rangle = (\lambda_0 - c)^{-1}\langle \Psi_0, \mathbf{u}_0 \rangle > 0$.

Suppose $\mathbf{f} \in \mathscr{S}$, that is, there exists $\mathbf{w} \in \mathscr{B}_1$ such that $\mathbf{f} = (L - \lambda_0 I - H)\,\mathrm{T}^c\,\mathbf{w} = \mathbf{w} - (\lambda_0 - c)\,\mathrm{T}^c\,\mathbf{w}$. It follows that

$$\langle \Psi_0, \mathrm{T}^c\,\mathbf{f} \rangle = \langle \Psi_0, \mathrm{T}^c\,\mathbf{w} - (\lambda_0 - c)\,T\,\mathrm{T}^c\,\mathbf{w} \rangle = \langle \Psi_0, \mathrm{T}^c\,\mathbf{w} \rangle - (\lambda_0 - c)\langle \Psi_0, T\,\mathrm{T}^c\,\mathbf{w} \rangle = 0.$$

This shows that $\mathbf{u}_0 \notin \mathscr{S}$. Moreover, we can define continuous projections P_0: $\mathscr{B}_1 \to \mathrm{span}\{\mathbf{u}_0\}$ and $I - P_0 : \mathscr{B}_1 \to \mathscr{S}$ in the following way:

(3.2) $$P_0\,\mathbf{f} = (\lambda_0 - c)\,\frac{\langle \Psi_0, \mathrm{T}^c\,\mathbf{f} \rangle}{\langle \Psi_0, \mathbf{u}_0 \rangle}\,\mathbf{u}_0 \quad \text{for } \mathbf{f} \in \mathscr{B}_1.$$

Clearly $P_0\,\mathbf{f} = \mathbf{0}$ for $\mathbf{f} \in \mathscr{S}$ and $P_0\,\mathbf{u}_0 = \mathbf{u}_0$.

Secondly we show that $\mathrm{T}^c(\mathscr{S}) \subset \mathscr{S}$. Let $\mathbf{u} \in \mathrm{T}^c(\mathscr{S})$. Then it follows that there exists $\mathbf{w} \in \mathscr{B}_1$ with $\mathbf{u} = \mathrm{T}^c(L - \lambda_0 I - H)\,\mathrm{T}^c\,\mathbf{w} = (L - \lambda_0 I - H)\,\mathrm{T}^c\,\mathrm{T}^c\,\mathbf{w} \in \mathscr{S}$. Hence one also has $\mathscr{R} \cap \mathscr{S} = \mathrm{T}^c(\mathscr{S})$ and

(3.3) $$\mathscr{R} = \mathrm{span}\{\mathbf{u}_0\} \oplus \mathrm{T}^c(\mathscr{S}).$$

(iib) The decomposition is invariant under $(L - \lambda I - H)$ for arbitrary λ.

If $\mathbf{w} \in (L - \lambda I - H)\,\mathrm{T}^c(\mathscr{S})$ then $\mathbf{w} = (L - \lambda I - H)\,\mathrm{T}^c(L - \lambda_0 I - H)\,\mathrm{T}^c\,\mathbf{u}$ for some $\mathbf{u} \in \mathscr{B}_1$. Since $(L - \lambda I - H)\,\mathrm{T}^c(L - \lambda_0 I - H)\,\mathrm{T}^c\,\mathbf{u} = (L - \lambda_0 I - H)\,\mathrm{T}^c(L - \lambda I - H)\,\mathrm{T}^c\,\mathbf{u}$, we find that

$$(L - \lambda I - H)\,\mathrm{T}^c(\mathscr{S}) \subset \mathscr{S}.$$

Clearly $(L - \lambda I - H)\mathbf{u}_0 = (\lambda_0 - \lambda)\mathbf{u}_0$.

Let $(L - \lambda I - H)_r$ denote the restriction of $(L - \lambda I - H)$ to $\mathrm{T}^c(\mathscr{S})$. There is $\delta_1 > 0$ such that for $\lambda < \lambda_0 + \delta_1$ the operator $(L - \lambda I - H)_r : \mathrm{T}^c(\mathscr{S}) \to \mathscr{S}$ is an isomorphism and, for $|\lambda - \lambda_0| < \delta_1$, $T_r^\lambda = (L - \lambda I - H)_r^{-1}$ is an analytical function of λ.

For $0 < |\lambda - \lambda_0| < \delta_1$, by setting $\mathbf{f} = P_0\,\mathbf{f} + \mathbf{f}_1$ with $\mathbf{f}_1 = (I - P_0)\mathbf{f} \in \mathscr{S}$, the solution $\mathbf{u}$ can be written as

(3.4) $$\mathbf{u} = \frac{-1}{\lambda - \lambda_0}\,P_0\,\mathbf{f} + T_r^\lambda\,\mathbf{f}_1.$$

(iic) Since $T_r^\lambda = T_r^{\lambda_0} \sum\limits_{n=0}^{\infty} ((\lambda - \lambda_0) T_r^{\lambda_0})^n$ and $T_r^{\lambda_0}: \mathscr{S} \to \mathscr{B}$ is compact, there exists $c_f > 0$ such that $\| T_r^\lambda \mathbf{f}_1 \|_w < c_f$ for all λ with $|\lambda - \lambda_0| < \frac{1}{2} \delta_1$. Hence

$$(3.5) \qquad\qquad -c_f \mathbf{u}_0 \leqq T_r^\lambda \mathbf{f}_1 \leqq c_f \mathbf{u}_0.$$

Let $P_0 \mathbf{f} = \alpha \mathbf{u}_0$. By (3.4) and (3.5) it follows that

$$\mathbf{u} \leqq \left(\frac{-\alpha}{\lambda - \lambda_0} + c_f \right) \mathbf{u}_0.$$

Set $\delta = \min\{\frac{1}{2} \delta_1, c_f^{-1} \alpha\}$. For $\lambda \in (\lambda_0, \lambda_0 + \delta)$ one finds $\mathbf{u} \ll \mathbf{0}$. $\quad\square$

4 When the coupling matrix is not essentially positive; the non cooperative case

One will not find a positivity result for a system with large 'negative coupling' or with small negative coupling and general positive boundary values.

Weinberger in [36] considers invariant sets for elliptic systems which are not necessarily the positive cone. Cosner and P.W. Schaefer in [16] consider restrictions of the cone for the right hand side that allow a decoupling. Results in a different direction are obtained by using the following result from potential theory.

Under sufficient regularity of the boundary and the elliptic operator L, there is $\varepsilon > 0$ such that for all $f > 0$:

$$(4.1) \qquad\qquad (I - \varepsilon (L)_0^{-1})(L)_0^{-1} f \gg 0;$$

$(L)_0^{-1}$ *is the inverse of L with zero Dirichlet boundary condition.*

This result (see (6.8) in the appendix) shows that $u \gg 0$ if $f > 0$ in

$$(4.2) \qquad\qquad Lu = f - \varepsilon v \quad \text{in} \ \ \Omega,$$

$$Lv = f \qquad\quad \text{in} \ \ \Omega,$$

$$u = v = 0 \qquad \text{on} \ \ \partial\Omega.$$

Results for this system can be found in [34, 8, 10].

The estimate is used for right hand side f with some components equal to zero in [11] to obtain a solution with all components positive.

We will consider the non cooperative case as a perturbation of the cooperative system (1.1) and allow all positive right hand sides f. Uniformly with respect to f, we get positivity of some fixed components of the solution.

Consider

$$(4.3) \qquad\qquad L\mathbf{u} = H\mathbf{u} - \varepsilon P\mathbf{u} + \mathbf{f} \quad \text{in} \ \ \Omega$$

$$\mathbf{u} = \mathbf{0} \qquad\qquad\quad \text{on} \ \ \partial\Omega;$$

with

(i) $P = \begin{pmatrix} 0 & \mathrm{P} \\ 0 & 0 \end{pmatrix}$, where P is a constant $m \times (k-m)$ matrix with $P_{ij} \geq 0$;

(ii) H is constant and $L_\nu = L_1$ for all $\nu \in \{1, 2, \dots, k\}$ with $c^1 \geq 0$;

(iii) if $n \geq 3$, the coefficients of L_1 are Hölder continuous and $\partial\Omega \in C^{1,1}$; if $n = 2$, the coefficients of L_1 are Lipschitz continuous and $\partial\Omega \in C^{2,\alpha}$.

Let α respectively β denote the first m, respectively the last $k-m$ components and rewrite (4.3) as:

$$L_\alpha \mathbf{u}_\alpha = H_{\alpha\alpha} \mathbf{u}_\alpha + H_{\alpha\beta} \mathbf{u}_\beta - \varepsilon \, \mathrm{P} \, \mathbf{u}_\beta + \mathbf{f}_\alpha \quad \text{in } \Omega,$$

$$L_\beta \mathbf{u}_\beta = H_{\beta\alpha} \mathbf{u}_\alpha + H_{\beta\beta} \mathbf{u}_\beta + \mathbf{f}_\beta \quad \text{in } \Omega,$$

$$\mathbf{u}_\alpha = 0 \quad \text{on } \partial\Omega,$$

$$\mathbf{u}_\beta = 0 \quad \text{on } \partial\Omega.$$

Set $\mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{w}_\alpha$ such that the system becomes:

(4.4)
$$L_\alpha \mathbf{w}_\alpha = H_{\alpha\alpha} \mathbf{w}_\alpha + \varepsilon \, \mathrm{P} \, \mathbf{u}_\beta \quad \text{in } \Omega,$$

$$L_\alpha \mathbf{v}_\alpha = H_{\alpha\alpha} \mathbf{v}_\alpha + H_{\alpha\beta} \mathbf{u}_\beta + \mathbf{f}_\alpha \quad \text{in } \Omega,$$

$$L_\beta \mathbf{u}_\beta = H_{\beta\alpha} \mathbf{v}_\alpha + H_{\beta\beta} \mathbf{u}_\beta - H_{\beta\alpha} \mathbf{w}_\alpha + \mathbf{f}_\beta \quad \text{in } \Omega,$$

$$\mathbf{w}_\alpha = 0 \quad \text{on } \partial\Omega,$$

$$\mathbf{v}_\alpha = 0 \quad \text{on } \partial\Omega,$$

$$\mathbf{u}_\beta = 0 \quad \text{on } \partial\Omega.$$

To solve for $\mathbf{w}_\alpha$, we have to consider the system

(4.5)
$$L_\alpha \tilde{\mathbf{u}} = H_{\alpha\alpha} \tilde{\mathbf{u}} + \tilde{\mathbf{f}} \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}} = 0 \quad \text{on } \partial\Omega,$$

which is not necessarily fully coupled. However, since $\chi = (L-H)_0^{-1} \mathbf{1} \gg 0$ we have that $(L_\alpha - H_{\alpha\alpha})\chi_\alpha = H_{\alpha\beta}\chi_\beta + \mathbf{1}_\alpha \gg 0$ and Condition 3 is satisfied on every fully coupled component. Use Theorem 1.1 on these subsets and one finds that $(L_\alpha - H_{\alpha\alpha})_0^{-1}$ is well defined and positive. Hence $(L_\alpha - H_{\alpha\alpha})_0^{-1} \mathrm{P}$ is positive and

$$\mathbf{w}_\alpha = \varepsilon (L_\alpha - H_{\alpha\alpha})_0^{-1} \mathrm{P} \, \mathbf{u}_\beta.$$

Set $\mathbf{v} = (\mathbf{v}_\alpha, \mathbf{u}_\beta)$ and define $B : (C_0(\bar\Omega))^k \to (C_0(\bar\Omega))^k$ by

(4.6)
$$B = \begin{pmatrix} 0 & 0 \\ 0 & H_{\beta\alpha}(L_\alpha - H_{\alpha\alpha})_0^{-1} \mathrm{P} \end{pmatrix}.$$

This yields a boundary value problem with the non local term B:

(4.7)
$$L\mathbf{v} = H\mathbf{v} - \varepsilon B\mathbf{v} + \mathbf{f} \quad \text{in } \Omega,$$

$$\mathbf{v} = 0 \quad \text{on } \partial\Omega.$$

The solution **u** is then as follows:

$$(4.8) \qquad \mathbf{u}_\alpha = \mathbf{v}_\alpha - \varepsilon (L_\alpha - H_{\alpha\alpha})^{-1} \, \mathrm{P} \, \mathbf{v}_\beta,$$

$$\mathbf{u}_\beta = \mathbf{v}_\beta.$$

We will show that we do have a uniform positivity result for **v**, and hence for $\mathbf{u}_\beta$. In general such a result can not be found for the other components of **u**.

From (4.7) we find that $\mathbf{v} = -\varepsilon(L-H)_0^{-1} B \mathbf{v} + (L-H)_0^{-1} \mathbf{f}$ and hence, since $(L-H)_0^{-1}$ and B are bounded operators in $(C_0(\bar{\Omega}))^k$, for ε small enough:

$$\mathbf{v} = \sum_{v=0}^{\infty} (-\varepsilon(L-H)_0^{-1} B)^v (L-H)_0^{-1} \mathbf{f}$$

is well defined. Moreover:

$$(4.9) \qquad \mathbf{v} = \sum_{v=0}^{\infty} (\varepsilon(L-H)_0^{-1} B)^{2v} (I - \varepsilon(L-H)_0^{-1} B)(L-H)_0^{-1} \mathbf{f}.$$

Since $(L-H)_0^{-1} B$ is a positive operator, it is sufficient to establish positivity of the operator $K_\varepsilon : (L_p(\Omega))^k \to (C_0(\bar{\Omega}))^k$ defined by:

$$(4.10) \qquad K_\varepsilon = (I - \varepsilon(L-H)_0^{-1} B)(L-H)_0^{-1}.$$

For $(L-H)_0^{-1}$ we can use the expression from Lemma 1.4 and with setting $\beta = 1$ we get $(L-H)_0^{-1} = \sum_{v=0}^{\infty} ((L \pm I)_0^{-1} (H+I))^v (L+I)_0^{-1}$.

Lemma 4.1 *Let $p \geqq n$. Assume the Conditions 1, 2 and 3 are satisfied for (1.1). Also let* (i) (ii) *and* (iii) *be true. Then there is a constant $c > 0$ such that for all $\mathbf{u} \in (L_p(\Omega))^k$ with $\mathbf{u} \geqq 0$:*

$$(4.11) \qquad c^{-1} \sum_{v=0}^{k-1} \mathcal{F}^{v+1} (H+I)^v \mathbf{u} \leqq (L-H)_0^{-1} \mathbf{u} \leqq c \sum_{v=0}^{k-1} \mathcal{F}^{v+1} (H+I)^v \mathbf{u}.$$

Moreover, if α is a subset of $\{1, 2, \ldots, k\}$, then there is a constant $c > 0$ such that for all $\mathbf{u}_\alpha \in (L_p(\Omega))^\alpha$ with $\mathbf{u}_\alpha \geqq 0$:

$$(4.12) \quad c^{-1} \sum_{v=0}^{k-1} \mathcal{F}^{v+1} (H_{\alpha\alpha}+I)^v \mathbf{u}_\alpha \leqq (L_\alpha - H_{\alpha\alpha})_0^{-1} \mathbf{u}_\alpha \leqq c \sum_{v=0}^{k-1} \mathcal{F}^{v+1} (H_{\alpha\alpha}+I)^v \mathbf{u}_\alpha.$$

Here $\mathcal{F}$ denotes the operator defined by $(\mathcal{F} \mathbf{u})(x) = \int_\Omega F_n(x, y) \mathbf{u}(y) \, dy$, F_n is defined in (6.3) and (6.4) of the appendix.

Proof. The left hand side inequality of (4.11) is a direct consequence of

$$(L-H)^{-1} = \sum_{v=0}^{\infty} ((L+I)^{-1}(H+I))^v (L+I)^{-1} \geqq \sum_{v=0}^{k-1} ((L+I)^{-1}(H+I))^v (L+I)^{-1},$$

and Theorem 6.1 and the remark thereafter. For $(L_\alpha - H_{\alpha\alpha})_0^{-1}$ the analogue holds. Proof of the right hand side:

(i) Decomposition

Because of assumption (ii) we can write

$$(L-H)^{-1} = \sum_{v=0}^{\infty} (L_1+1)^{-(v+1)}(H+I)^v.$$

Let $\varphi \in C_0(\bar{\Omega})$ be the first eigenfunction of L_1 with eigenvalue λ_0. Since $H+I$ is irreducible, $(H+I)^{k-1}$ contains just positive elements and the finite dimensional version of the Krein-Rutman Theorem gives an eigenvector $\zeta \in \mathbb{R}^k$, with $\zeta_i > 0$ for all i and with eigenvalue $\rho(H+I)$. Set $\mu_0 = \rho(H+I) - 1$. Then:

$$(4.13) \qquad (L-H)^{-1} \varphi\zeta = \sum_{v=0}^{\infty} (\lambda_0+1)^{-(v+1)}(\mu_0+1)^v \varphi\zeta = \frac{1}{\lambda_0-\mu_0}\varphi\zeta.$$

From Condition 3 it follows that $\lambda_0 > \mu_0$.

Since $\zeta \gg 0$ there are positive numbers c_1 and c_2 such that for every $j \in \{1, 2, \ldots, k\}$, with $\mathbf{e}_j$ the j-th unit vector:

$$(4.14) \qquad\qquad c_1 (H+I)^{k-1} \mathbf{e}_j \leqq \zeta \leqq c_2 (H+I)^{k-1} \mathbf{e}_j.$$

For the second part let $\tilde{\mathbf{e}}_j$ be the unit vector in $\mathbb{R}^\alpha$ and ζ_α the restriction of ζ to $\mathbb{R}^\alpha$. Let $i \in \alpha(j)$ if $((H_{\alpha\alpha}+I)^{k-1}\tilde{\mathbf{e}}_j)_i \neq 0$ $(j \in \alpha(j) \subset \alpha)$, and let $\pi_{\alpha(j)}$ denote the projection on the $\alpha(j)$ components. Then, for some $c_1, c_2 > 0$:

$$(4.15) \qquad\qquad c_1 (H_{\alpha\alpha}+I)^{k-1} \tilde{\mathbf{e}}_j \leqq \pi_{\alpha(j)}\zeta_\alpha \leqq c_2 (H_{\alpha\alpha}+I)^{k-1} \tilde{\mathbf{e}}_j,$$

and

$$(4.16) \qquad (H_{\alpha\alpha}+I)\pi_{\alpha(j)}\zeta_\alpha = \pi_{\alpha(j)}(H_{\alpha\alpha}+I)\pi_{\alpha(j)}\zeta_\alpha$$
$$\leqq \pi_{\alpha(j)}(H_{\alpha\alpha}+I)\zeta_\alpha \leqq \pi_{\alpha(j)}(\mu_0+1)\zeta_\alpha.$$

(ii) Componentwise

Applying $(L-H)^{-1}$ to $u_j \mathbf{e}_j$, with $u_j \geqq 0$, one gets:

$$(L-H)^{-1} u_j \mathbf{e}_j = \sum_{v=0}^{\infty} ((L+I)^{-1}(H+I))^v (L+I)^{-1} u_j \mathbf{e}_j$$
$$= \sum_{v=0}^{k-2} (L_1+1)^{-(v+1)} u_j (H+I)^v \mathbf{e}_j$$
$$+ (L_1+1)^{1-k} \sum_{v=0}^{\infty} (L_1+1)^{-(v+1)} u_j (H+I)^v (H+I)^{k-1} \mathbf{e}_j.$$

Since all the involved operators are positive, (4.14) shows:

$$(4.17) \qquad c_1(L_1+1)^{1-k} \sum_{v=0}^{\infty} (L_1+1)^{-(v+1)} u_j(H+I)^v(H+I)^{k-1} \mathbf{e}_j$$

$$\leqq (L_1+1)^{1-k} \sum_{v=0}^{\infty} (L_1+1)^{-(v-1)} u_j(H+I)^v \zeta$$

$$\leqq (L_1+1)^{1-k} \sum_{v=0}^{\infty} (L_1+1)^{-(v+1)}(\mu_0+1)^v u_j \zeta$$

$$= (L_1+1)^{1-k}(L_1-\mu_0)^{-1} u_j \zeta$$

$$\leqq c_2(L_1+1)^{1-k}(L_1-\mu_0)^{-1}(H+I)^{k-1} u_j \mathbf{e}_j.$$

Hence there is $c>0$ such that

$$(4.18) \quad (L-H)^{-1} u_j \mathbf{e}_j$$

$$\leqq c\left(\sum_{v=0}^{k-2}(L_1+1)^{-(v+1)}(H+I)^v+(L_1+1)^{1-k}(L_1-\mu_0)^{-1}(H+I)^{k-1}\right)u_j \mathbf{e}_j.$$

Using (4.15) and (4.16) in (4.17), with $\pi_{\alpha(j)}\zeta_\alpha$ instead of ζ and $H_{\alpha\alpha}$ instead of H, we get the analogous inequality for $(L_\alpha - H_{\alpha\alpha})^{-1} u_j \tilde{\mathbf{e}}_j$. Since this is true for every component, we can replace $u_j \mathbf{e}_j$ by $\mathbf{u}$ in (4.18), respectively $u_j \tilde{\mathbf{e}}_j$ by $\mathbf{u}_\alpha$.

(iii) Estimates for the Green functions

Using the result stated in Theorem 6.1 and the remark thereafter there is $c>0$ such that

$$\sum_{v=0}^{k-2}(L_1+1)^{-(v+1)}(H+I)^v \mathbf{u}+(L_1-\mu_0)^{-1}(L_1+1)^{1-k}(H+I)^{k-1} \mathbf{u}$$

$$\leqq c \sum_{v=0}^{k-1} \mathscr{F}^{v+1}(H+I)^v \mathbf{u}.$$

With (4.18) it shows the right hand side of (4.11). Similarly we obtain (4.12). $\square$

By using Lemma 4.1 the sufficient condition for positivity will be merely algebraic. Define the diagonal matrix E_α by:

$$(E_\alpha)_{ii} = \begin{cases} 1 & \text{if } i \in \alpha, \\ 0 & \text{if } i \notin \alpha. \end{cases}$$

Theorem 4.2 *Assume the Conditions* 1, 2 *and* 3 *are satisfied for* (1.1). *Also let* (i), (ii) *and* (iii) *be satisfied. Suppose that for some* $c > 0$:

$$(4.19) \qquad E_\beta (HE_\alpha)^\nu P \leqq c \sum_{s=0}^{\nu+1} H^s \quad for \ \nu \in \{1, 2, ..., k-3\}.$$

Then there is $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in [0, \varepsilon_0)$ *and* $\mathbf{f} \in (L_p(\Omega))^k$ *with* $\mathbf{f} \geqq \mathbf{0}$, *the solution* $\mathbf{v}$ *of* (4.7) *satisfies* $\mathbf{v} \gg \mathbf{0}$ *or* $\mathbf{v} = \mathbf{0} = \mathbf{f}$. *Hence respectively* $\mathbf{u}_\beta \gg \mathbf{0}$ *or* $\mathbf{u} = \mathbf{0} = \mathbf{f}$.

Remark 4.1 For 3×3 systems condition (4.19) is void.

Remark 4.2 Although for every $\mathbf{f} \geqq \mathbf{0}$ with $f_i > 0$ for $i \in \alpha$, there is $\varepsilon_f > 0$ such that for every $\varepsilon \in [0, \varepsilon_f)$ the solution $\mathbf{u}$ satisfies $\mathbf{u} \gg \mathbf{0}$, one cannot expect a uniform result for $\mathbf{u}$ as Theorem 4.2 states for $\mathbf{v}$. Hence, the result above cannot be used for the parabolic case. By discretizing the time variable and solving the elliptic problem for every time step one looses positivity of the inhomogeneous term after the first step.

Proof. We will show that the operator K_ε, defined in (4.10), is positive for ε positive but small enough. By Lemma 4.1 we have appropriate estimates for $(L-H)_0^{-1}$.

In order to show that $\varepsilon (L-H)_0^{-1} B(L-H)_0^{-1} \leqq (L-H)_0^{-1}$ for some $\varepsilon > 0$ it is sufficient, by Lemma 4.1, that for some $c > 0$

$$(4.20) \qquad \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \sum_{\tau=0}^{k-1} \mathscr{F}^{\mu+\nu+\tau+3} (H+I)^\mu E_\beta HE_\alpha ((H+I)E_\alpha)^\nu P(H+I)^\tau$$

$$\leqq c \sum_{\nu=0}^{k-1} \mathscr{F}^{\nu+1} (H+I)^\nu.$$

Notice that $(H+I)^{k-1}$ has strictly positive entries. Since we know from Corollary 6.2 of the appendix that there is $c > 0$ such that $\mathscr{F}^2 \leqq c\mathscr{F}$ (and for any $c \in \mathbb{R}: \mathscr{F} \nleqq c\mathscr{F}^2$) we may show just as well that there is $c > 0$ with:

$$\sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \sum_{\tau=0}^{k-1} \mathscr{F}^{\mu+\nu+\tau+3} H^\mu E_\beta HE_\alpha (HE_\alpha)^\nu PH^\tau \leqq c \sum_{\nu=0}^{k-1} \mathscr{F}^{\nu+1} (H+I)^\nu.$$

This last inequality is true if and only if there is $c > 0$ with

$$\mathscr{F}^{\nu+3} E_\beta (HE_\alpha)^{\nu+1} P \leqq c \sum_{\mu=0}^{k-1} \mathscr{F}^{\mu+1} (H+I)^\mu \quad for \ all \ \nu \in \{0, 1, ..., k-1\}.$$

Again since $\mathscr{F}^2 \leqq c\mathscr{F}$ and $\mathscr{F} \nleqq c\mathscr{F}^2$ it is sufficient that:

$$\mathscr{F}^{\nu+3} E_\beta (HE_\alpha)^{\nu+1} P \leqq c \sum_{\mu=0}^{\nu+2} \mathscr{F}^{\mu+1} (H+I)^\mu \quad for \ all \ \nu \in \{0, 1, ..., k-1\}$$

or equivalently, there is $c > 0$ such that:

$$E_\beta (HE_\alpha)^{v+1} P \leq c \sum_{\mu=0}^{v+2} H^\mu \quad \text{for all} \quad v \in \{0, 1, \ldots, k-1\}.$$

Finally, since H is irreducible we have that the matrix $I + H + \ldots + H^{k-1}$ contains only strictly positive elements. Which means that it suffices to check the last inequality for $v \in \{0, 1, \ldots, k-4\}$. $\square$

We will end this section with an example.

Let Ω be a bounded, sufficiently smooth domain in $\mathbb{R}^n$. Let $\mathbf{u}$ be a solution of

(4.21)

$$(-\Delta + 1)\mathbf{u}_1 = \mathbf{u}_2 - \varepsilon \mathbf{u}_4 + \mathbf{f}_1 \quad \text{in} \quad \Omega,$$

$$(-\Delta + 1)\mathbf{u}_2 = \mathbf{u}_3 - \varepsilon \mathbf{u}_4 + \mathbf{f}_2 \quad \text{in} \quad \Omega,$$

$$(-\Delta + 1)\mathbf{u}_3 = \mathbf{u}_4 + \mathbf{f}_3 \quad \text{in} \quad \Omega,$$

$$(-\Delta + 1)\mathbf{u}_4 = \mathbf{u}_1 + \mathbf{f}_4 \quad \text{in} \quad \Omega,$$

$$\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_3 = \mathbf{u}_4 = 0 \quad \text{on} \quad \partial\Omega.$$

The conditions of Theorem 4.2 are satisfied for this system. Hence there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0)$ the following holds. For $\mathbf{f} > 0$ the solution $\mathbf{u}$ satisfies $\mathbf{u}_3 \gg 0$ and $\mathbf{u}_4 \gg 0$.

5 Appendix 1

In this paper we used the following results on a Banach lattice E with $\dim(E) > 1$.

Krein-Rutman Theorem [27] *Let $T \in \mathscr{L}(E)$ be a compact and positive operator with $r = r(T) > 0$. Then there is $0 < u \in E$ with $Tu = ru$.*

De Pagter Theorem [29] *Let $T \in \mathscr{L}(E)$ be a positive, irreducible and compact operator. Then $r(T) > 0$.*

Corollary (of [33, Theorem V.5.2]) *Let $T \in \mathscr{L}(E)$ be a positive and irreducible operator with $r = r(T) > 0$ as an eigenvalue and $T'\phi = r\phi$ for some $0 < \phi \in E'$. Then:*
(i) r is the unique eigenvalue of T with a positive eigenvector;
(ii) $\{u \in E; Tu = ru\}$ has dimension one.

6 Appendix 2

In Sect. 4 we used pointwise estimates for Green functions. With sufficient regularity of the coefficients and of the boundary of the domain all Green functions for second order elliptic operators (zero Dirichlet boundary conditions) have a similar behaviour. Results of this type have been obtained for $n > 2$ by Hueber and Sieveking [25, 26] and independently by Zhao [37] and Cranston et al. [18]. Results for $n \geq 2$ are published by Ancona in [4].

Theorem 6.1 (Hueber-Sieveking) *Let L be a strictly elliptic operator on $\mathbb{R}^n$ with $n > 2$:*

(6.1)
$$L = -\sum_{ij}^{n} a_{ij}(\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i}^{n} b_i(\cdot)\frac{\partial}{\partial x_i} + c(\cdot)$$

with Hölder-continuous coefficients and $c \geq 0$. Let Ω be a bounded $C^{1,1}$-domain and let G denote the Green function (zero Dirichlet boundary condition). Then there is $\alpha > 0$ such that:

(6.2)
$$\alpha^{-1} F_n(x, y) \leq G(x, y) \leq \alpha F_n(x, y) \quad \text{for } x, y \in \Omega,$$

with

(6.3)
$$F_n(x, y) = |x - y|^{-n} \min(d(x, \partial\Omega)\, d(y, \partial\Omega), |x - y|^2).$$

In Theorem 8 of [4] Ancona states a similar estimate. Let L_1 and L_2 denote second order strictly elliptic operators as in (6.1). He assumes that L_1 and L_2 have Lipschitz continuous coefficients and that the domain (in $\mathbb{R}^n$ with $n \geq 2$) has a $C^{2,\alpha}$ boundary except at a closed set Φ. At Φ the boundary is Lipschitz and satisfies some technical conditions. If L_1 and L_2 have the same principal part on Φ there is $c > 0$ such that the corresponding Green functions satisfy

$$c^{-1} \leq \frac{G_1(x, y)}{G_2(x, y)} \leq c \quad \text{for all } x, y \in \Omega.$$

By using a result of Zhao in [38] we obtain (6.2) for $n = 2$ with

(6.4)
$$F_2(x, y) = \log(1 + d(x, \partial\Omega)\, d(y, \partial\Omega)\, |x - y|^{-2}).$$

As a consequence of Theorem 6.1 one gets:

Corollary 6.2 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ with $n \geq 3$. Let L_1 and L_2 be two elliptic operators satisfying the conditions of Theorem 6.1. Let G_1 and G_2 denote the Green functions. Then there is $\beta > 0$, depending on Ω, such that:*

(6.5)
$$\frac{G_1(x, z)\, G_2(z, y)}{G_1(x, y)} \leq \beta(|x - z|^{2-n} + |y - z|^{2-n}) \quad \text{for } x, y, z \in \Omega.$$

Remark 6.1 Cranston et al. [18] showed a theorem related to Corollary 6.2 (which they call the $3G$ Theorem) to obtain the estimates for the Green functions.

Zhao showed in [38] a similar estimate for $n = 2$ and $L = -\Delta$. Using the result of Ancona the equivalent of (6.5) for $n = 2$ is:

(6.6)
$$\frac{G_1(x, z)\, G_2(z, y)}{G_1(x, y)} \leq c(\max(-\log|x - z|, 1) + \max(-\log|y - z|, 1)).$$

It follows from (6.5) respectively (6.6) that there is $M \in \mathbb{R}$ such that:

(6.7)
$$\int_{\Omega} \frac{G_1(x, z)\, G_2(z, y)}{G_1(x, y)}\, dz \leq M \quad \text{for all } x, y \in \Omega.$$

Then for $\varepsilon < M^{-1}$ and $0 \neq f \geq 0$ it follows that:

(6.8)
$$((I - \varepsilon(L_2)_0^{-1})(L_1)_0^{-1}f)(x)$$
$$= \int_\Omega G_1(x, y) f(y) \, dy - \varepsilon \int_\Omega \int_\Omega G_2(x, z) \, G_1(z, y) f(y) \, dy \, dz$$
$$= \int_\Omega G_1(x, y) \left(1 - \varepsilon \int_\Omega \frac{G_1(x, z) \, G_2(z, y)}{G_1(x, y)} \, dz \right) f(y) \, dy$$
$$\geq \int_\Omega G_1(x, y)(1 - \varepsilon M) f(y) \, dy > 0 \quad \text{for } x \in \Omega.$$

Remark 6.2 If $L_1 = L_2 = -\Delta$ then:

(6.9)
$$\int_\Omega \frac{G(x, z) \, G(z, y)}{G(x, y)} \, dz = E_y^x \tau_\Omega.$$

E_y^x is the expectation for Brownian motion killed outside Ω, starting in x and conditioned to converge to y. The path lifetime is τ_Ω. See e.g. [20].

Remark 6.3 Without using the relation with conditioned Brownian motion or potential theory, bounds for (6.9) have been considered in [34, 8, 10]. In the one dimensional, respectively the radially symmetric case on the ball in $\mathbb{R}^n$ one finds the following expression for the smallest bound M.

(6.10)
$$M = \sum_{n=1}^\infty (\lambda_n)^{-1},$$

where $\{\lambda_n\}$ is the set of all eigenvalues of

$$-\Delta \phi = \lambda \phi \quad \text{in } \Omega$$
$$\phi = 0 \quad \text{on } \partial \Omega.$$

See [34] and [10].

The series in (6.10) only converge in a basically one dimensional domain.

Remark 6.4 For $x \neq y$ and $x, y \notin \partial \Omega$ one finds immediately that

$$\int_\Omega \frac{G_1(x, z) \, G_2(z, y)}{G_1(x, y)} \, dz > 0.$$

However, there is no uniform positivity result for $-u$ when ε is large, where u is the expression in (6.8). Since one can show that

$$\lim_{x \to y} \int_\Omega \frac{G_1(x, z) \, G_2(z, y)}{G_1(x, y)} \, dz = 0,$$

there is for every $\varepsilon > 0$ a function $f \in C(\bar{\Omega})$, with $f \geq 0$, and $x \in \Omega$ such that $u(x) > 0$.

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