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Global Solution of the System of Wave and Klein-Gordon Equations

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1. Introduction

The first purpose of this work is to prove a new $L^\infty - L^2$ weighted estimate for the linear wave equation in space dimension $n=3$ with weights associated with the generators of the Poincaré group. The second purpose is to apply this estimate together with the results of W. von Wahl [12], Klainerman [9], Hörmander [6] and prove the global existence of small amplitude solution to the non-linear system of wave and Klein-Gordon equations.

We study the Cauchy problem

$$\begin{aligned} &(\partial_t^2 - \Delta)\varphi_j = F_j(t, x, \varphi, \nabla\varphi), \quad j=1,\dots,N, \\ &(\partial_t^2 - \Delta + M_j^2)\varphi_j = F_j(t, x, \varphi, \nabla\varphi), \quad j=N+1,\dots,K, \\ &\varphi_j = \varepsilon f_j, \quad \partial_t\varphi_j = \varepsilon g_j \quad \text{for} \quad t=0, \quad j=1,\dots,K, \end{aligned} \tag{1.1}$$

where Δ is the Laplace operator in $\mathbb{R}^3$, $\varphi=(\varphi_1,\dots,\varphi_K)$, $(\nabla\varphi)=(\partial_j\varphi)$, $j=0,1,2,3$, $\partial_0=\partial_t$, $\partial_k=\partial_{x_k}$, $k=1,2,3$ and $M_j>0$ for $j=N+1,\dots,K$.

The investigation of the Cauchy problem (1.1) is important, since the coupled system of wave and Klein-Gordon equations gives a model of interacting mass and massless classical fields.

In the case, when $M_j=0$, Klainerman [10] introduce the notion of the null condition for the non-linear quadratic terms and prove that this condition leads to the existence of global solution to the system. The approach in [10] essentially uses the generators of the Poincaré group together with the radial vector field $t\partial_t + x_1\partial_1 + x_2\partial_2 + x_3\partial_3$.

The case, when $M_j>0$, is rather different and leads to some essential difficulties. First, the commutation relations of the radial fector field and the operator of the Klein-Gordon equation show that the radial vector field is not convenient for the investigation of the Klein-Gordon equation. Secondly, the null condition of Klainerman [10] works well if the radial vector field is included in the Sobolev norms. To overcome this difficulty we introduce a stronger version of the null condition of Klainerman [10] for the quadratic nonlinearity in (1.1). The definition of the strong null condition, given in Sect. 2, is fulfilled for some important physical examples of interacting mass and massless fields. We discuss these examples in Sect. 2 in details. Our main result is the following.

Theorem 1. *Suppose* $f_j, g_j \in C_0^\infty(\mathbb{R}^3), j=1,\dots,K$ *and assume the non-linear terms in* (1.1) *satisfy the strong null condition. Then there exists a sufficiently small* $\varepsilon_0>0$, *such that the Cauchy problem* (1.1) *has a unique smooth solution* $\varphi(t,x)$ *for* $0<\varepsilon\leqq\varepsilon_0$.

In the case of space dimension $n>5$ and nonlinearity of quadratic type the existence of solution to the wave equation can be obtained by the aid of L^p estimates of W. von Wahl [12]. The same estimates work in the cases $n=3,4,5$ and nonlinearity of cubic type. We refer to [11], where the Dirichlet problem has been studied. The estimates in [12] are proved by using the Kirchhoff representation of the solution to the wave and Klein-Gordon equations.

For the Klein-Gordon equation in space dimension $n=3$ Klainerman [9], Hörmander [6], Bachelot [1], [2] applied suitable estimates of Sobolev norms associated with the Poincaré group.

Since we deal with the coupled system of wave and Klein-Gordon equations, we have to estimate the weighted norms for the solution to the wave equation assuming the weights are connected only with the Poincaré group. On the other hand, the estimates for the wave equation obtained in [8], [10] include the radial vector field. The approach developed in [9] shows that we have to neglect this field when we study the Klein-Gordon equation. Thus, to prove the global existence of solution to the Cauchy problem (1.1) we need a new estimate of the wave equation with norms associated with the generators of the Poincaré group only.

To estimate the solution to the wave equation

$$(\partial_t^2-\Delta)u=F \tag{1.2}$$

with zero initial data, we denote by $\Gamma_1,\dots,\Gamma_{10}$ the generators ∂_j, $j=0,1,2,3$, $\Omega_{jk}=x_j\partial_k-x_k\partial_j$, $1\leqq j<k\leqq 3$, $\Omega_{0j}=t\partial_j+x_j\partial_t$, $j=1,2,3$, of the Poincaré group. Then for any $u(t,x)\in C^\infty(\mathbb{R};C_0^\infty(\mathbb{R}^3))$ and any non-negative integer k we introduce the seminorms (see [1], [6], [9])

$$|u(t,x)|_k=\sum_{|\alpha|\leqq k}|\Gamma^\alpha u(t,x)|, \qquad \Gamma^\alpha=\Gamma_1^{\alpha_1}\Gamma_2^{\alpha_2}\dots\Gamma_{10}^{\alpha_{10}}, \tag{1.3}$$

$$\|u(t)\|_k=\left(\int_{\mathbb{R}^3}|u(t,x)|_k^2\,dx\right)^{1/2}. \tag{1.4}$$

Theorem 2. *Suppose* $u(t,x)$ *is a smooth solution to the wave equation* (1.2) *with zero initial data and assume* $\operatorname{supp}F\subseteq\{(t,x);|x|\leqq t+R\}$ *for some positive R. Then for* $t\geqq 0$ *we have the estimate*

$$|u(t,x)|_k\leqq\frac{C}{1+t}\left(\sum_{r=0}^{\infty}\sup_{s\in I_r\cap[0,t]}2^{3r/2}\|F(s)\|_{k+3}\right),$$

where $I_r=[2^{r-1},2^{r+1}]$ *for* $r>0$, $I_0=[0,2]$.

The above estimate is one of the main tools in the proof of Theorem 1.

A similar result for the solution $u(t,x)$ to the Klein-Gordon equation

$$(\partial_t^2 - \Delta + 1)u = F$$

has been announced by Hörmander [6]. More precisely, assuming

$$u(0,x) = \partial_t u(0,x) = 0$$

and

$$\operatorname{supp} F \subseteq \{(t,x); |x| \leqq t + R\},$$

one can write the estimate (see [6])

$$|u(t,x)|_k \leqq \frac{C}{(1+t)^{3/2}} \left(\sum_{r=0}^{\infty} \sup_{s \in I_r \cap [0,t]} 2^r \|F(s)\|_{k+5} \right).$$

The main idea in the proof of Theorem 2 is to represent the solution to the wave equation by oscillatory integral over the isotropic cone. The stationary phase method leads to an estimate of this oscillatory integral by the Radon transform of the right-hand side $F(t,x)$ of the wave equation. A suitable estimate of the Radon transform leads to the desired estimate.

The plan of the work is the following. In Sect. 2 we give some preliminary results, the definition of the strong null condition and we recall the representation of the solution to the wave equation by an oscillatory integral over the isotropic cone. The estimate of these integrals is given in Sect. 3 by means of the Radon transform in $\mathbb{R}^3$. The proof of Theorem 2 is given in Sect. 4. Finally, in Sect. 5 we prove the stability of the solution and complete the proof of Theorem 1.

2. The Strong Null Condition and Preliminary Results

The non-linear terms in (1.1) will be supposed to have the form

$$F_j(t,x,\varphi,\nabla\varphi) = Q_j(t,x,\nabla\varphi) + C_j(\varphi,\nabla\varphi),$$

where $C_j(\varphi,\psi)$ are smooth functions,

$$C_j(\varphi,\psi) = O(|\varphi|^3 + |\psi|^3) \quad \text{near} \quad (\varphi,\psi) = (0,0)$$

and $Q_j(t,x,\nabla\varphi) = Q_j(t,x,\nabla\varphi,\nabla\varphi)$ is given by the sesquilinear form

$$Q_j(t,x,\nabla\varphi,\nabla\psi) = \sum_{m,k,r,s} q_j^{mkrs}(t,x)\,\partial_r\varphi_m \overline{\partial_s\psi_k} \tag{2.1}$$

with q_j^{mkrs} being smooth function of (t,x) and $\bar{z}$ is the complex conjugate to z.

The quadratic part of the non-linear term is assumed to satisfy a stronger version of the null condition introduced by Klainerman [10].

Definition 1. The sesquilinear form (2.1) satisfies the *strong null condition* if

(a) $$|\partial_t^p \partial_x^\alpha q_j^{mkrs}(t,x)| = 0\,((|t|+|x|)^{-|\alpha|-p}) \quad \text{as} \quad |t|+|x| \to +\infty,$$

(b) $$\sum_{r,s=0}^{3} q_j^{mkrs}(t,x)\eta_r\eta_s = 0 \quad \text{for any} \quad \eta = (\eta_0,\eta_1,\eta_2,\eta_3),$$

$\eta \in \mathbb{R}^4$, $j,m,k = 1,\dots,K$.

Example 1. Let

$$Q(\nabla\varphi_1, \nabla\varphi_2) = \partial_1\varphi_1\overline{\partial_2\varphi_2} - \partial_2\varphi_1\overline{\partial_1\varphi_2},$$

Then Q satisfies the strong null condition. This is a typical non-linear term for interacting fields.

Example 2. Consider the sesquilinear form

$$\mathrm{L}(\psi_1, \psi_2) = \langle\gamma^0\gamma^5\psi_1, \psi_2\rangle$$

connected with the pseudoscalar model of Yukawa (see [1], [2]). Here $\psi_1, \psi_2 \in \mathbb{C}^4$, $\langle\,,\rangle$ is the inner product in $\mathbb{C}^4$, $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\gamma^0, \dots, \gamma^3$ are the Dirac matrices. The above non-linear term takes part in the non-linear Dirac-Klein-Gordon equation studied in [1], [2]. Theorem 4.2 in [1] shows that L is compatible in the sense of B. Hanouzet and J. Joly [4] with the operator $P+M$, where

$$P = \gamma^0\partial_t + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3$$

is the Dirac operator. An important role in [1] is played by the following sesquilinear form associated with L:

$$Q(\nabla\varphi, \nabla\psi) = L(P\varphi, P\psi) + \sum_{\lambda,\mu=0}^{3} \eta^{\lambda\mu} L(\partial_\lambda\varphi, \partial_\mu\psi)$$

with $(\eta^{\lambda\mu}) = \mathrm{diag}(1, -1, -1, -1)$ being the metric in the Minkowski space $\mathbb{R}^4$. Taking advantage of the properties

$$\gamma^\lambda\gamma^5 = -\gamma^5\gamma^\lambda, \qquad \gamma^\lambda\gamma^\mu + \gamma^\mu\gamma^\lambda = 2\eta^{\lambda\mu}$$

of the Dirac matrices and the fact that $\gamma^0\gamma^\lambda$ are Hermitian matrices, one can check that the sesquilinear form Q satisfies the strong null condition.

Next, we aim to clarify the role of the strong null condition.

Lemma 2.1. *If the sesquilinear form*

$$Q(t, x, \nabla\varphi, \nabla\psi) = \sum_{m,k,r,s} q^{mkrs}(t, x)\partial_r\varphi_m\overline{\partial_s\psi_k}$$

satisfies the strong null condition, then for $t \geqq 1$, $|x| \leqq t+R$ *and any integer* $k \geqq 0$ *we have*

$$|Q(t, x, \nabla\varphi, \nabla\psi)|_k \leqq Ct^{-1}|\varphi(t, x)|_{k+1}|\psi(t, x)|_{k+1}.$$

Proof. The condition (b) in the definition of the strong null condition means that $q^{mkrs} + q^{mkrs} = 0$. Hence, the sesquilinear form $\tilde{Q}$ with coefficients $\tilde{q}^{mkrs} = \Gamma^\alpha q^{mkrs}$ satisfies also the strong null condition for any $\alpha \in \mathbb{Z}^{10}$. Thus, it suffices to study only the case $k=0$. Given any $j = 1, 2, 3$ from

$$\Omega_{0j} = t\partial_j + x_j\partial_0$$

we obtain the equality

$$\partial_j = (t)^{-1}(\Omega_{0j} - x_j\partial_0).$$

Then the strong null condition yields

$$Q = R_1 + R_2 + R_3,$$

where

$$R_1 = (t)^{-2} \sum_{m,k,r,s} q^{mkrs} \Omega_{0r} \varphi_m \overline{\Omega_{0s} \psi_k},$$

$$R_2 = -(t)^{-2} \sum_{m,k,r,s} q^{mkrs} \Omega_{0r} \varphi_m x_s \overline{\partial_0 \psi_k},$$

$$R_3 = -(t)^{-2} \sum_{m,k,r,s} q^{mkrs} x_r \partial_0 \varphi_m \overline{\Omega_{0s} \psi_k},$$

and the sum is over $\{1,2,3\}$ for r, s. The assumptions $t \geqq 1$, $|x| \leqq t+R$ imply that

$$|R_1| + |R_2| + |R_3| \leqq C(t)^{-1} |\varphi(t,x)|_1 |\psi(t,x)|_1$$

and this completes the proof of the Lemma. □

Next, we recall some estimates and results concerning the solution to the wave and Klein-Gordon equations. First, consider the wave equation

$$(\partial_t^2 - \Delta) u = F. \tag{2.2}$$

Since the generators of the Poincaré group commute with the D'Alembertian, we shall use in the sequel the following result due to Klainerman [10].

Theorem 2.2. [10] *Suppose $u(t,x) \in C^\infty(\bar{\mathbb{R}}_+ \times \mathbb{R}^3)$ solves* (2.2) *and assume*

$$\operatorname{supp} u \cup \operatorname{supp} F \subset \{|x| \leqq t+R\}$$

for some real $R > 0$. Then for any integer $k \geqq 1$ we have

$$\|u(t)\|_k \leqq C\left(\|u(s)\|_{k|s=0} + \int_0^t (1+s) \|F(s)\|_{k-1} ds \right),$$

$$\sum_{j=0}^3 \|\partial_j u(t)\|_k \leqq C\left(\sum_{j=0}^3 \|\partial_j u(s)\|_{k|s=0} + \int_0^t \|F(s)\|_k ds \right).$$

More precisely, only the first of the above inequalities is proved in [10]. The second one, can be obtained by multiplying the wave equation by $\partial_0 u$.

The next goal is to represent the solution to the wave equation (2.2) provided

$$F(t,x) \in C^\infty(\bar{\mathbb{R}}_+; S(\mathbb{R}^3)), \quad u(0,x) = \partial_t u(0,x) = 0. \tag{2.3}$$

Lemma 2.3. *Suppose $u(t,x)$ is real-valued smooth solution to* (2.2) *and assume the conditions* (2.3) *are fulfilled. Then we have*

$$u(t,x) = (2\pi)^{-3} I(t,x),$$

where

$$I(t,x) = \operatorname{Im} \int_{\mathbb{R}^3} \hat{F}_t(|\xi|, \xi) e^{i[t|\xi| + x.\xi]} |\xi|^{-1} d\xi$$

and $\hat{F}_t(\tau, \xi)$ is the Fourier transform of $F\chi(0 \leqq s \leqq t)$, i.e.

$$\hat{F}_t(\tau, \xi) = \int_0^t \int_{\mathbb{R}^3} F(s,y) e^{-i[s\tau + y.\xi]} dy\, ds.$$

Proof. The solution to (2.2) is

$$u(t,x)=(2\pi)^{-3}\int_0^t\int_{\mathbb{R}^3}\tilde{F}(s,\xi)|\xi|^{-1}\sin(|\xi|(t-s))e^{ix\cdot\xi}d\xi\,ds,$$

where

$$\tilde{F}(s,\xi)=\int_{\mathbb{R}^3}F(s,y)e^{-iy\cdot\xi}dy$$

is the partial Fourier transform. Comparing the above representation with the needed equality and taking advantage of the fact that

$$\hat{F}_t(-\tau,-\xi)=\overline{\hat{F}_t(\tau,\xi)}$$

for any real-valued function $F(s,y)$, we complete the proof of the Lemma. □

Next, we turn our attention to the Klein-Gordon equation

$$(\partial_t^2-\Delta+M^2)u=F. \tag{2.4}$$

Multiplying the above equation by $\partial_t u$ and taking advantage of the fact the generators of the Poincaré group commute with the D'Alembertian we obtain the estimate

Lemma 2.4. *Suppose $M>0$, u and F are smooth functions satisfying* (2.4), *so that*

$$u(t,x),\qquad F(t,x)\in C^\infty(\mathbb{R};S(\mathbb{R}^3)).$$

Then for any $k\geqq 0$ we have

$$\sum_{|\alpha|\leqq 1}\|\partial^\alpha u(t)\|_k\leqq C\left(\sum_{|\alpha|\leqq 1}\|\partial^\alpha u(s)\|_{k|s=0}+\int_0^{|t|}\|F(s)\|_k ds\right).$$

The next estimate is obtained by Hörmander [6] and will play an essential role in our investigations.

Theorem 2.5. [6] *Suppose $u(t,x)$ is a smooth solution to* (2.4) *and assume*

$$\operatorname{supp}u\cup\operatorname{supp}F\subseteq\{|x|\leqq t+R\}$$

for some $R\geqq 0$. Then for any $t\geqq 0$ we have

$$(1+t)^{3/2}|u(t,x)|_k\leqq C\left(\|u(0)\|_{k+4}+\sum_{r=0}^{\infty}\sup_{s\in I_r\cap[0,t]}2^r\|F(s)\|_{k+5}\right).$$

where $I_r=[2^{r-1},2^{r+1}]$ for $r\geqq 1$ and $I_0=[0,2]$.

An important role in our analysis will be played by the space of the dual variables $\tau\in\mathbb{R}$, $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$.

The operators corresponding to Ω_{jk} are

$$\hat{\Omega}_{jk}=\xi_j\hat{\partial}_k-\xi_k\hat{\partial}_j,\qquad \hat{\Omega}_{0j}=-\tau\hat{\partial}_j-\xi_j\hat{\partial}_\tau, \tag{2.5}$$

where $\hat{\partial}^k=\partial/\partial\xi_k$. Then we have the relations

$$\hat{\Omega}_{kj}F(H)=F(\Omega_{kj}H), \tag{2.6}$$

where $F(H)=\hat{H}$ is the Fourier transform of the function $H(x)\in S(\mathbb{R}^4)$. Therefore, the corresponding generators $\hat{\Gamma}_j$ in the space of the dual variables has the same form as the generators of the Poincaré group in the coordinate space. For this reason we shall omit the hats for the operators in the space of the dual variables.

3. Estimates of Oscillatory Integrals over the Isotropic Cone

Consider the oscillatory integral

$$I(t,x)=\operatorname{Im}\int_{\mathbb{R}^3}\hat{F}_t(|\xi|,\xi)e^{i[t|\xi|+x.\xi]}|\xi|^{-1}d\xi\,, \tag{3.1}$$

taking part in the representation of the solution to the wave equation. Here $\hat{F}_t(\tau,\xi)$ is the Fourier transform of the function $F(s,y)\chi(0\leqq s\leqq t)$, i.e.

$$\hat{F}_t(\tau,\xi)=\int_0^t\int_{\mathbb{R}^3}F(s,y)e^{-i[\tau s+y.\xi]}dyds\,.$$

Sometimes we shall omit the index t and shall write simply $\hat{F}(\tau,\xi)$.

Lemma 3.1. *Suppose*

$$F(s,y)\in C^\infty(\mathbb{R};S(\mathbb{R}^3))\,. \tag{3.2}$$

Then for $|x|\geqq 1$ *the oscillatory integral* $I(t,x)$ *in* (3.1) *can be represented by*

$$I(t,x)=-2\pi|x|^{-1}\sum_{\sigma=\pm}\sigma\operatorname{Re}\int_0^\infty\hat{F}(\varrho,\sigma\varrho x/|x|)e^{i\varrho[\sigma|x|+t]}d\varrho$$
$$+|x|^{-1}\sum_{1\leqq r<k\leqq 3}\operatorname{Re}\int_{\mathbb{R}^3}|\xi|^{-2}\frac{c_{kr}(x,\xi)}{\left|\frac{x}{|x|}\times\frac{\xi}{|\xi|}\right|}\hat{F}_{kr}(|\xi|,\xi)e^{i[t|\xi|+x.\xi]}d\xi\,,$$

where $c_{kr}(x,\xi)$ *are bounded functions, homogeneous of degree* 0 *with respect to* x,ξ, $\hat{F}_{kr}=\Omega_{kr}\hat{F}$ *and* $a\times b$ *denotes the vector product of the vectors* $a,b\in\mathbb{R}^3$.

Proof. We lose no generality assuming

$$x=(0,0,|x|)\,.$$

Introduce polar coordinates

$$\xi_1=\varrho\sin\theta\sin\varphi,\ \xi_2=\varrho\sin\theta\cos\varphi,\ \xi_3=\varrho\cos\theta\,. \tag{3.3}$$

Then $I(t,x)$ becomes

$$I(t,x)=\operatorname{Im}\int_0^{2\pi}\int_0^\infty J(\varrho,\varphi)e^{i\varrho t}\varrho d\varrho d\varphi\,, \tag{3.4}$$

where

$$J(\varrho,\varphi)=\int_0^\pi\hat{F}e^{i\varrho|x|\cos\theta}\sin\theta d\theta\,.$$

Integrating by parts with respect to θ into $J(\varrho,\varphi)$, we get

$$J(\varrho,\varphi)=-\sum_{\sigma=\pm} i\sigma(|x|\varrho)^{-1}\hat{F}(\varrho,\sigma\varrho x/|x|)e^{i\varrho\sigma|x|}$$
$$+\int_0^\pi (i|x|\varrho)^{-1}\partial_\theta \hat{F}e^{i|x|\varrho\cos\theta}d\theta. \tag{3.5}$$

On the other hand, a direct calculation shows that

$$\partial_\theta \hat{F}(\varrho,\varrho\sin\theta\sin\varphi,\varrho\sin\theta\cos\varphi,\varrho\cos\theta)$$
$$=\sin\varphi\Omega_{31}\hat{F}+\cos\varphi\Omega_{32}\hat{F}.$$

Thus, setting

$$c_{21}=0,\quad c_{32}=-\cos\varphi,\quad c_{31}=-\sin\varphi$$

and taking advantage of the identity

$$\sin\theta=\left|\frac{x}{|x|}\times\frac{\xi}{|\xi|}\right|$$

together with (3.4) and (3.5) we obtain the needed equality.

This completes the proof. □

Corollary 3.2. *Suppose the assumptions of Lemma 3.1 are fulfilled and assume F is a real-valued function. Then the oscillatory integral $I(t,x)$ in* (3.1) *satisfies the estimate*

$$|I(t,x)|\leqq C|x|^{-1}\sum_{|\beta|\leqq 1}\max_{\tau,\omega}\left|\int_0^t R(\Omega^\beta F)(\tau-s,\omega,s)ds\right|,$$

where the maximum is taken over $\omega\in\mathbb{S}^2$, $|\tau|\leqq t+|x|$ and

$$R(F)(p,\omega,s)=\int_{x.\omega=p}F(s,x)dS_x$$

is the Randon transform of $F(s,.)$.

Proof. Given any unit vector $\eta\in\mathbb{S}^2$ and any bounded function $g(\omega)$ on $\mathbb{S}^2$ we have the estimate

$$\left|\int_{\mathbb{S}^2} g(\omega)|\eta\times\omega|^{-1}d\omega\right|\leqq C\max_{\omega\in\mathbb{S}^2}|g(\omega)|.$$

From this estimate and Lemma 3.1 we obtain

$$|I(t,x)|\leqq C|x|^{-1}\sum_{|\beta|\leqq 1}\max_{\omega\in\mathbb{S}^2}\left|\mathrm{Re}\int_0^\infty \Omega^\beta\hat{F}(\varrho,\varrho\omega)e^{i\varrho[t+x.\omega]}d\varrho\right|. \tag{3.6}$$

On the other hand, for any real number τ we have

$$\mathrm{Re}\int_0^\infty \hat{F}_t(\varrho,\varrho\omega)e^{i\varrho\tau}d\varrho=\mathrm{Re}\int_0^t\int_0^\infty \tilde{F}(s,\varrho\omega)e^{i\varrho(\tau-s)}d\varrho ds,$$

where

$$\tilde{F}(s,\xi)=\int_{\mathbb{R}^3}F(s,y)e^{-iy\cdot\xi}dy$$

is the partial Fourier transform.

The assumption that F is real-valued yields

$$\tilde{F}(s,-\xi)=\overline{\tilde{F}(s,\xi)}$$

and we find

$$2\,\mathrm{Re}\int_0^\infty \hat{F}_t(\varrho,\varrho\omega)e^{i\varrho\tau}d\varrho=\int_0^t\int_{-\infty}^\infty \tilde{F}(s,\varrho\omega)e^{i\varrho(\tau-s)}d\varrho ds.$$

Now the relation (see [5])

$$\int_{-\infty}^\infty \tilde{F}(s,\varrho\omega)e^{i\varrho\tau}d\varrho=R(F)(\tau,\omega,s)$$

leads to the identity

$$2\,\mathrm{Re}\int_0^\infty \hat{F}_t(\varrho,\varrho\omega)e^{i\varrho\tau}d\varrho=\int_0^t R(F)(\tau-s,\omega,s)ds. \tag{3.7}$$

From this identity with $\tau=t+x.\omega$ and (3.6) we derive the desired estimate.

This proves the corollary. □

Lemma 3.3. *Suppose the assumption* (3.2) *of Lemma 3.1 is fulfilled and*

$$1\leqq|x|\leqq t/2.$$

Then we have

$$I(t,x)=\mathrm{Re}\sum_{|\beta|\leqq 1}\int_0^\infty\int_{\mathbb{S}^2}(t+x.\omega)^{-1}C_\beta(\omega)F(\Gamma^\beta F)_t(\varrho,\varrho\omega)e^{i\varrho(t+x.\omega)}d\omega d\varrho$$

$$-\mathrm{Re}\sum_{j=1}^3\int_0^\infty\int_{\mathbb{S}^2}(t+x.\omega)^{-1}\omega_j(\tilde{F}_j(t,\varrho\omega)-e^{i\varrho t}\tilde{F}_j(0,\varrho\omega))e^{i\varrho x.\omega}d\omega d\varrho,$$

where $F_j(s,y)=y_jF(s,y)$,

$$\tilde{H}(s,\xi)=\int_{\mathbb{R}^3}H(s,y)e^{-iy.\xi}dy$$

is the partial Fourier transform,

$$F(H)_t=\hat{H}_t(\tau,\xi)=\int_0^t\int_{\mathbb{R}^3}H(s,y)e^{-i(\tau s+y.\xi)}dyds$$

and $C_\beta(\omega)$ *are smooth functions on* $\mathbb{S}^2$.

Proof. Introduce polar coordinates

$$\varrho=|\xi|,\qquad \omega=\xi/|\xi|\in\mathbb{S}^2$$

in $I(t,x)$. Thus we have

$$I(t,x)=\mathrm{Im}\int_0^\infty\int_{\mathbb{S}^2}\hat{F}_t(\varrho,\varrho\omega)e^{i\varrho(t+x.\omega)}\varrho d\omega d\varrho.$$

It is possible to integrate by parts with respect to ϱ since

$$|t+x.\omega|\geqq t-|x|\geqq t/2 \tag{3.8}$$

according to the assumptions of the Lemma.

On the other hand, we have the relation

$$\varrho\partial_\varrho \hat{F}(\varrho, \varrho\omega) = -\sum_{j=1}^{3} \omega_j \Omega_{0j} \hat{F}(\varrho, \varrho\omega).$$

Moreover, from

$$\hat{F}_t(\tau, \xi) = \int_0^t \int_{\mathbb{R}^3} F(s, y) e^{-i(\tau s + \xi \cdot y)} dy\, ds$$

we get with $F_j(t, y) = y_j F(t, y)$

$$\Omega_{0j} \hat{F}_t(\tau, \xi) = F(\Omega_{0j} F)_t(\tau, \xi) - e^{-it\tau} \tilde{F}_j(t, \xi) + \tilde{F}_j(0, \xi).$$

Combining the above relations with (3.8) and integrating by parts with respect to ϱ in $I(t, x)$, we complete the proof. □

Following the proof of Corollary 3.2, from Lemma 3.3 we deduce

Corollary 3.4. *Suppose the assumption* (3.2) *of Lemma 3.1 fulfilled and assume F is a real-valued function. Then for*

$$|x| \leqq t/2, \quad t \geqq 1$$

we have the estimate

$$|I(t, x)| \leqq C t^{-1} \max_{\tau, \omega} \sum_{|\beta| \leqq 1} \left| \int_0^t R(\Gamma^\beta F)(\tau - s, \omega, s) ds \right|$$
$$+ C t^{-1} \sum_{j=1}^{3} \max_{\tau, \omega} \max_{0 \leqq s \leqq t} |R(y_j F)(\tau, \omega, s)|,$$

where the maximum is taken over $\omega \in \mathbb{S}^2$ and $\tau \in \mathbb{R}$.

4. Proof of Theorem 2

After a small translation in time and applying the $L^\infty - L^1$ estimates of W. von Wahl [12], we see that without loss of generality we can assume

$$\operatorname{supp} u \cup \operatorname{supp} F \subset \{|x| \leqq t - 1\}. \tag{4.1}$$

The estimates derived in Corollaries 3.2 and 3.4 suggest us to estimate the Radon transform

$$R(F)(p, \omega, s) = \int_{y \cdot \omega = p} F(s, y) dS_y. \tag{4.2}$$

Lemma 4.1. *If* $\operatorname{supp} F(s, y) \subset \{|y| \leqq s - 1\}$ *and F is a smooth function, then we have*

$$|R(F)(p, \omega, s)| \leqq C(1 + s)^{1/2} \sum_{|\beta| \leqq 2} \left(\int_{\mathbb{R}^3} |\Omega^\beta F(s, y)|^2 dy \right)^{1/2}.$$

Proof. From (4.2) and the assumption of the Lemma it follows that

$$\operatorname{supp}_p R(F)(p, \omega, s) \subseteq \{p\,;\, 0 \leqq p \leqq s - 1\}.$$

Starting with the inequality

$$|R(F)(p,\omega,s)| \leqq \int_0^s |\partial_p R(F)(p,\omega,s)|\,dp$$

and applying the Cauchy-Schwartz inequality we get

$$|R(F)(p,\omega,s)| \leqq C(1+s)^{1/2}\left(\int_0^s |\partial_p R(F)(p,\omega,s)|^2\,dp\right)^{1/2}.$$

Since the vector fields $\Omega_{jk}, j, k=1,2,3$, form a basis of the vector fields L tangent to the sphere $\mathbb{S}^2$, for any smooth vector field $L(\omega,\partial_\omega)\in T(\mathbb{S}^2)$ we have the identity

$$LR(F)(p,\omega,s) = \sum_{j,k=1}^{3} c^{jk}(\omega)R(\Omega_{jk}F)(p,\omega,s),$$

where $c^{jk}(\omega)$ are smooth functions on $\mathbb{S}^2$. Thus, the Sobolev inequality on $\mathbb{S}^2$ implies that

$$|\partial_p R(F)(p,\omega,s)|^2 \leqq C \sum_{|\beta|\leqq 2} \int_{\mathbb{S}^2} |\partial_p R(\Omega^\beta F)(p,\omega,s)|^2\,d\omega.$$

Combining the above estimates with the Plancherel identity (see [5], chapter I)

$$\int_{\mathbb{R}}\int_{\mathbb{S}^2} |\partial_p R(\Omega^\beta F)(p,\omega,s)|^2\,d\omega\,dp = c\int_{\mathbb{R}^3} |\Omega^\beta F(s,y)|^2\,dy,$$

we complete the proof of the Lemma. □

Combining the above Lemma with Corollaries 3.2 and 3.4, we finish the proof of the Theorem 2. □

5. Global Existence of Solution to the System of Mass and Massless Wave Equations

In this section we shall prove the global existence of a solution to the system (1.1). Since the strong null condition is invariant under the change of variable $t\to -t$, it is sufficient to prove the existence and uniqueness of solution in the class $C^\infty(\bar{\mathbb{R}}_+\times\mathbb{R}^3)$.

For any integer $k\geqq 0$, real number a and non-negative t introduce the weighted L^∞-norm

$$|u(t)|_{k,a} = \sup_{0\leqq s\leqq t}\ \sup_{y\in\mathbb{R}^3} (1+s)^{-a} \sum_{|\alpha|\leqq k} |\Gamma^\alpha u(s,y)|.$$

For any smooth $\mathbb{C}^K$-valued function $\varphi(t,x)$ decaying sufficiently fast as $|x|\to\infty$, any integer k and any real δ set

$$E_k(\varphi;t) = \sum_{j=1}^{N} |\varphi_j(t)|_{k,-1} + \sum_{j=N+1}^{K} |\varphi_j(t)|_{k,-3/2}$$

and

$$W_{k,\delta}(\varphi;t) = \max_{0\leqq s\leqq t} \sum_{j=1}^{K} (1+s)^{-\delta}\|A\varphi_j(s)\|_k, \qquad A(\varphi) = (\varphi,\partial_0\varphi,\ldots,\partial_3\varphi).$$

Let $R=\max\{|x|;\ x\in\operatorname{supp} f\cup\operatorname{supp} g\}$ and $T>0$ be fixed. Consider the truncated cone $K=\{(t,x);|x|\leqq t+R\}$. Let $X(k,\delta,T)$ be the Banach space formed by the functions

$$\varphi(t,x)\in C^{k+1}([0,T]\times\mathbb{R}^3;\mathbb{C}^K)\cap\left(\bigcap_{j=0}^{2k+1} C^{2k+1-j}([0,T];H^j(\mathbb{R}^3;\mathbb{C}^K))\right)$$

supported in K with norm

$$\|\varphi\|_X=\|\varphi\|_{X(k,\delta,T)}=E_{k+1}(\varphi;T)+W_{2k,\delta}(\varphi;T).$$

Here k is an integer and $\delta>0$. Define the following complete metric space $L=L(\sigma,k,\varepsilon,\delta,T)$

$$L=\{\varphi\in X(k,\delta,T);\ \varphi(0,x)=\varepsilon f,\ \partial_t\varphi(0,x)=\varepsilon g,\ \|\varphi\|_X\leqq\sigma\}.$$

The metric in L is determined by $\varrho(\varphi,\psi)=\|\varphi-\psi\|_X$. Given any $\varphi(t,x)\in L$, we define the map

$$\phi:\varphi\to\phi(\varphi)\in X$$

so that $\chi(t,x)=\phi(\varphi)$ is solution to the Gauchy problem

$$\begin{aligned}&(\partial_t^2-\Delta)\chi_j=F_j(\varphi,\nabla\varphi), \quad j=1,\dots,N, \quad 0\leqq t\leqq T,\\ &(\partial_t^2-\Delta+M_j^2)\chi_j=F_j(\varphi,\nabla\varphi), \quad j=N+1,\dots,K, \quad 0\leqq t\leqq T,\\ &\chi_j=\varepsilon f_j, \quad \partial_t\chi_j=\varepsilon g_j \quad \text{for} \quad t=0, \quad j=1,\dots,K.\end{aligned}\tag{5.1}$$

We shall show that $\phi=\phi(\sigma,k,\varepsilon,\delta,T)$ is a contraction map from L into itself provided $\delta>0$, $\sigma>0$, $\varepsilon>0$ are chosen sufficiently small and independent of T.

First, we note that the assumptions concerning the non-linear functions $C_j(\varphi,\nabla\varphi)$ guarantee that

$$\|C_j(\varphi,\nabla\varphi)(t)\|_k\leqq(1+t)^{-2}\|A\varphi(t)\|_k P(F,k,|\varphi(t)|_{[k/2]+1,-1}),\tag{5.2}$$

where $t\geqq 0$ and $P(F,k,M)=O(|M|^2)$ near $M=0$. Moreover the quadratic part $Q(\nabla\varphi)=Q(t,x,\nabla\varphi)$ of the non-linear function F in (1.1) satisfies the estimate

$$\|Q(\nabla\varphi)(t)\|_k\leqq C(1+t)^{-1}\|A\varphi(t)\|_k|\varphi(t)|_{[k/2]+1,-1}.\tag{5.3}$$

Further, we combine Theorem 2.2 and Lemma 2.4 and derive from (5.1) the inequality

$$W_{2k,0}(\chi;t)\leqq C\left(\varepsilon+\sum_{j=1}^{K}\int_0^t((1+s)\|F_j(s)\|_{2k-1}+\|F_j(s)\|_{2k})ds\right).$$

From (5.2), (5.3) and Lemma 2.1 we find

$$W_{2k,0}(\chi;t)\leqq C\left(\varepsilon+\int_0^t(1+s)^{-1}W_{2k,0}(\varphi;s)P(E_{k+1}(\varphi;s))ds\right),\tag{5.4}$$

where $P(z)=O(|z|)$ near $z=0$. Applying the assumption $\varphi\in L$ we get

$$W_{2k,\delta}(\chi;t)\leqq C(\varepsilon+\sigma^2/\delta).\tag{5.5}$$

The application of Lemma 2.1 gives

$$\|Q(\nabla\varphi)(t)\|_{2k-1} \leqq C(1+t)^{-2}\|\varphi(t)\|_{2k}|\varphi(t)|_{k+1,-1}.$$

The estimates of Theorem 2 and Theorem 2.5 together with (5.2) yield

$$E_{k+1}(\chi;t) \leqq C(\varepsilon + W_{2k,\delta}(\varphi;t)P(E_{k+1}(\varphi;t)) \tag{5.6}$$

provided $\delta < 1/2$ and $k+1+5<2k$, i.e. $k>6$. From (5.4), (5.6) and the definition of the space L we deduce

$$\|\chi\|_X \leqq C'(\varepsilon + \sigma^2/\delta)$$

with some constant C' independent of $T, \varepsilon, \sigma, \delta$. Thus, choosing

$$\delta < 1/2, \quad \sigma \leqq \delta/(4C'), \quad \varepsilon \leqq \sigma/(4C'), \tag{5.7}$$

we conclude that $\|\chi\|_X \leqq \sigma/2$ and hence $\chi = \phi(\varphi) \in L$ provided $\varphi \in L$.

To show that ϕ is a contraction map we take $\varphi, \psi \in L$. Then

$$\chi^* = \phi(\varphi) - \phi(\psi)$$

is a solution to the Cauchy problem

$$\begin{aligned}
&(\partial_t^2 - \Delta)\chi_j^* = F_j(\varphi, \nabla\varphi) - F_j(\psi, \nabla\psi), \quad j=1,\dots,N, \quad 0 \leqq t \leqq T,\\
&(\partial_t^2 - \Delta + M_j^2)\chi_j^* = F_j(\varphi, \nabla\varphi) - F_j(\psi, \nabla\psi), \quad j=N+1,\dots,K, \quad 0 \leqq t \leqq T,\\
&\chi_j^* = 0, \quad \partial_t \chi_j^* = 0 \quad \text{for} \quad t=0, \quad j=1,\dots,K.
\end{aligned}$$

To estimate the quadratic part Q of F we apply the relation

$$Q(\nabla\varphi, \nabla\varphi) - Q(\nabla\psi, \nabla\psi) = Q(\nabla\varphi - \nabla\psi, \nabla\varphi) + Q(\nabla\psi, \nabla\varphi - \nabla\psi)$$

and via Lemma 2.1 we get

$$\begin{aligned}
&\|(Q(\nabla\varphi) - Q(\nabla\psi))(t)\|_{k-1}\\
&\qquad \leqq C(1+t)^{-2}\|(\varphi-\psi)(t)\|_k(|\varphi(t)|_{[k/2]+1,-1} + |\psi(t)|_{[k/2]+1,-1})\\
&\qquad + C(1+t)^{-2}|(\varphi-\psi)(t)|_{[k/2]+1,-1}(\|\varphi(t)\|_k + \|\psi(t)\|_k)\\
&\qquad \leqq (1+t)^{-2+\delta}\|\varphi-\psi\|_X P(F,k,\|\varphi\|_X + \|\psi\|_X), \quad \delta > 0.
\end{aligned}$$

Similarly, we have

$$\|(Q(\nabla\varphi) - Q(\nabla\psi))(t)\|_{2k} \leqq C(1+t)^{-1+\delta}\|\varphi-\psi\|_X(\|\varphi\|_X + \|\psi\|_X).$$

Now we replace the estimation of the cubic term C_j given in (5.2) by the following inequality

$$\begin{aligned}
&\|(C_j(\varphi, \nabla\varphi) - C_j(\psi, \nabla\psi)(t)\|_{2k}\\
&\qquad \leqq (1+t)^{-2+\delta}\|\varphi-\psi\|_X P(F,k,\|\varphi\|_X + \|\psi\|_X),
\end{aligned}$$

$P(F,k,\lambda) = O(|\lambda|)$. Repeating the application of L^∞ and L^2 estimates for the wave and Klein-Gordon equation we get

$$\|\chi^*\|_X = \|\phi(\varphi) - \phi(\psi)\|_X \leqq C\sigma\delta^{-1}\|\varphi-\psi\|_X$$

with some constant C independent of T, σ, δ. Taking $\sigma \leqq \delta/(2C)$ we conclude that ϕ is a contraction map.

The application of the contraction mapping principle yields the existence and uniqueness of solution to (1.1) in the complete metric space $L(\sigma, k, \varepsilon, \delta, T)$ provided $k > \delta$ and the positive numbers σ, ε, δ are chosen sufficiently small and independent of T. To show that the solution $u \in L$ to (1.1) is a smooth function we use the inclusion

$$L(\sigma, k, \varepsilon, \delta, T) \subset C^{k+1}([0, T] \times \mathbb{R}^3; \mathbb{C}^K).$$

From the inclusion $L(\sigma, k+1, \varepsilon, \delta, T) \subset L(\sigma, k, \varepsilon, \delta, T)$ and $L(\sigma, k, \varepsilon, \delta, T_1) \subseteq L(\sigma, k, \varepsilon, \delta, T_2)$, $T_1 \geqq T_2$, and the uniqueness of the solution in L we conclude that there exists a global smooth solution u supported in $\{|x| \leqq t+R\}$.

Turning to the uniqueness of the solution, we see that it is sufficient to show that for any smooth solution $u(t,x)$ to (1.1) one can prove

$$u \in L(\sigma, k, \varepsilon, \delta, T). \tag{5.8}$$

First, we note that the usual energy estimates and the Gronwall lemma show that if $\varphi(t,x)$ is a smooth solution to the Cauchy problem

$$(\square + M)\varphi = F(t, x, \varphi, \nabla\varphi), \quad \text{for} \quad |x - x^0| \leqq t^0 - t, \qquad M \geqq 0,$$
$$\varphi(0,x) = \partial_t \varphi(0,x) = 0 \quad \text{for} \quad |x - x^0| \leqq t^0,$$

then $\varphi(t,x) = 0$ for $|x - x^0| \leqq t^0 - t$. This observation shows that the solution to (1.1) is supported into the cone $\{|x| \leqq t+R\}$. To prove (5.8) it remains to check the inequality

$$\|\varphi\|_X \leqq \sigma.$$

This follows from the estimates (5.4) and (5.6) applied with $\chi = \varphi$. In fact, from (5.4) with $\chi = \varphi$ we derive

$$W_{2k,0}(\chi; t) \leqq C\left(\varepsilon + \int_0^t (1+s)^{-1} W_{2k,0}(\chi; s) P(E_{k+1}(\chi; s)) ds\right).$$

Applying the Gronwall lemma we deduce

$$W_{2k,0}(\chi; t) \leqq C\varepsilon \exp\left(\ln(2+t) \sup_{0 \leqq s \leqq t} P(E_{k+1}(\chi; s))\right)$$

so

$$W_{2k,\delta}(\chi; t) \leqq C\varepsilon \sup_{0 \leqq s \leqq t} (1+s)^{-\delta + \sup\{P(E_{k+1}(\chi;\tau)); 0 \leqq \tau \leqq s\}}.$$

On the other hand from (5.6) and the Gronwall lemma we get with $f(t) = E_{k+1}(\chi; t)$ the following estimates

$$f(t) \leqq C\varepsilon\left(1 + h(f; t) \sup_{0 \leqq s \leqq t} (1+s)^{h(f;s) - \delta}\right), \qquad t \geqq 0, \qquad f(0) \leqq C\varepsilon. \tag{5.9}$$

Here $h(f; t) = \sup\{g(f(\tau)); 0 \leqq \tau \leqq t\}$, $g(u)$ is a nonnegative function satisfying the estimate

$$g(u) \leqq C|u| \quad \text{for} \quad |u| \leqq 1 \tag{5.10}$$

and C is a constant independent of $t, \varepsilon, \delta, f$.

To finish the proof it suffices to apply the following.

Lemma 5.1. *Suppose $f(t) \geqq 0$ is a continuous function, $g(u)$ is a bounded function and the estimate* (5.10) *is fulfilled. Then for any δ, $0 < \delta < C$, one can find $\varepsilon = \varepsilon(\delta) > 0$, such that* (5.9) *implies $f(t) \leqq 2C\varepsilon$, $t \geqq 0$.*

Proof. Let $T = T(\delta, C, f)$ be determined by

$$T = \sup\{t \geqq 0; f(\tau) \leqq \delta/C, 0 \leqq \tau \leqq t\}.$$

If $C^2\varepsilon \leqq \delta/2$, then the estimates (5.9) yield $f(0) \leqq C\varepsilon \leqq \delta/(2C)$. Hence, $T > 0$ is correctly defined. For $0 \leqq t < T$ we have $f(t) \leqq 1$, since $\delta \leqq C$, so (5.10) implies with $h(f; t) = \sup\{g(f(\tau)); 0 \leqq \tau \leqq t\}$

$$(1+t)^{h(f;t)-\delta} \leqq (1+t)^{C\delta/C-\delta} = 1.$$

Applying the first estimate in (5.9) together with (5.10) we obtain

$$f(t) \leqq C\varepsilon(1 + CF(t)), \quad 0 \leqq t < T, \quad F(t) = \sup\{f(\tau); 0 \leqq \tau \leqq t\}.$$

Hence, $F(t) \leqq C\varepsilon(1 + CF(t))$ and choosing $C^2\varepsilon \leqq 1/2$ we find

$$F(t) = \sup\{f(\tau); 0 \leqq \tau \leqq t\} \leqq 2C\varepsilon, \quad 0 \leqq t < T. \tag{5.11}$$

With $2C\varepsilon \leqq \delta/(2C)$ we have $f(t) \leqq \delta/(2C)$ and from the definition of T we conclude that $T = \infty$. Thus, (5.11) completes the proof of the Lemma. □

The application of the above Lemma guarantees that $\|\chi\|_X$ is small and this completes the proof of (5.8). The application of the contraction mapping principle yields the uniqueness of the solution.

This completes the proof of Theorem 1. □

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