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Invariant metrics and peak functions on pseudoconvex domains of homogeneous finite diagonal type

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Introduction

For a certain class of pseudoconvex domains in $\mathbb{C}^n$ we want to study the boundary behavior of the invariant metrics of Caratheodory and Kobayashi. Let us first recall some definitions: For a domain $D \subset \mathbb{C}^n$ the *Caratheodory* pseudometric is defined as

$$\mathrm{Cara}_D(z, X) = \sup\{|(\partial f(z), X)| \mid f: D \to \Delta_1(0, 1), f \text{ holomorphic}\},$$

and the *Kobayashi* pseudometric is, by definition,

$$\mathrm{Kob}_D(z, X) = \inf\{R > 0 \mid \text{There exists a holomorphic map } f: \Delta_1(0, R^{-1}) \to D \text{ such that } f(0) = z, f'(0) = X\}$$

for $(z, X) \in D \times \mathbb{C}^n$.

Both pseudometrics are distance decreasing under holomorphic mappings. Therefore they can be used in the study of the boundary behavior of proper holomorphic mappings. [B-F1, H, D-F1].

Since the paper of Graham [Gr], the boundary behavior of the pseudometrics of Caratheodory and Kobayashi on bounded strictly pseudoconvex domains is well understood, (see also [H]). In the weakly pseudoconvex case Catlin obtained precise estimates for the growth of Cara_D and Kob_D, when D is a pseudoconvex domain in $\mathbb{C}^2$ of finite type [Ca].

Except for some very special cases the precise boundary behavior of these invariant metrics on a weakly pseudoconvex domain in $\mathbb{C}^n$, $n \geqq 3$, is not known. In this paper we want to generalize Catlin's result to the class of pseudoconvex domains which are of homogeneous finite diagonal type; by this we mean the following:

Definition. Let P be a real-valued plurisubharmonic polynomial in $\mathbb{C}^{n-1}$ without pluriharmonic terms (here $n \geqq 2$). Then the domain

$$\Omega = \{r = \mathrm{Re}\, z_1 + P(z') < 0\}$$

is called of *homogeneous finite diagonal type*, if

(0.1) The polynomial P is *weighted homogeneous*, i.e. there exist positive integers $m_2, \ldots, m_n$, such that for all $t>0$:

$$P(t^{1/2m_2} z_2, \ldots, t^{1/2m_n} z_n) = t\, P(z')$$

(*where* $z'=(z_2, \ldots, z_n)$).

(0.2) For a small positive number s the polynomial $P(z') - 2s \sum_{j=2}^{n} |z_j|^{2m_j}$ is also plurisubharmonic in $\mathbb{C}^{n-1}$.

Observe that no holomorphic support function needs to exist in such a domain as was first shown by Kohn and Nirenberg, [K-N], in a domain in $\mathbb{C}^2$ which has properties (0.1) and (0.2).

We are going to study the boundary behavior of the invariant pseudometrics Cara_Ω and Kob_Ω for domains of homogeneous finite diagonal type only under the additional assumption that at most two of the variables $z_2, \ldots, z_n$ appear at a time in the Taylor terms of P.

In order to be able to state our results we need to introduce the following auxiliary functions: Let Ω be a domain of homogeneous finite diagonal type. For $2 \leqq j \leqq n$, $2 \leqq l \leqq 2m_j$ let

$$A_{lj}(z) = \max\left\{ \left| \frac{\partial^l r(z)}{\partial z_j^\nu \, \partial \bar{z}_j^\mu} \right| \;\middle|\; \nu, \mu \geqq 1,\ \nu + \mu = l \right\} \tag{0.3}$$

and

$$\mathscr{C}_j(z) = \sum_{l=2}^{2m_j} \left(\frac{A_{lj}(z)}{-r(z)} \right)^{\frac{1}{l}} \tag{0.4}$$

for $z \in \Omega$. On $\Omega \times \mathbb{C}^n$ we define the metric

$$M_\Omega(z, X) = \frac{|(\partial r(z), X)|}{|r(z)|} + \sum_{j=2}^{n} \mathscr{C}_j(z)\, |X_j|. \tag{0.5}$$

In Theorem 1 we compare Cara_Ω, Kob_Ω, and M_Ω.

Theorem 1 *Suppose* $\Omega = \{r = \operatorname{Re} z_1 + P(z') < 0\}$ *is a domain of homogeneous finite diagonal type and* P *is of the form*

$$P(z') = \sum_{j=2}^{n} P_j(z_j) + \sum_{j<k} P_{jk}(z_j, z_k), \tag{0.6}$$

where P_j *and* P_{jk} *are real-valued polynomials and, for* $j<k$, P_{jk} *has the form*

$$P_{jk}(z_j, z_k) = \sum_{\nu, \mu, \kappa, \lambda} c_{\nu\mu\kappa\lambda}\, z_j^\nu \bar{z}_j^\mu z_k^\kappa \bar{z}_k^\lambda,$$

where the sum is extended over finitely many indices ν, μ, κ, λ *with* $\nu+\mu\geqq 1$, $\kappa+\lambda\geqq 1$. *Then, with a universal constant* $C_1>0$, *we have*

$$(0.7)\qquad 1/C_1\, M_\Omega(z,X)\leqq \mathrm{Cara}_\Omega(z,X)\leqq \mathrm{Kob}_\Omega(z,X)\leqq C_1\, M_\Omega(z,X)$$

for $(z,X)\in\Omega\times\mathbb{C}^n$.

We recall that, if $D\subset\mathbb{C}^n$ is a domain such that the Bergman kernel function on the diagonal is positive everywhere on D, then $\log K_D(z,\bar z)$ is the potential of the Bergman metric B_D^2 on $D\times\mathbb{C}^n$. It is well-known, [Ha], that $B_D^2\geqq(\mathrm{Cara}_D)^2$. On the other hand, the functional $b_D^2=K_D B_D^2$, viewed as a domain functional, is increasing, when D decreases, i.e.

(0.8) *If* $D'\subset D$ *then* $b_{D'}^2\,|D'\times\mathbb{C}^n\geqq b_D^2|\,D'\times\mathbb{C}^n$.

In [He2] the author shows:

Theorem. *If* Ω *and* P *are as in Theorem* 1, *then, with a certain constant* $C>0$, *we have*

$$(0.9)\qquad \frac{1}{C}\leqq K_\Omega(z,\bar z)\,|r(z)|^2\Big/\prod_{j=2}^{n}\mathscr{C}_j(z)^2\leqq C.$$

From (0.8) and (0.9) we can then easily deduce that on the class of domains as in Theorem 1 the metrics of Caratheodory, Kobayashi, and Bergman have equivalent growth at the boundary. (In case that all the P_{jk} are zero the estimates (0.7) and (0.9) are contained in McNeal [M1, Theorems 1 and 2].) Notice that the equivalence of these metrics cannot be expected to hold for general pseudoconvex domains. For counterexamples see [D-F2] and [D-F-H].

A qualitative estimate for the Kobayashi metric on uniformly extendable bounded pseudoconvex domains was deduced in [D-F1] and [D-L], and for the Bergman metric on pseudoconvex domains with a subelliptic $\bar\partial$-Neumann problem in [D-F-H]. McNeal proved in [M2], that the Bergman metric grows in each direction at least as $\mathrm{dist}(\cdot, b\Omega)^{-2\varepsilon+\delta}$ (with arbitrarily small $\delta>0$) in a neighborhood of a point $q\in b\Omega$, such that a subelliptic estimate of order ε holds near q for the $\bar\partial$-Neumann problem on (0, 1)-forms. For the Caratheodory metric of certain two-dimensional scaling invariant non-pseudoconvex domains, and of real-analytically bounded pseudoconvex domains in $\mathbb{C}^2$, see [B-F1]. Range estimated in [Ra2] the Caratheodory metric on a real-analytic bounded convex domain in $\mathbb{C}^n$.

It is well-known that the problem of estimating the Caratheodory metric is related to the problem of constructing peak functions which is also still open in the weakly pseudoconvex case with $n\geqq 3$. The method of proving Theorem 1 also gives an answer to the peak function problem for the class of domains considered in Theorem 1.

Theorem 2 *If* P *is a polynomial as in Theorem* 1, *then each boundary point* $q\in b\Omega$ *is a peak point for the algebra* $A^0(\Omega)$ *of functions which are holomorphic on* Ω *and continuous on* $\bar\Omega$.

A peak function is already known to exist at 0, if P satisfies conditions (0.1) and (0.2) with equal weights $m_2,\ldots,m_n$, see [B-F2]. The case $n=3$ is also contained in [B-F2] (see Theorem 3.8). Recently Noell [N] showed that 0 is a peak point in case that P satisfies (0.1) with equal weights $m_2, \ldots, m_n$, and is not harmonic on any complex line passing through the origin in $\mathbb{C}^{n-1}$.

An immediate consequence of Theorem 2 is:

Corollary. *A domain of homogeneous finite diagonal type as in Theorem 2 is complete with respect to the distance functions of Caratheodory, Kobayashi, and Bergman.*

The method of proving Theorems 1 and 2 is based on the construction of certain analytic polyhedra D_M associated with positive numbers M. They generalize the polyhedra introduced by Nagel et al. in [N-S-W]. The geometric key lemma is a local bumping which generalizes the bumping lemma of Fornaess and Sibony [F-S]. The analytic part of the proof consists in a construction of holomorphic auxiliary functions which have large derivatives at a prescribed point while the L^∞-norm over Ω is bounded independently of the boundary distance of that point. This in principle gives the left-hand side of (0.7). The middle inequality is known. The last one follows easily from the Schwarz-Pick lemma. The same sort of holomorphic auxiliary functions as constructed for the estimation of the Caratheodory metric will, together with an argument due to Bishop [Bi], give us a local peak function at a given point of the boundary. It can be globalized in an elementary way (Lemma 5 in Sect. 2).

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Notational convention. In $\mathbb{C}^d$ we will denote by $B_d(a, R)$ (resp. $\Delta_d(a, R)$) the *ball* (resp. the *polydisc*) around the point a with radius R.

1 A local pseudoconvex supporting domain at a boundary point

Let $\Omega = \{r = \operatorname{Re} z_1 + P(z') < 0\}$ be a pseudoconvex domain as in Theorem 1 and 2. We choose a boundary point $q \in b\Omega$ of the form $q = (-P(q'), q')$, *where* $q' = (q_2, \ldots, q_n)$, $|q'| < 1$. If we expand P in a Taylor series at q' we will obtain

$$(1.0) \qquad P(z') = P(q') + \operatorname{Re} h(q', z'-q') + \sum_{j=2}^{n} \hat{P}_j(q', z'-q') + \sum_{j<k} \hat{P}_{jk}(q_j, q_k; z_j - q_j, z_k - q_k)$$

with real-valued polynomials $\hat{P}_j(q', \cdot)$ in $\mathbb{C}$, and some (real-valued) polynomials $\hat{P}_{jk}$ such that $\hat{P}_{jk} = O(|z_j - q_j|\,|z_k - q_k|)$, for $2 \leqq j < k \leqq n$. Furthermore,

$$h(q', w') = 2 \sum_{\alpha \in S} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} P}{\partial z'^{\alpha}}(q') w'^{\alpha}$$

where the sum is extended over $S = \left\{ \alpha = (\alpha_2, \ldots, \alpha_n) \,\middle|\, \sum_{j=2}^{n} \alpha_j / 2m_j \leqq 1 \right\}$.

Let

$$(1.1)\qquad \tilde{r}_q(v)=\operatorname{Re} v_1+\sum_{j=2}^{n}\hat{P}_j(q',v_j)+\sum_{j<k}\hat{P}_{jk}(q_j,q_k;v_j,v_k)$$

for $v\in\mathbb{C}^n$, *and*

$$F(q,z)=(z_1+P(q')+h(q',z'-q'),\,z'-q'),$$

for $z\in\mathbb{C}^n$. Then Ω is mapped under $F(q,\cdot)$ biholomorphically onto the domain

$$(1.2)\qquad \tilde{\Omega}_q=\{v\in\mathbb{C}^n\,|\,\tilde{r}_q(v)<0\}.$$

Here $F(q,q)=0$. In the sequel all work will be done on $\tilde{\Omega}_q$. We will need a slightly generalized version of the bumping lemma in [F-S, Sect. 3].

Lemma 1 *There exist positive numbers* r_0, A, B, *and* C, *and continuous functions* $S_2(q',\cdot),\ldots,S_n(q',\cdot)$ *on the complex plane* $\mathbb{C}$, *such that for each* $j\in\{2,\ldots,n\}$ *the following all hold:*

(I) *For any* $u,w\in\mathbb{C}$, *and* $R\in(0,1)$, *such that* $|w|\leqq(1+|u|/R)^{-2m_j}R$ *one has*

$$S_j(q',u+w)\leqq S_j(q',u)+C\sum_{j=2}^{n}\|\hat{P}_{j,l}(q',\cdot)\|\;R^l$$

(II) *The function* $S_j(q',\cdot)$ *is subharmonic on* $\Delta_1(0,r_0)$.

(III) *For* $w\in\Delta_1(0,r_0)$ *we have the inequality*

$$(1.3)\qquad -B\sum_{l=2}^{2m_j}\|\hat{P}_{j,l}(q',\cdot)\|\,|w|^l\leqq S_j(q',w)-\sum_{l=2}^{2m_j}\hat{P}_{j,l}(q',w)$$

$$\leqq -A\sum_{l=2}^{2m_j}\|\hat{P}_{j,l}(q',\cdot)\|\,|w|^l.$$

Here $\hat{P}_{j,l}(q',\cdot)$ *is the homogeneous part of degree* l *appearing in* $\hat{P}_j$, *and* $\|\hat{P}_{j,l}(q',\cdot)\|$ *is the maximum of the moduli of the coefficients of* $\hat{P}_{j,l}(q',\cdot)$.

Proof. Let us argue for $j=2$. We abbreviate by $\mathscr{E}(i)$, for an even integer i, the following statement.

$\mathscr{E}(i)$: There exist positive numbers r_i, A_i, B_i, and C_i with the following property: If $q'\in\mathbb{C}^{n-1}$, $|q'|<1$, and $r_*<r_i$ is a positive number such that for a suitable $\nu\in\{2,\ldots,i\}$ we have the estimate

$$LE(\nu,i)\quad \|\hat{P}_{2,\nu}(q',\cdot)\|\,r_*^\nu\geqq C_i\max_{l\neq\nu}\|\hat{P}_{2,\nu}(q',\cdot)\|\,r_*^l$$

then there exists a continuous function $S_2^{(\nu)}(q',\cdot)$ on $\mathbb{C}$ with

(I′) *If* $u, w \in \mathbb{C}$ *and* $0<R<1$ *satisfy* $|w| \leqq (1+|u|/R)^{-2m_2} R$, *then the estimate*

$$S_2^{(\nu)}(q', u+w) \leqq S_2^{(\nu)}(q', u) + C_i \sum_{l=2}^{2m_2} \|\hat{P}_{2,l}(q', \cdot)\| R^l$$

holds.

(II′) *The function* $S_2^{(\nu)}(q', \cdot)$ *is subharmonic on* $\Delta_1(0, r_*)$.

(III′) *On* $\Delta_1(0, r_*)$ *it satisfies the estimate*

$$-B_i \sum_{l=2}^{2m_2} \|\hat{P}_{2,l}(q', \cdot)\| \, |w|^l \leqq S_2^{(\nu)}(q', w) - \sum_{l=2}^{2m_2} \hat{P}_{2,l}(q', w)$$

$$\leqq -A_i \sum_{l=2}^{2m_2} \|\hat{P}_{2,l}(q', \cdot)\| \, |w|^l.$$

Roughly speaking, $\mathscr{E}(i)$ is a variant of Lemma (3, 3, i) of [F-S] adapted to our situation. The difference between $\mathscr{E}(i)$ and that Lemma consists in the appearance of property (I) (resp. (I′)) which is not discussed in [F-S], and of the family

$$\left(\left\{r_{q_2, q''}(v_1, v_2) = \operatorname{Re} v_1 + \sum_{k=2}^{2m_2} \hat{P}_{2,k}(q', v_2) < 0\right\}\right)_{|q''|<1}$$

of two-dimensional pseudoconvex domains which is parametrized by $q'' = (q_3, \ldots, q_n)$. The coefficients of $r_{q_2, q''}$ depend smoothly on q''. Now, pursuing the constructions in the proof of Lemma (3, 3, i) of [F-S] step by step, we can prove $\mathscr{E}(i)$ for all even integers $i \leqq 2m_2$ by induction on i. Also the proof of the existence of a radius $r_0 < r_{2m_2}$ such that the hypotheses of $\mathscr{E}(2m_2)$ is fulfilled uniformly in q' with $r_* = r_0$ is the same.

Next we treat the coupling terms $\hat{P}_{jk}(q_j, q_k; \cdot, \cdot)$ appearing in formula (1.1). First let us note that

$$\hat{P}_{jk}(q_j, q_k; v_j, v_k) = \sum_{\nu, \mu, \kappa, \lambda} T^{jk}_{\nu\mu\kappa\lambda}(q', v') \tag{1.4}$$

where the sum is extended over a finite set of indices ν, μ, κ, and λ such that $\nu+\mu$, $\kappa+\lambda$, $\nu+\kappa$, and $\mu+\lambda$ are all positive, and

$$T^{jk}_{\nu\mu\kappa\lambda}(q', v') = \tilde{a}^{jk}_{\nu\mu\kappa\lambda}(q_j, q_k) v_j^\nu \bar{v}_j^\mu v_k^\kappa \bar{v}_k^\lambda. \tag{1.5}$$

The functions $\tilde{a}^{jk}_{\nu\mu\kappa\lambda}(q_j, q_k)$ are polynomials in q_j, q_k of the form

$$\tilde{a}^{jk}_{\nu\mu\kappa\lambda}(q_j, q_k) = \sum_{A,B,C,D} b^{jk,\nu\mu\kappa\lambda}_{ABCD} q_j^A \bar{q}_j^B q_k^C \bar{q}_k^D \tag{1.6}$$

where we sum over nonnegative integers for which

$$\frac{A+B+\nu+\mu}{2m_j} + \frac{C+D+\kappa+\lambda}{2m_k} = 1. \tag{1.7}$$

The numbers $b^{jk,\nu\mu\kappa\lambda}_{ABCD}$ do not depend on q'. With the abbreviations

(1.8) $$\mathscr{B}_j(q', w) = \sum_{l=2}^{2m_j} \|\hat{P}_{j,l}(q', \cdot)\| \, |w|^l$$

and

(1.9) $$\sigma_j(q_j, w) = |q_j|^{2m_j-2} |w|^2 + |w|^{2m_j},$$

for $w \in \mathbb{C}$ and $2 \leqq j \leqq n$ we can state our estimates for the $T^{jk}_{\nu\mu\kappa\lambda}$ in

Lemma 2 *There is a positive number N, depending only on the coefficients of P, such that, given a positive number $\delta < 1/2$, we have for all j, k and all $(\nu,. \mu, \kappa, \lambda)$ as in* (1.4) *the estimate*

$$|T^{jk}_{\nu\mu\kappa\lambda}(q', v')| \leqq \delta(\mathscr{B}_j(q_j, v_j) + \mathscr{B}_k(q_k, v_k)) + \delta^{-N}(\sigma_j(q_j, v_j) + \sigma_k(q_k, v_k)).$$

For the proof of this we make use of the following sublemma which is not hard to prove.

Sublemma 2.1 (a) *If $M \in \mathbb{Z}^+$ is given, then there exists a positive-number c_M such that for any M-tuple $(a_1, \ldots, a_M) \in \mathbb{C}^M$ one has*

$$\sup_{0 \leqq \theta \leqq 2\pi} \left| \sum_{\nu=1}^{M} a_\nu e^{i(\nu-1)\theta} \right| \geqq c_M \sum_{\nu=1}^{M} |a_\nu|.$$

(b) *Let k be a positive integer and $F(u, \bar{u}) = \sum_{\nu,\mu} a_{\nu\mu} u^\nu \bar{u}^\mu$ a real-valued polynomial of degree $2k$ in $\mathbb{C}$ such that for a certain positive radius R one has $F(u, \bar{u}) \geqq 0$ on $\Delta_1(0, R)$; then, with a positive constant c_k (which does not depend on the $a_{\nu\mu}$) the estimate*

$$\sum_{m \text{ odd}} \Big(\sum_{\nu+\mu=m} |a_{\nu\mu}| \Big) |u|^m \leqq c_k \sum_{m \text{ even}} \Big(\sum_{\nu+\mu=m} |a_{\nu\mu}| \Big) |u|^m$$

exists for each $u \in \Delta_1(0, R)$.

Proof of Lemma 2 We begin by estimating all the terms $T^{jk}_{\nu\mu\kappa\lambda}$ for which $\nu + \mu \geqq 2$, $\kappa + \lambda \geqq 2$. From (1.5) and (1.6) we see that we have to consider terms of the form

$$|q_j|^{A+B} |v_j|^{\nu+\mu} |q_k|^{C+D} |v_k|^{\kappa+\lambda},$$

where the exponents satisfy (1.7). Since $\nu + \mu \geqq 2$, we have

$$\begin{aligned}(|q_j|^{A+B} |v_j|^{\nu+\mu})^{m_j} &\leqq |q_j|^{(A+B+\nu+\mu)(m_j-1)} |v_j|^{A+B+\nu+\mu} + |v_j|^{(A+B+\nu+\mu)m_j} \\ &\leqq 2\sigma_j^{A+B+\nu+\mu/2}.\end{aligned}$$

Correspondingly

$$(|q_k|^{C+D} |v_k|^{\kappa+\lambda})^{m_k} \leqq 2\sigma_k^{C+D+\kappa+\lambda/2}.$$

Therefore, because of (1.7),

$$|T^{jk}_{\nu\mu\kappa\lambda}(q', v')| \leqq 2S(\sigma_j(q_j, v_j) + \sigma_k(q_k, v_k)),$$

where S is the sum of the moduli of the coefficients $b_{ABCD}^{jk,\nu\mu\kappa\lambda}$ from (1.6). For large enough N_1 it is less than $2^{N_1} < \delta^{-N_1}$.

Next we estimate all the terms $T^{jk}_{\nu\mu 01}$ with ν, $\mu \geqq 1$. For this we fix a vector $v' \in \mathbb{C}^{n-1}$ and a number $\delta_1 \in (0, 1/2)$. For $u \in \mathbb{C}$ let

$$F_{jk}(u) = |v_j|^2 \, |u|^4 \, \frac{\partial^2 \tilde{r}_q}{\partial v_j \, \partial \bar{v}_j} \left(v_j u^2 e_j + v_k \frac{u}{\delta_1} e_k \right)$$

where we denote by e_i the i-th unit vector in $\mathbb{C}^{n-1}$. Then $F_{jk} \geqq 0$ everywhere, and thus

$$\begin{aligned}
(1.10) \qquad 0 \leqq F_{jk}(u) = {} & |v_j|^2 \, |u|^4 \, \frac{\partial^2 \hat{P}_j}{\partial v_j \, \partial \bar{v}_j} (q', v_j u^2) \\
& + \sum_{\nu, \mu \geqq 1} \nu \mu \, \tilde{a}^{jk}_{\nu\mu 10}(q_j, q_k) \frac{1}{\delta_1} v_j^\nu \bar{v}_j^\mu v_k u^{2\nu+1} \bar{u}^{2\mu} \\
& + \sum_{\nu, \mu \geqq 1} \nu \mu \, \tilde{a}^{jk}_{\nu\mu 01}(q_j, q_k) \frac{1}{\delta_1} v_j^\nu \bar{v}_j^\mu \bar{v}_k u^{2\nu} \bar{u}^{2\mu+1} \\
& + \sum_{\substack{\nu, \mu \geqq 1 \\ \kappa + \lambda \geqq 2}} \frac{\partial^2}{\partial v_j \, \partial \bar{v}_j} T^{jk}_{\nu\mu\kappa\lambda} \left(q', v_j u^2 e_j + v_k \frac{u}{\delta_1} e_k \right).
\end{aligned}$$

Now apply Sublemma 2.1 b to $F = F_{jk}$ and $u = 1$. This gives, with the abbreviation $\tilde{a}^{jk}_{\nu\mu 10} = \tilde{a}^{jk}_{\nu\mu 10}(q_j, q_k)$, $\tilde{a}^{jk}_{\nu\mu 01} = \tilde{a}^{jk}_{\nu\mu 01}(q_j, q_k)$:

$$\begin{aligned}
(1.11) \sum_{\nu, \mu \geqq 1} & (|\tilde{a}^{jk}_{\nu\mu 10}| + |\tilde{a}^{jk}_{\nu\mu 01}|) \, |v_j|^{\nu+\mu} \, |v_k| \\
& \leqq \delta_1 \frac{\partial^2 \hat{P}_j}{\partial v_j \, \partial \bar{v}_j} (q', v_j) \, |v_j|^2 + \delta_1 \sum_{\substack{\nu, \mu \geqq 1 \\ \kappa + \lambda \geqq 2}} \left| \frac{\partial^2}{\partial v_j \, \partial \bar{v}_j} T^{jk}_{\nu\mu\kappa\lambda} \left(q', v_j e_j + \frac{1}{\delta_1} v_k e_k \right) \right| \\
& \leqq \delta_1 \mathscr{B}_j(q', v_j) + \delta_1^{-N_2} (\sigma_j(q_j, v_j) + \sigma_k(q_k, v_k)),
\end{aligned}$$

with a large constant N_2 independent of q. Similarly we can estimate the absolute values of the $T^{jk}_{10\kappa\lambda}$ and $T^{jk}_{01\kappa\lambda}$, where κ, $\lambda \geqq 1$, by

$$\delta_1 \mathscr{B}_k(q', v_k) + \delta_1^{-N_2} (\sigma_j(q_j, v_j) + \sigma_k(q_k, v_k)).$$

Finally the terms $T^{jk}_{\nu 001}$ and $T^{jk}_{100\lambda}$, ν, $\lambda \geqq 1$, must be estimated.

Let $0 < \delta_2 < 1$, and for $\theta \in \mathbb{R}$, $v^\theta = e^{i\theta} v'$. For j, $k \in \{2, \ldots, n\}$ we abbreviate $w^\theta_{jk} = \delta_2^2 v_j^\theta e_j + v_k^\theta e_k$. Computing the mixed partial derivative of $\tilde{r}_q$ at w^θ_{jk}, we obtain

$$\begin{aligned}
(1.12) \quad v_j \bar{v}_k \frac{\partial^2 \tilde{r}_q}{\partial v_j \, \partial \bar{v}_k} (w^\theta_{jk}) = {} & \sum_{\nu=2}^{2m_j} \tilde{a}^{jk}_{\nu 001} \delta_2^{2\nu-2} e^{i(\nu-1)\theta} v_j^\nu \bar{v}_k \\
& + \sum_{\lambda=1}^{2m_k} \tilde{a}^{jk}_{100\lambda} e^{-i(\lambda-1)\theta} v_j \bar{v}_k^\lambda + v_j \bar{v}_k \frac{\partial^2 T^{jk}}{\partial v_j \, \partial \bar{v}_k} (w^\theta_{jk}),
\end{aligned}$$

where T^{jk} denotes the sum of all the terms $T^{jk}_{\nu\mu\kappa\lambda}$, for which $\nu+\lambda\geqq 1$, or $\mu+\kappa\geqq 1$. These have already been estimated in the desired way. Now, taking the supremum over all $\theta\in\mathbb{R}$, we obtain from Sublemma 2.1a:

$$\begin{aligned}(1.13)\qquad &\sum_{\lambda=1}^{2m_k}|\tilde a^{jk}_{100\lambda}|\,|v_j|\,|v_k|^\lambda+\sum_{\nu=2}^{2m_j}\delta_2^{2\nu-2}|\tilde a^{jk}_{\nu001}|\,|v_j|^\nu\,|v_k|\\ &\leqq c_1\sup_{\theta\in\mathbb{R}}\left|\sum_{l=2}^{2m_j}\tilde a^{jk}_{l001}\,\delta_2^{2l-2}\,e^{i(l-1)\theta}v_j^l\bar v_k+\sum_{\lambda=1}^{2m_k}\tilde a^{jk}_{100\lambda}e^{i(1-\lambda)\theta}v_j\bar v_k^\lambda\right|\\ &\leqq c_1\sup_{\theta\in\mathbb{R}}|v_j|\,|v_k|\left|\frac{\partial^2\tilde r_q}{\partial v_j\,\partial\bar v_k}(w^\theta_{jk})\right|+c_1\,\delta_2(\mathscr{B}_j(q',v_j)+\mathscr{B}_k(q',v_k))\\ &\quad+c_1\,\delta_2^{-N_2}(\sigma_j(q_j,v_j)+\sigma_k(q_k,v_k))\end{aligned}$$

by (1.12). The second term on left side of (1.13) can be dropped. So we will get the desired estimate for $T^{jk}_{100\lambda}$, if we can estimate

$$S_{jk}=\sup_{\theta\in\mathbb{R}}|v_j|\,|v_k|\left|\frac{\partial^2\tilde r_q}{\partial v_j\,\partial\bar v_k}(w^\theta_{jk})\right|$$

in a suitable way. From the plurisubharmonicity of $\tilde r_q$ it follows, that, for each $\theta\in\mathbb{R}$,

$$\begin{aligned}|v_j|\,|v_k|\left|\frac{\partial^2\tilde r_q}{\partial v_j\,\partial\bar v_k}(w^\theta_{jk})\right|&\leqq\left(|v_j|^2\frac{\partial^2\tilde r_q}{\partial v_j\,\partial\bar v_j}(w^\theta_{jk})\right)^{1/2}\left(|v_k|^2\frac{\partial^2\tilde r_q}{\partial v_k\,\partial\bar v_k}(w^\theta_{jk})\right)^{1/2}\\ &\leqq\frac{1}{\delta_2}|v_j|^2\frac{\partial^2\tilde r_q}{\partial v_j\,\partial\bar v_j}(w^\theta_{jk})+\delta_2\,|v_k|^2\frac{\partial^2\tilde r_q}{\partial v_k\,\partial\bar v_k}(w^\theta_{jk}).\end{aligned}$$

But

$$|v_j|^2\frac{\partial^2\tilde r_q}{\partial v_j\,\partial\bar v_j}(w^\theta_{jk})=|v_j|^2\frac{\partial^2\hat P_j}{\partial v_j\,\partial\bar v_j}(\delta_2^2 v_j^\theta)+T^j,$$

where

$$T^j=|v_j|^2\sum_{\nu,\mu\geqq 1}\frac{\partial^2}{\partial v_j\,\partial\bar v_j}T^{jk}_{\nu\mu\kappa\lambda}(w^\theta_{jk}).$$

From this we conclude that

$$\begin{aligned}\frac{1}{\delta_2}|v_j|^2\frac{\partial^2\tilde r_q}{\partial v_j\,\partial\bar v_j}(w^\theta_{jk})&\leqq 2\,\delta_2\,\mathscr{B}_j(q',v_j)+\delta_2\,\mathscr{B}_k(q',v_k)\\ &\quad+\delta^{-N_3}(\sigma_j(q_j,v_j)+\sigma_k(q_k,v_k)),\end{aligned}$$

with a large constant N_3 independent of q. The corresponding estimate is valid also for $\delta_2\,|v_k|^2\,\partial^2\tilde r_q/\partial v_k\,\partial\bar v_k$ at the point w^θ_{jk}. Taking the supremum over $\theta\in\mathbb{R}$, we get the desired estimate for $|T^{jk}_{100\lambda}|$. Similarly we can treat the terms $|T^{jk}_{\nu001}|$. Thus the lemma follows, if we choose $\delta_1=\delta_2=\delta/10$, $N=N_1+N_2+N_3$.

We are now ready to introduce the defining function for a pseudoconvex bumping for $\tilde{\Omega}_q$ at q. Let for $v \in \mathbb{C}^n$

$$\Phi_1(v) = \operatorname{Re} v_1 + \sum_{j=2}^{n} S_j(q'; v_j) \tag{1.14}$$

$$\Phi_2(v) = \tilde{r}_q(v) - s \sum_{j=2}^{n} W_j(q_j, v_j),$$

where we define for $2 \leqq j \leqq n$

$$W_j(q_j, v_j) = |q_j + v_j|^{2m_j} - |q_j|^{2m_j} - 2 \operatorname{Re} \frac{\partial |q_j|^{2m_j}}{\partial q_j} v_j.$$

Finally, let for $0 < a < 1$:

$$\varphi = a\,\Phi_1 + (1-a)\,\Phi_2. \tag{1.15}$$

Then φ has the following properties.

Lemma 3 (1) *The function Φ_1 is plurisubharmonic on the tube $T_{r_0} = \mathbb{C} \times \Delta_{n-1}(0, r_0)$ and continuous on $\mathbb{C}^n$*
(2) *The function Φ_2 is plurisubharmonic and smooth on $\mathbb{C}^n$*
(3) *For sufficiently small a, the function φ satisfies, with a universal constant $A > 0$*

$$-\frac{1}{A} \sum_{j=2}^{n} \mathscr{B}_j(q', v_j) + \tilde{r}_q(v) \leqq \varphi(v) \leqq \tilde{r}_q(v) - \frac{A}{2} \sum_{j=2}^{n} \mathscr{B}_j(q', v_j) \tag{1.16}$$

(4) *There exists a positive constant A_1 with the following property: If $v' = (v_2, \ldots, v_n)$ and $w' = (w_2, \ldots, w_n)$ are vectors in $\mathbb{C}^{n-1}$ and $R_2, \ldots, R_n$ are positive numbers, such that (with $m = m_2 + \ldots + m_n$)*

$$|w_j| \leqq R_j \left(1 + \sum_{i=2}^{n} \frac{|v_i|}{R_i}\right)^{-2m}$$

for $j = 2, \ldots, n$, then

$$\varphi(0, v' + w') \leqq \varphi(0, v') + A_1 \sum_{j=2}^{n} \mathscr{B}_j(q', R_j).$$

Proof. Properties (1) and (2) are obvious because of Lemma 1 and assumption (0.2) about P. In order to show (3), we write

$$\Phi_1(v) - \tilde{r}_q(v) = \sum_{j=2}^{n} \left(S_j(q', v_j) - \sum_{l=2}^{2m_j} \hat{P}_{j,l}(q', v_j)\right) - \sum_{j<k} \hat{P}_{jk}(q'; v_j, v_k). \tag{1.17}$$

If $\delta \in (0, 1/2)$ is small, we can estimate by Lemma 2

$$(1.18) \qquad \sum_{j<k} \hat{P}_{jk}(q'; v_j, v_k) \leqq (2m)^4 \sum_{j=2}^{n} \delta \mathscr{B}_j(q', v_j) + \delta^{-N} \sigma_j(q_j, v_j).$$

The definition of the functions $\mathscr{B}_j(q', v_j)$ together with (0.2) implies, with a positive constant A_2, independent of q, that

$$\sigma_j(q_j, v_j) \leqq A_2 \mathscr{B}_j(q', v_j).$$

We substitute (1.18) and (1.3) into (1.17). What we obtain, is, with a positive constant $B_1 > B$:

$$(1.19) \qquad -B_1 \sum_{j=2}^{n} \mathscr{B}_j(q', v_j) \leqq \Phi_1(v) - \tilde{r}_q(v)$$

$$\leqq -(A-(2m)^4 \delta) \sum_{j=2}^{n} \mathscr{B}_j(q', v_j) + (2m)^4 \delta^{-N} \sum_{j=2}^{n} \sigma_j(q_j, v_j).$$

Now let $\delta = A(2m)^{-5}$. In order to obtain (3) we therefore only need to show that, for a suitable $A_3 > 0$, independent of q, the estimate

$$(1.20) \qquad -\frac{1}{A_3} \sum_{j=2}^{n} \sigma_j(q_j, v_j) \leqq \Phi_2(v) - \tilde{r}_q(v) \leqq -A_3 \sum_{j=2}^{n} \sigma_j(q_j, v_j)$$

holds. But Lemma 5.3 in [Ra1] implies, with a constant $A_4' > 0$:

$$\frac{1}{A_4'} \sigma_j \leqq W_j \leqq A_4' \sigma_j.$$

From this and from (1.14) we obtain (1.20). Thus condition (3) is satisfied by φ if we choose $a := A_3 \delta^N / (A_3 \delta^N + (2m)^6)$.

Let us now prove (4). By Lemma 1, (I), the assertion is true for Φ_1. So we need to show it only for Φ_2, or, equivalently, for $\tilde{r}_q$ and the functions W_j. (See (1.14)). Let $v' \in \mathbb{C}^{n-1}$ be fixed, and

$$\hat{R}_j = R_j \left(1 + \sum_{i=2}^{n} \frac{|v_i|}{R_i}\right)^{-2m}.$$

For any $w' = (w_2, \ldots, w_n) \in \mathbb{C}^{n-1}$, such that $|w_j| \leqq \hat{R}_j$, for all $j = 2, \ldots, n$, we write

$$(1.21) \qquad \tilde{r}_q(0, w'+v') - \tilde{r}_q(0, v') = \sum_{j=2}^{n} \left(\sum_{l=2}^{2m_j} T_{jl}(v', w') + \sum_{j<k} S_{jk}(v', w') \right),$$

with the abbreviations

$$T_{jl}(v', w') = \hat{P}_{j,l}(q', v_j + w_j) - \hat{P}_{j,l}(q', v_j)$$

and

$$S_{jk}(v', w') = \hat{P}_{jk}(q_j, q_k; v_j + w_j, v_k + w_k) - \hat{P}_{jk}(q_j, q_k; v_j, v_k).$$

Now, because of the homogeneity of $\hat{P}_{j,l}(q';\cdot)$ we have

$$\begin{aligned}|T_{jl}(v',w')| &\leqq A_5\,\|\hat{P}_{j,l}(q',\cdot)\|\,|w_j|(|v_j|+|w_j|)^{l-1}\\ &\leqq A_5\,\|\hat{P}_{j,l}(q',\cdot)\|\,\hat{R}_j(|v_j|+R_j)^{l-1}\\ &\leqq A_5\,\|\hat{P}_{j,l}(q',\cdot)\|\,\hat{R}_j R_j^{l-1}\left(1+\frac{|v_j|}{R_j}\right)^{l-1}\\ &\leqq A_5\,\|\hat{P}_{j,l}(q',\cdot)\|\,R_j^l\\ &\leqq A_5\,\mathscr{B}_j(q',R_j),\end{aligned}$$

because

$$\hat{R}_j\left(1+\frac{|v_j|}{R_j}\right)^{l-1}\leqq R_j \quad \text{for } 2\leqq l\leqq 2m_j.$$

Similarly, $|S_{jk}|$ is less than a sum of at most $(2m)^4$ terms of the form

$$T=\text{const}\left|\frac{\partial^{a+b+c+d}\hat{P}_{jk}(q';0)}{\partial v_j^a\,\partial\bar{v}_j^b\,\partial v_k^c\,\partial\bar{v}_k^d}\right|(|w_j|(|v_j|+|w_j|)^{a+b-1}(|v_k|+|w_k|)^{c+d}$$
$$+|w_k|(|v_j|+|w_j|)^{a+b}(|v_k|+|w_k|)^{c+d-1}).$$

If we use

$$|v_i|+|w_i|\leqq R_i\left(1+\frac{|v_i|}{R_i}\right)$$

and

$$|w_i|\leqq\hat{R}_i\leqq R_i\left(1+\frac{|v_j|}{R_j}+\frac{|v_k|}{R_k}\right)^{-2m_j-2m_k}$$

for $i\in\{j,k\}$, we obtain

$$T\leqq\text{const}\left|\frac{\partial^{a+b+c+d}\hat{P}_{jk}}{\partial v_j^a\,\partial\bar{v}_j^b\,\partial v_k^c\,\partial\bar{v}_k^d}(q';0)\right|R_j^{a+b}R_k^{c+d}.$$

This implies because of Lemma 2 (with $\delta=1/2$)

$$T\leqq A_6(\mathscr{B}_j(q',v_j)+\mathscr{B}_k(q',v_k)),$$

where the constants A_5, A_6 are again universal. The functions W_j are estimated in the same way as the T_{jl} above. From this the inequality (4) follows, and the proof of Lemma 3 is complete.

In particular, we have found an optimal exterior domain of comparison for $\tilde{\Omega}_q$ at 0, namely

$$\hat{\Omega}_q=\left\{v\in T_{3r_0/4}\,\middle|\,\varphi(v)+\frac{A}{4}\sum_{j=2}^{n}\mathscr{B}_j(q',v_j)<0\right\}. \tag{1.22}$$

We have

$$\tilde{\Omega}_q\cap T_{3r_0/4}\subset\hat{\Omega}_q.$$

Also note that $\varphi(v)=\operatorname{Re} v_1+\tilde{\varphi}(v')$, where $\tilde{\varphi}(v')=\varphi(0, v')$. It will be on $\hat{\Omega}_q$ that we construct bounded holomorphic auxiliary functions which will be needed for the proof of Theorem 1 and 2. In the next section we prove a comparison lemma for holomorphic functions from which it will follow that, in order to prove Theorem 1, it is enough to estimate the Caratheodory metric of $\hat{\Omega}_q$ on the interior normal at 0. For the proof of Theorem 2 we will construct a peak function on $\hat{\Omega}_q$, which gives us a local peak function for Ω at q. By means of the comparison lemma we finally make out of that local peak function a global one.

2 Bounded holomorphic auxiliary functions

We begin with a lemma which is based on the L^2-theory for the $\bar{\partial}$ operator due to Hörmander [Hö]. It is essential for the construction of holomorphic functions which satisfy L^2 estimates. These will imply an L^∞ estimate via the mean value inequality. This idea was also used in [B-F1; Ca; F-S, Sect. 2; Ra2].

Lemma 4 *Let $G \subset \mathbb{C}^d$ be a pseudoconvex domain and $G' \subset G$ be open. Suppose U and ψ are plurisubharmonic on G, and ψ is strictly plurisubharmonic on G'. Let a $\bar{\partial}$-closed $(0,1)$-form α on G be given with smooth coefficients such that* $\operatorname{supp}(\alpha) \subset G'$, *and*

$$I(\alpha)=\int_G |\alpha|^2_{\partial\bar{\partial}\psi}\, e^{-U-\psi}\, d\lambda_{2d} \text{ is finite.} \tag{2.1}$$

Then there exists a smooth solution $u_\alpha \in C^\infty(G)$ for the equation $\bar{\partial} u_\alpha=\alpha$ with the following properties:

$$\int_G |u_\alpha|^2 e^{-U-\psi}\, d\lambda_{2d} \leqq 2I(\alpha). \tag{2.2.1}$$

If $x \in G \setminus \overline{G'}$ and $\hat{r}_1(x), \ldots, \hat{r}_d(x)>0$ are radii, such that

$$\hat{\Delta}(x)=\{y \in \mathbb{C}^d \mid |y_j-x_j| \leqq \hat{r}_j(x) \text{ for all } 1 \leqq j \leqq d\} \subset G \setminus \bar{G}',$$

then

$$|u_\alpha(x)|^2 \leqq (\hat{r}_1(x) \cdot \ldots \cdot \hat{r}_d(x))^{-2} (\max_{\hat{\Delta}(x)} e^{U+\psi}) \cdot I(\alpha). \tag{2.2.2}$$

Here, $d\lambda_{2d}$ denotes the Lebesgue measure in $\mathbb{C}^d$, and $|\alpha|_{\partial\bar{\partial}\psi}$ is the length of α with respect to the Kähler metric on G' with potential ψ.

Proof. The existence of a solution $u_\alpha \in C^\infty(G)$ of $\bar{\partial} u_\alpha=\alpha$ satisfying (2.2.1) follows from a slight modification of Theorem (2.2.1′) in [Hö]. If $x \in G \setminus \overline{G'}$ then u_α is holomorphic near $\hat{\Delta}(x)$. So we obtain by the mean value inequality, applied to u_α^2

$$|u_\alpha(x)|^2 \leqq \mathrm{Vol}(\hat{\Delta}(x))^{-1} \int_{\hat{\Delta}(x)} |u_\alpha(y)|^2\, d\lambda_{2d}(y)$$
$$\leqq \pi^{-d}(\hat{r}_1(x)\cdot \ldots \cdot \hat{r}_d(x))^{-2} \max_{\hat{\Delta}(x)} e^{U+\psi} \int_G |u_\alpha|^2 e^{-U-\psi}\, d\lambda_{2d}.$$

This together with (2.2.1) will imply estimate (2.2.2).

We next show a comparison lemma for holomorphic functions on general pseudoconvex domains of homogeneous finite diagonal type.

Lemma 5 (Comparison Lemma) *Let $\Omega' = \{r' = \mathrm{Re}\, z_1 + P'(z') < 0\}$ be a general domain of homogeneous finite diagonal type (No assumptions about P' but* (0.1), *and* (0.2)!). *Let $0 < \rho_1$, $\delta < 1$ and $q \in b\Omega'$. Assume there exists a pseudoconvex domain $\hat{\Omega}'_q$ with the properties*

(2.3) $$\Omega' \cap B_n(q, \tfrac{9}{8}\rho_1) \subset \hat{\Omega}'_q \cap B_n(q, \tfrac{9}{8}\rho_1)$$

(2.4) $$\{r' < \delta\} \cap (B_n(q, \tfrac{9}{8}\rho_1) \setminus \bar{B}_n(q, \tfrac{6}{8}\rho_1)) \subset \hat{\Omega}'_q.$$

Further, let E be a finite set in $\Omega' \cap B_n(q, \frac{1}{2}\rho_1)$ with $\# E$ elements. Then there exists a positive constant γ depending on ρ_1, δ, and $\# E$, such that the following holds:

Let χ be a smooth cut-off function, $0 \leqq \chi \leqq 1$, $\chi(x) = 1$, for $x \leqq (\frac{7}{8})^2$ and $\chi(x) = 0$ for $x \geqq 1$, let $L \in \mathbb{N}_0$, and $f_0 \in \mathcal{O}(\hat{\Omega}'_q \cap B_n(q, \frac{9}{8}\rho_1))$ a function such that

$$\|f_0\|_{L^\infty(\hat{\Omega}'_q \cap (B_n(q, \frac{9}{8}\rho_1) \setminus B_n(q, \frac{3}{4}\rho_1)))} \leqq 1.$$

Then there exists a smooth function u_E^L on $\Omega'_\delta = \{r' < \delta\}$ satisfying

(2.5) $$\hat{f} = \chi\left(\frac{|z-q|^2}{\rho_1^2}\right) f_0(z) - u_E^L \in \mathcal{O}(\Omega')$$

(2.6.1) $$\|u_E^L\|_{L^\infty(\Omega')} \leqq \gamma,$$

and

(2.6.2) u_E^L *vanishes of L^{th} order at the points of E.*

Proof. We apply Lemma 4 to the form

$$\alpha = \begin{cases} \bar{\partial}(\chi(|z-q|^2/\rho_1^2) f_0(z)), & \text{on } \Omega' \cap A' \\ 0, & \text{on } \Omega' \setminus A' \end{cases}$$

where $A' = B_n(q, \rho_1) \setminus B_n(q, \frac{7}{8}\rho_1)$. Note that $\Omega'_\delta \cap A' \subset \hat{\Omega}'_q$. Further, in the situation of Lemma 4, we have $d = n$, $G = \Omega'_\delta$. We also need to choose the right plurisubharmonic weight functions on G. They were constructed by the author in [He1, Satz 5].

Lemma 5.1 *For small $\delta>0$ there exist plurisubharmonic functions V_E and ψ' on Ω'_δ with the following properties:*

With suitable positive constants $c_0, c_1, \ldots, c_3$, depending only on ρ_1, δ, and $\# E$:

(2.7) $0 \leqq \psi' \leqq c_2$, $\psi' \in C^\infty(\Omega'_\delta)$, *and* $\partial\bar{\partial}(\psi' - c_0|z|^2) \geqq 0$ *on* $\Omega'_\delta \cap B_n(0, 2\rho_1)$

(2.8) $V_E \in C^2(\Omega'_\delta \setminus E)$, *and V_E is strictly plurisubharmonic on $\Omega'_\delta \setminus E$. We have* $V_E(z) \leqq \sum_{e\in E} \log|z-e|^2 + c_1$ *for any* $z \in \Omega'_\delta$.

(2.9) *On Ω'_δ one has $V_E \leqq c_2$, and $V_E \geqq -c_3$ on $\Omega'_\delta \setminus B_n(q, \frac{3}{4}\rho_1)$.*

Proof. Satz 5 in [He1] implies the lemma for $\delta=0$ and $\# E=1$. But the arguments of the proof given there go through also for a small positive δ and a finite set E.

To continue the proof of Lemma 5 we introduce two functions, namely

$$U = r' - s'\sigma' + (n+L)V_E,$$

and

$$\psi = \psi',$$

where $\sigma'(z') = \sum_{j=2}^{n} |z_j|^{2m'_j}$, and $(m'_2, \ldots, m'_n)$ is the set of weights associated to P' according to (0.1) and s' is the number appearing in (0.2) for P'. Let u_E^L be the function u_α from Lemma 4. (It is obvious that $I(\alpha)$ is finite.) Then (2.5) follows immediately. Property (2.6.2) is implied by (2.2.1) for u_E^L, since $\exp(-U - \psi')$ behaves like $|z-e|^{-2(n+L)}$ near any point $e \in E$. For the proof of (2.6.1) we distinguish three cases:

(a) Let $z \in \Omega' \cap (B_n(q, \frac{17}{16}\rho_1) \setminus \bar{B}_n(q, \frac{13}{16}\rho_1))$; then, with a small radius $r_1>0$ depending only on ρ_1 and δ, we have

$$\Delta_n(z, r_1) \subset \Omega'_\delta \cap (B_n(q, \tfrac{9}{8}\rho_1) \setminus \bar{B}_n(q, \tfrac{3}{4}\rho_1)).$$

Thus by the mean value inequality (with $c = (\pi r_1^2)^{-n}$)

$$\begin{aligned} |\hat{f}(z)|^2 &\leqq c \int_{\Delta_n(z,r_1)} |\hat{f}|^2 \, d\lambda_{2n} \\ &\leqq 2c \int_{\Delta_n(z,r_1)} (|f_0|^2 + |u_E^L|^2) \, d\lambda_{2n} \\ &\leqq 2c(1 + (\max_{\Delta_n(z,r_1)} e^{U+\psi}) \int_{\Omega'_\delta} |u_E^L|^2 e^{-U-\psi} \, d\lambda_{2n}) \\ &\leqq 2c(1 + 2\exp(\delta + (n+1+L)c_2) I(\alpha). \end{aligned}$$

Since $I(\alpha) \leqq c_4 e^{(n+L)c_3}$ with a constant c_4 independent of E, we obtain

$$|\hat{f}(z)|^2 \leqq c_5(\rho_1, \delta, \# E)$$

with a universal constant $c_5(\rho_1, \delta, \# E)$, and finally $|u_E^L(z)|^2 \leqq 2|\hat{f}(z)|^2 + 2 \leqq \gamma_1 := 2(c_5+1)$.

(b) Suppose now $z \in \Omega' \cap \bar{B}_n(q, \frac{13}{16}\rho_1)$. Then for small enough $r_2 > 0$ (depending only on ρ_1, and δ) u_E^L is holomorphic on $\Delta_n(z, r_2)$. Apply (2.2.2) of Lemma 4 with $G' = \Omega'_\delta \cap (B_n(q, \frac{33}{32}\rho_1) \setminus \bar{B}_n(q, \frac{27}{32}\rho_1))$ ($\supseteq \mathrm{supp}(\alpha)$), and $\hat{r}_i(z) = r_2$, $i = 1, \ldots, n$.
(c) Finally, if $z \in \Omega' \cap (\mathbb{C}^n \setminus B_n(q, \frac{17}{16}\rho_1))$, we choose, with a small positive constant c independent of z and E

$$\hat{r}_1(z) = \frac{c}{4}\delta \quad \text{and} \quad \hat{r}_2(z) = \ldots = \hat{r}_n(z) = \frac{c}{4}\delta(1 + |z'|^2)^{-2m_2 - \ldots - 2m_n}.$$

Then the corresponding polydisc $\hat{\Delta}(z)$ is contained in $\Omega'_\delta \setminus G'$. Applying (2.2.2) of Lemma 4, we get, with a constant $c_6 = c_6(\rho_1, \delta, \# E)$,

$$|u_E^L(z)|^2 \leqq c_6(1 + |z'|^2)^{2(n-1)m} e^{-s'\sigma'(z')} \leqq c_7$$

for a large constant c_7 uniformly in z. Here we abbreviated $m = m_2 + \ldots + m_n$. Hence condition (2.6.1) is satisfied for $\gamma = (\gamma_1 + c_7)^{1/2} r_2^{-n}$.

In order to construct the auxiliary functions which are relevant for the proof of Theorems 1 and 2 we introduce some more notations.

For $M > 0$ let $R_j(M)$ be the solution of the equation

(2.10) $$\mathscr{B}_j(q', R_j(M)) = 1/2M$$

for $2 \leqq j \leqq n$, and

$$Q_M(v') = \frac{5}{r_0^2} \sum_{j=2}^{n} \frac{|v_j|^2}{R_j(M)^2}.$$

Let us also denote by ∂' (resp. $\bar{\partial}'$) the operators ∂ (resp. $\bar{\partial}$) in the space $\mathbb{C}^{n-1}_{(v_2, \ldots, v_n)}$. From now on we assume that M is so large that $R_j(M) \leqq 1/2$ for $2 \leqq j \leqq n$ (note that $R_j(M) \lesssim M^{-1/2m_j}$). Then the "ellipsoid" $\{v' | Q_M(v') \leqq 5\}$ is contained in $\Delta_{n-1}(0, \frac{1}{2}r_0)$.

Lemma 6 *With a certain positive constant A_7 we have for each $M > 0$:*

$$M|\tilde{\varphi}(v')| \leqq A_7,$$

for any $v' \in D'_M = \Delta_1(0, R_2(M)) \times \ldots \times \Delta_1(0, R_n(M))$.

Proof. Follows immediately from (1.16) in Lemma 3.

Lemma 7 *With a suitable positive constant A_8 the following holds: There exist for any large enough M holomorphic functions ${}_1\mathscr{J}_M, \ldots, {}_n\mathscr{J}_M$ on $\Delta_{n-1}(0, r_0)$ such that*

(2.11) $${}_1\mathscr{J}_M(0) = 1, \qquad \partial'_1 \mathscr{J}_M(0) = 0$$

(2.12) $$\partial'_l \mathscr{J}_M(0) = d v_l / R_l(M) \quad \text{for } 2 \leqq l \leqq n$$

(2.13) $$|{}_l\mathscr{J}_M(v')| \leqq A_8(1 + Q_M(v'))^{m_1} e^{M\tilde{\varphi}(v')/2}$$

for all $v' \in \Delta_{n-1}(0, \frac{3}{4}r_0)$ and $1 \leqq l \leqq n$; here $m_1 = (m_2 + \ldots + m_n) \cdot n$.

Proof. We choose a smooth cut-off function χ_1 on $\mathbb{R}$, $0 \leqq \chi_1 \leqq 1$, $|\chi'_1| \leqq 2$, such that $\chi_1(x) = 1$ for $x \leqq \frac{1}{4}$, and $\chi_1(x) = 0$ for $x \geqq 1$. For $M > 0$ we define the functions

${}_1\tilde{g}_M \equiv 1$, ${}_j\tilde{g}_M(v') = v_j/R_j(M)$, $2 \leqq j \leqq n$. Then we solve on $G := \Delta_{n-1}(0, r_0)$ the $\bar{\partial}'$-equation according to Lemma 4 (where $d = n-1$) given by

$$\bar{\partial}'\, {}_ju_M = {}_j\alpha_M := \bar{\partial}'({}_j\tilde{g}_M\, \chi_1 \circ Q_M). \tag{2.14}$$

The plurisubharmonic weight functions U and ψ are defined by

$$\psi = \psi_M = \log(1 + Q_M)$$

and

$$U = U_M = \log(1 + |v'|^2) + n \log Q_M + M\,\tilde{\varphi}.$$

We estimate the L^2 integral $I({}_j\alpha_M)$ associated to the form ${}_j\alpha_M$. Since $\operatorname{supp}({}_j\alpha_M) \subset \{\frac{1}{4} \leqq Q_M \leqq 1\}$, and $\partial' \bar{\partial}' \psi_M \geqq \frac{1}{4}\, \partial' Q_M \wedge \bar{\partial}' Q_M$ on $\operatorname{supp}({}_j\alpha_M)$ we obtain by virtue of Lemma 6 the estimates $|{}_j\alpha_M|^2_{\partial'\bar{\partial}'\psi_M} \leqq 16\, r_0^2$, and

$$I({}_j\alpha_M) \leqq A_8'\, R_2(M)^2 \cdot \ldots \cdot R_n(M)^2. \tag{2.15}$$

By Lemma 4 we obtain a smooth function ${}_ju_M$ on $\Delta_{n-1}(0, r_0)$ such that the function ${}_j\tilde{f}_M := \chi_1 \circ Q_M \cdot {}_jg_M - {}_ju_M$ is holomorphic on $\Delta_{n-1}(0, r_0)$. Because of the term $n \log Q_M$ occuring in U_M it satisfies (2.11) and (2.12). Let us discuss (2.13). We fix a $v' \in \Delta_{n-1}(0, \frac{3}{4} r_0)$ and distinguish two cases:

(a) $Q_M(v') \geqq 5$. If we set $\hat{r}_j(v') = (r_0/5\,n)(1 + Q_M(v'))^{-m_1/n} R_j(M)$, $2 \leqq j \leqq n$, and $\hat{\Delta}(v') = \Delta_1(v_2, \hat{r}_2(v')) \times \ldots \times \Delta_1(v_n, \hat{r}_n(v'))$, then $\hat{\Delta}(v')$ is relatively compact in $\Delta_{n-1}(0, r_0) \setminus \operatorname{supp}({}_j\alpha_M)$, and (2.2.2) of Lemma 4 applies. Since ${}_j\tilde{f}_M(v') = -{}_ju_M(v')$, we obtain

$$|{}_j\tilde{f}_M(v')|^2 \leqq (\hat{r}_2(v') \cdot \ldots \cdot \hat{r}_n(v'))^{-2}\, I({}_j\alpha_M) \max_{\hat{\Delta}(v')} e^{U_M + \psi_M}.$$

From (2.15), the definition of the $\hat{r}_j(v')$ and (4) of Lemma 3 the estimates

$$I({}_j\alpha_M)(\hat{r}_2(v') \cdot \ldots \cdot \hat{r}_n(v'))^{-2} \leqq A_8''(1 + Q_M(v'))^{2m_1(1-1/n)}$$

and

$$\max_{\hat{\Delta}(v')} e^{U_M + \psi_M} \leqq (2e)^{n+6}\, Q_M(v')^n\, e^{M\tilde{\varphi}(v')}$$

follow and imply (2.13).

(b) Assume $Q_M(v') < 5$. Now, by the maximum principle we can estimate

$$|{}_j\tilde{f}_M(v')| \leqq \max_{w' : Q_M(w') = 5} |{}_j\tilde{f}_M(w')|.$$

The right-hand side of this is less than a positive constant A_8''' independent of q and M, (this follows from part (a)). Since on $\{Q_M < 5\}$ we have $e^{M\tilde{\varphi}/2} \geqq A_8''''$ uniformly in q and M, now (2.13) is completely shown.

3 Proof of the Theorems

Proof of Theorem 1

Lower estimate for the Caratheodory metric

Any point $z \in \Omega$ can be written as $z=(-P(z')+i\,\mathrm{Im}\,z_1, z')+r(z)e_1$, where e_1 is the first unit vector in $\mathbb{C}^n$. We may assume for the estimation of $\mathrm{Cara}_\Omega(z, X)$, $X \in \mathbb{C}^n$, that $\mathrm{Im}\,z_1=0$, and $|z'|<1$, for Ω is invariant under translation by vectors of the form $(ia, 0, \ldots, 0)$, a real, and the scaling map $S_\lambda(v):=(\lambda v_1, \lambda^{1/2m_2} v_2, \ldots, \lambda^{1/2m_n} v_n)$, λ positive. We now write $q=(-P(z'), z')$, and let $t=-r(z)$. Then we apply Lemma 7 with $M=\frac{n}{t}$, and consider the functions

$$ {}_jh_t(v)=\exp\left(\frac{n}{2t} v_1\right) {}_j\tilde{f}_M(v') $$

on $\hat{\Omega}_q$. From (2.13) we see that

$$ |{}_jh_t(v)| \leqq A_8(1+Q_M(v'))^{m_1} e^{\frac{n}{2t}\varphi(v)}. \tag{3.1} $$

Our claim is that ${}_jh_t \in H^\infty(\hat{\Omega}_q)$, and $\|{}_jh_t\|_{L^\infty(\hat{\Omega}_q)} \leqq A_9$, independently of q and t.

Let $v \in \hat{\Omega}_q$. If $Q_M(v') \leqq 1$, then $|{}_jh_t(v)| \leqq 2^{m_1} A_8$. If $Q_M(v')>1$, then there exists a $j \in \{2, \ldots, n\}$

$$ \frac{|v_j|^2}{R_j(M)^2} \geqq \frac{r_0^2}{5n} Q_M(v'). $$

So, (3) of Lemma 3, together with $\mathscr{B}_j(q', v_j)=\mathscr{B}_j\left(q', R_j(M)\frac{v_j}{R_j(M)}\right) \geqq (r_0^2/10\,nM)\,Q_M(v')$ implies

$$ \varphi(v) \leqq -\frac{A}{4}\mathscr{B}_j(q', v_j) \leqq -\frac{r_0^2}{40n} A\,t\,Q_M(v'). $$

Substitute this in (3.1). This gives $|{}_jh_t(v)| \leqq A_9' := A_8 \sup_{x \geqq 1}(1+x)^{m_1} e^{-bx}$ with $b=r_0^2 A/80$.

Let $F(q, \cdot)$ be the biholomorphic mapping introduced at the beginning of Sect. 1 (between (1.1) and (1.2)). We want to apply the Comparison Lemma, Lemma 5, to $\Omega'=\Omega$, and $\hat{\Omega}_q'=F(q,\cdot)^{-1}(\hat{\Omega}_q)$. Certainly we can find positive numbers ρ_1 and δ, independent of q and t such that (2.3) and (2.4) hold. Let ${}_j\hat{h}_z$ be the functions associated by the comparison lemma to $f_0={}_jh_t \circ F(q,\cdot)$, where $E=\{z\}$, and $L=2$. Then $\|{}_j\hat{h}_z\|_{L^\infty(\Omega)} \leqq A_{10}$ uniformly in z. Further, for any vector $X \in \mathbb{C}^n$, we have, since $F(q, z)=-t\,e_1$, because of (2.11), and (2.12):

$$ (\partial_1 \hat{h}_z(z), X)=(\partial_1 h_t(-t\,e_1), F(q, z)'X)=\frac{n}{2t} e^{-n/2}(\partial r(z), X) $$

and

$$ (\partial_j \hat{h}_z(z), X)=e^{-n/2}\frac{|X_j|}{R_j(n/t)}, \quad 2 \leqq j \leqq n. $$

Using the definition of $\mathrm{Cara}_\Omega(z, X)$ we obtain from this

(3.2) $$\mathrm{Cara}_\Omega(z, X) \geqq \frac{e^{-n/2}}{2nA_{10}} \left(\frac{|(\partial r(z), X)|}{t} + \sum_{j=2}^{n} \frac{|X_j|}{R_j(n/t)} \right).$$

Since with a universal constant $A_{11} > 0$

(3.3) $$\frac{1}{A_{11}} \leqq R_j(n/t)\, \mathscr{C}_j(z) \leqq A_{11}$$

the right-hand side of (3.2) is greater than or equal to $(2nA_{10}A_{11}e)^{-1} M_\Omega(z, X)$, as was to be shown.

Upper estimate for the Kobayashi metric

Let z and X be as before, likewise q and t. From the definition of the radii $R_j(n/t)$, $2 \leqq j \leqq n$, it follows easily that

$$\Delta_t := \Delta_1\left(-t, \frac{t}{2}\right) \times \Delta_1(0, R_2(n/t)) \times \ldots \times \Delta_1(0, R_n(n/t)) \subset \tilde{\Omega}_q.$$

Consequently

$$\begin{aligned} \mathrm{Kob}_\Omega(z, X) &= \mathrm{Kob}_{\tilde{\Omega}_q}(-t e_1, F(q, z)' X) \leqq \mathrm{Kob}_{\Delta_t}(-t e_1, F(q, z)' X) \\ &= \max\left\{ \frac{2|(\partial r(z), X)|}{t}, |X_j|/R_j(n/t), j = 2, \ldots, n \right\} \\ &\leqq A_{12} M_\Omega(z, X) \end{aligned}$$

with A_{12} independent of z and X. The proof of Theorem 1 is now complete.

Proof of Theorem 2

The principal tool for the proof of Theorem 2 is

Lemma 8 *Let q be a boundary point of Ω, such that $\mathrm{Im}\, q_1 = 0$, and $|q'| < 1$. (This assumption causes no loss of generality.) Then on $\hat{\Omega}_q$ there exists a peak function at 0 in the algebra $A^0(\hat{\Omega}_q)$.*

Proof. For $M > 0$ let $f_M(v) = \exp(M v_1/2)\, {}_1\tilde{f}_M(v')$ on $\hat{\Omega}_q$. Here ${}_1f_M$ is the function constructed in Lemma 7. We prove at first that f_M is, in a sense, an "almost" peak function. Note that $f_M(0) = 1$. For $\delta > 0$ let $U_\delta = \{v \in \bar{\hat{\Omega}}_q \mid |\mathrm{Re}\, v_1| + |v'| \geqq \delta\}$. Our claim is that for any positive δ there exists a number $M_\delta > 0$, such that for all $M \geqq M_\delta$ one has

(3.4) $$\sup_{U_\delta} |f_M| \leqq \tfrac{1}{4}.$$

If $v \in U_\delta$ and $Q_M(v') \geqq 1$, then, similarly as in the proof of Theorem 1 we have $M\varphi(v) \leqq -b\,Q_M(v')$, where b is a positive universal constant. Thus $M\varphi(v) \leqq -(b\,Q_M(v') + M\varphi(v))/2$. Now (2.13) implies

$$(3.5) \qquad |f_M(v)| \leqq A_8(1+Q_M(v'))^{m_1} \exp\left(-\frac{b}{4}\,Q_M(v') + \frac{M}{4}\,\varphi(v)\right).$$

Obviously (3.5) remains valid, after enlarging A_8, if necessary, also for $v \in U_\delta$ with $Q_M(v') \leqq 1$.

With a universal constant $A_{13} > 0$ we have, as we can see from (1.16):

$$(3.6) \qquad |\varphi(v) - \operatorname{Re} v_1| \leqq A_{13}\,|v'|^2.$$

We let $c = 1/(4+A_{13})$ and distinguish two cases: Case (I): $|v'| \leqq c\,\delta$. Since $\varphi(v) < 0$ on $\hat{\Omega}_q$, we see from (3.6) that $\operatorname{Re} v_1 \leqq c^2 A_{13}\,\delta^2$. Since $v \in U_\delta$, $\operatorname{Re} v_1$ cannot be positive (note that c is small!). So $\operatorname{Re} v_1 = -(|\operatorname{Re} v_1| + |v'|) + |v'| \leqq -\delta + c\,\delta \leqq -3\,\delta/4$. Now (3.6) implies that $\varphi(v) < -\delta/2$. Case (II): $|v'| > c\,\delta$. In this case we must have $Q_M(v') \geqq A_{14} M^\varepsilon c^2 \delta^2$, with $\varepsilon = 1/(2m_2 + \ldots + 2m_n)$. Thus (3.5) implies that one can choose in any case a number M_δ such that (3.4) is satisfied for each $M \geqq M_\delta$.

Let A_{16} be a positive number such that $\|f_M\|_{L^\infty(\hat{\Omega}_q)} \leqq A_{16}$ for all M. Then we can apply the arguments given in [Bi, pp. 633–634], in a slightly modified way to the functions $g_M = f_M/A_{16}$, and find a strictly increasing sequence $(M_k)_k$ of positive numbers such that

$$f_q = \beta \sum_{k=1}^{\infty} (1-(\gamma_2-\gamma_1))^k g_{M_k}$$

is the desired peak function for $\hat{\Omega}_q$ at 0 with respect to $A^0(\hat{\Omega}_q)$, where $\gamma_1 = 1/A_{16}$, $\gamma_2 = 3\gamma_1$, and $\beta = A_{16}(\gamma_2-\gamma_1)/1-(\gamma_2-\gamma_1)$.

We apply the comparison lemma to $\Omega' = \Omega$, $\hat{\Omega}'_q = F(q,\cdot)^{-1}(\Omega''_q)$, where $\Omega''_q = \left\{v \,\middle|\, \varphi(v) + \frac{A}{3} \sum_{j=2}^{n} \mathscr{B}_j(q'; v_j) < 0\right\}$, and the function $f_0 = 1/(f_q \circ F(q,\cdot) - 1)$. Here, f_q is the peak function from Lemma 8 for $\hat{\Omega}_q$. We choose $E = \varnothing$ and $L=1$. What we obtain, is a function $\hat{f}_q \in \mathcal{O}(\Omega)$, whose real part is less than some constant A_{17} and which blows up at q. Hence

$$g_q = \frac{\hat{f}_q - A_{17} + 1}{\hat{f}_q - A_{17} - 1}$$

is the desired peak function for Ω at q with respect to $A^0(\Omega)$, for $\hat{f}_q$ is of the form $\hat{f}_q = \chi f_0 - u$, with a certain cut-off function χ which vanishes outside the domain of definition of f_0 and a function $u \in C^\infty(\bar{\Omega})$. Furthermore, since $\chi \cdot f_0 \in C^0(\bar{\Omega} \setminus \{q\})$, we have $\hat{f}_q \in C^0(\bar{\Omega} \setminus \{q\})$, and finally $g_q \in C^0(\bar{\Omega})$.

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