Math. Z. 203, 553–567 (1990)

Mathematische Zeitschrift

Symplectic Topology and Hamiltonian Dynamics II

Ivar Ekeland[1,*] and Helmut Hofer[2,**]

[1] Ceremade, Université Paris-9, Dauphine, F-75775 Paris, Cedex 16, France
[2] Ruhr-Universität Bochum, D-4630 Bochum 1, Federal Republic of Germany

I. Symplectic Capacities

Consider the real vectorspace $\mathbb{C}^n$ which we equip with the standard symplectic form ω defined by

$$\omega(\xi, \eta) = \mathrm{Im}(\xi, \eta).$$

Here $(\xi, \eta) = \sum_{k=1}^{n} \xi_k \bar{\eta}_k$. A map $T\colon \mathbb{C}^n \to \mathbb{C}^n$ is called symplectic if it is $\mathbb{R}$-linear and preserves ω, that is

$$T^* \omega = \omega.$$

Symplectic maps in $\mathbb{C}^n$ build a group which we denote by $Sp(n)$. A smooth map $f\colon \mathbb{C}^n \to \mathbb{C}^n$ is called a symplectic diffeomorphism if f is a diffeomorphism and

$$f^* \omega = \omega.$$

We denote the symplectic diffeomorphism group by $\mathscr{D}(n)$. Denote by $\mathscr{P}(\mathbb{C}^n)$ the power set of $\mathbb{C}^n$.

Definition 1. A symplectic capacity on $\mathbb{C}^n$ is a map $c\colon \mathscr{P}(\mathbb{C}^n) \to [0, +\infty) \cup \{+\infty\}$ having the following properties.

(M) 1) $c(S) \leqq c(T)$ if $S \subseteq T$
2) $c(f(S)) = c(S)$ for $f \in \mathscr{D}(n)$
3) $c(\alpha S) = \alpha^2 c(S)$ for $\alpha \in \mathbb{R}$

(N) 1) $c(B^{2n}(1)) > 0$
2) $c(B^2(1) \times \mathbb{C}^{n-1}) < +\infty$

* Supported by U.S. Army contract DAJA 45-88-C-0009
** Supported by the Alfred P. Sloan Foundation and NSF grant DMS 88-03496

Here B^{2n} is the Euclidean $2n$-ball in $\mathbb{C}^n$. Axioms (*M*) 1) and 2) say that c is a monotonic symplectic invariant. (*M*) 3) says that it has conformal behavior with respect to conformal symplectic maps. Note that (*M*) would be satisfied by the map

$$S \to (\text{outer measure}\,(S))^{1/n}. \tag{1}$$

In order to exclude such invariants, which are more related to the volume preserving character of symplectic maps rather than to their "symplectic character", we impose condition (*N*) (N = nontriviality). (*N*) 1) says that c is locally nontrivial and (*N*) 2) excludes the pathology (1) if $n \geqq 2$.

In an earlier paper [3] we proved the existence of a symplectic capacity c_1 satisfying in addition to (*M*) and (*N*) the normalisation property

$$c_1(B^{2n}(1)) = c_1(B^2(1) \times \mathbb{C}^{n-1}), \tag{2}$$

using Hamiltonian dynamics. The first such invariant was introduced by Gromov in [8] using first order elliptic systems.

In this paper we shall do away with the requirement (2). In fact we shall construct a sequence c_k, $k=1, \ldots,$ of distinct symplectic capacities with

$$c_1 \leqq c_2 \leqq c_3 \ldots . \tag{3}$$

Each of the c_k has a remarkable representation property which can be described as follows. Given a bounded domain U with smooth boundary ∂U of restricted contact type the capacity $c_k(U)$ can be represented by a closed characteristic on ∂U. We refer the reader to [3] for background information. We just recall that a compact smooth hypersurface Δ in $\mathbb{C}^2$ is said to be of restricted contact type if there exists a 1-form λ on $\mathbb{C}^n$ such that $d\lambda = \omega$ and $\lambda(x, \xi) \neq 0$ for every nonzero $(x, \xi) \in \ker(\omega \,|\, \Delta)$, where

$$\ker(\omega \,|\, \Delta) := \{(x, \xi) \in T\Delta \,|\, \xi \perp_\omega T_x \Delta\}.$$

A closed characteristic on Δ is a compact leaf of the foliation associated to the distribution $\ker(\omega \,|\, \Delta) \to \Delta$ on Δ.

We shall compute $c_k(\Omega)$ in some simple cases and we shall prove an embedding result.

Let us also note that the existence of a symplectic capacity c satisfying (*M*) and (*N*) already characterizes symplecticity in the following sense. Consider the group G_Ω of linear maps preserving $\Omega := \omega^n$. Then $G_\Omega \supset Sp(n)$. Now assume G_c is the subgroup of G_Ω consisting of all linear maps in G_Ω which preserve the capcity of linear ellipsoids. Here a linear ellipsoid is of the form

$$\{x \in \mathbb{C}^n \,|\, q(x) < 1\}$$

where $q: \mathbb{C}^n \to \mathbb{R}$ is a positive definite quadratic form.

Then, following the arguments in [3],

- $G_c = Sp(n)$ if n is odd
- $G_c = Sp(n) \cup \varphi \circ Sp(n)$ if n is even.

Here $\varphi: \mathbb{C}^n \to \mathbb{C}^n$ is antisymplectic: $\varphi^* \omega = -\omega$.

Using the continuity properties of a symplectic capacity as derived from the axioms in Proposition 3 of Chap. II one can derive again C^0-stability results as proved in [3]. This is one aspect of the importance of symplectic capacity. Somewhat surprisingly, capacities can be seen as a consequence of an additional structure in a variant of Floer-Homology for Hamiltonian systems as was shown in [7]. Moreover, the capacities introduced here are interrelated by a product property, [7]. There are also alternative definitions for capacities based on Hamiltonian dynamics which can be defined for every symplectic manifold, [11].

II. Construction of the c_k

We start with the functional analytic framework which is the same as in [3]. We introduce the Hilbert space E consisting of all $x \in L^2(\mathbb{R}/\mathbb{Z}, \mathbb{C}^n)$ such that its Fourier series

$$x = \sum_{k \in \mathbb{Z}} e^{2\pi i k t} x_k, \qquad x_k \in \mathbb{C}^n \tag{1}$$

satisfies

$$\sum |k| \, |x_k|^2 < \infty. \tag{2}$$

The inner product in E is defined by

$$(x, y) := \langle x_0, y_0 \rangle + 2\pi \sum |k| \langle x_k, y_k \rangle \tag{3}$$

where $\langle \cdot, \cdot \rangle: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ is the real scalar product defined by $\langle \cdot, \cdot \rangle = \mathrm{Re}(\cdot, \cdot)$. We denote by $\|x\|$ the norm corresponding to (3). E has a natural orthogonal splitting:

$$\begin{aligned} E &= E^- \oplus E^0 \oplus E^+ \\ E^- &= \{x \in E \mid x_k = 0 \text{ for } k \geq 0\} \\ E^0 &= \{x \in E \mid x_k = 0 \text{ for } k \neq 0\} = \mathbb{C}^n \\ E^+ &= \{x \in E \mid x_k = 0 \text{ for } k \leq 0\}. \end{aligned}$$

We denote by P^+, P^0 and P^- the corresponding orthogonal projection. We introduce the action on E as the quadratic form defined by

$$A(x) = -\tfrac{1}{2} \|x^-\|^2 + \tfrac{1}{2} \|x^+\|^2 \tag{4}$$

where $x^* = P^* x$ for $* \in \{+, -, 0\}$. If $x \in E$ is smooth one easily sees that

$$A(x) = \tfrac{1}{2} \int_0^1 \langle -i\dot{x}, x \rangle \, dt. \tag{5}$$

On E there is a natural S^1-action by phase shift defined by

$$(\theta * x)(t) = x(t+\theta)$$

for $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ and $x \in E$. Finally we need a specific set of homeomorphisms of E which we will denote by Γ. We say $h \in \Gamma$ provided $h: E \to E$ is a homeomorphism and

$$h(x) = e^{\gamma^+(x)} x^+ + x^0 + e^{\gamma^-(x)} x^- + K(x).$$

Here γ^+ and γ^-: $E \to \mathbb{R}$ are required to be continuous, S^1-invariant, mapping bounded sets into bounded sets, while $K: E \to E$ is continuous, S^1-equivariant, mapping bounded sets into precompact sets. In addition there must be a $\rho > 0$ such that $A(x) \leqq 0$ or $\|x\| \geqq \rho$ implies that $\gamma^+(x) = \gamma^-(x) = 0$ and $K(x) = 0$. It is easily checked that Γ is a group. Γ is similar to the group Γ introduced in [3] where the S^1-equivariance was not required. We construct now a pseudo-index theory in the sense of Benci, [1], associated to the Fadell-Rabinowitz-Index [5]. We recall the relevant points of the F-R-Index. Given a paracompact S^1-space X we build a free S^1-space $X \times S^\infty$ by letting S^1 act through the diagonal action. Here $S^\infty = \bigcup S^{2n-1}$, $S^{2n-1} \subset \mathbb{C}^n$. Taking the quotient with respect to the S^1-action we obtain a principal S^1-bundle

$$X \times S^\infty \to (X \times S^\infty)/S^1.$$

The classifying map

$$f: (X \times S^\infty)/S^1 \to \mathbb{C}P^\infty$$

induces an homomorphism

$$f^*: \bar{H}(\mathbb{C}P^\infty) \to \bar{H}((X \times S^\infty)/S^1) =: \bar{H}_{S^1}(X)$$

in Alexander-Spanier-Cohomology with rational coefficients. Here $\bar{H}_{S^1}$ is the well-known Borel construction for an S^1-equivariant cohomology theory. We know that $\bar{H}(\mathbb{C}P^\infty) = \mathbb{Q}[t]$, the generator t being of degree 2. We define the index of X denoted by $\alpha(X)$ to be the largest number k such that $f^*(t^{k-1}) \neq 0$. If $f^*(t^k) \neq 0$ for all k we define $\alpha(X) := +\infty$. If $X = \phi$ we put $\alpha(X) = 0$. Next we define the index of an S^1-equivariant subset ξ of E by

$$\operatorname{ind}(\xi) = \inf_{h \in \Gamma} \alpha(h(\xi) \cap S^+)$$

where S^+ is the unit sphere in E^+. We need the following:

Proposition 1. *Let X be a finite dimensional S^1-invariant subspace of E^+. Then*

$$\operatorname{ind}(E^- \oplus E^0 \oplus X) = \tfrac{1}{2} \dim X.$$

We shall sketch the proof for the convenience of the reader although it is given in principle (modulo notation) in [1] for a somewhat smaller group of homeomorphisms.

Proof. Taking $h=Id$ we see that

$$\operatorname{ind}(E^-\oplus E^0\oplus X)\leqq\alpha(X\cap S^+)=\tfrac{1}{2}\dim X$$

by a result in [5]. Next pick $h\in\Gamma$ such that with $F=E^-\oplus E^0\oplus X$

$$\alpha(h(F)\cap S^+)=\operatorname{ind}(F). \tag{6}$$

Arguing indirectly let us assume that $\operatorname{ind}(F)<\frac{1}{2}\dim(X)$. By the continuity property of the α-index as proved in [5] we find an open neighborhood U of $h(F)\cap S^+$ such that $\alpha(U)=\operatorname{ind}(F)$. Denote by Q_k: $E\to E_k=\{x+E\,|\,x_j=0 \text{ for } |j|>k\}$, the orthogonal projection. We show first that for large k

$$(Q_k h(F\cap E_k))\cap S^+\subset U. \tag{7}$$

If (7) does not hold we find a sequence (x_k) such that

$$\begin{aligned} &x_k\in E_k\cap F\\ &Q_k h(x_k)\in S^+\\ &Q_k h(x_k)\notin U. \end{aligned} \tag{8}$$

Using the special form of h we can write the two last conditions in the equivalent form

$$x_k^0+x_k^-+(e^{-\gamma^-(x_k)}P^-+P^0)\,Q_k K(x_k)=0$$

$$\|Q_k h(x_k)\|=1,\qquad Q_k h(x_k)\notin U.$$

By the properties of γ^- and K the sequence $(x_k^0+x_k^-)$ is precompact. Let us show that (x_k^+) remains bounded. Since $x_k^+\in X$ this fact will imply that (x_k) is precompact. If (x_k^+) is unbounded we may assume after taking a subsequence that $\|x_k^+\|\to+\infty$. Then by using the properties of h we see that $h(x_k)=x_k$ for k large enough. In particular we obtain the contradiction

$$\begin{aligned} +\infty&=\lim\|x_k^+\|\\ &=\lim\|Q_k x_k\|\\ &=\lim\|Q_k h(x_k)\|\\ &=1. \end{aligned}$$

Hence, summing up (x_k) is precompact. So we may assume eventually taking a subsequence that $x_k\to x_\infty$ as $k\to+\infty$. Taking the limit in (8) gives

$$\begin{aligned} &x_\infty\in F\\ &h(x_\infty)\in S^+\\ &h(x_\infty)\notin U, \end{aligned}$$

contradicting that fact that U was a neighborhood of $h(F)\cap S^+$. The map $x\to Q_k h(x)$ is the identity in E_k for $\|x\|\geqq\rho$. Moreover this map is also the identity on $E^0 \cap E_k = E^0$, which is the fixed point for the action since $A(x) \leqq 0$ for $x \in E^0$. By proposition 3.3 in [6] we have (for k large enough)

$$\alpha(Q_k h(F \cap E_k) \cap (S^+ \cap E_k)) \geqq \tfrac{1}{2} \dim X,$$

which completes the proof of the proposition, since by (7) this implies $\alpha(U) \geqq \frac{1}{2} \dim X$. □

Denote by $\mathscr{H}$ the set of all smooth Hamiltonians $H\colon \mathbb{C}^n \to (0, +\infty)$ such that

(9)
- $H \mid U \equiv 0$ for some open subset U of $\mathbb{C}^n$
- $H(x) = a\,|x|^2$ for $|x|$ large, where $a > \pi$, $a \notin \mathbb{N}\pi$.

We define now for $k \in \mathbb{N}$ and $H \in \mathscr{H}$ a number $c_{H,k} \in (0, +\infty) \cup \{\infty\}$ by

$$c_{H,k} = \inf\{\sup \Phi_H(\xi) \mid \xi \subset E \text{ is } S^1\text{-equivariant and } \operatorname{ind}(\xi) \geqq k\}, \tag{10}$$

where $\Phi_H\colon E \to \mathbb{R}$ is defined by

$$\Phi_H(x) = A(x) - \int_0^1 H(x(t))\,dt.$$

That $c_{H,k} > 0$ follows from the following observation. Let $H \in \mathscr{H}$ be flat at U and pick $x^0 \in U$. Arguing as in [3], find $\varepsilon > 0$ small such that

$$\Phi|_{x^0 + \varepsilon S^+} \geqq \beta > 0 \tag{11}$$

for some positive β. The group Γ gives enough freedom to find an $\tilde{h} \in \Gamma$ such that

$$\tilde{h}(S^+) = x^0 + \varepsilon S^+$$

Hence if $\operatorname{ind}(\xi) \geqq 1$ we infer that

$$\tilde{h}^{-1}(\xi) \cap S^+ \neq \emptyset$$

or equivalently

$$\emptyset \neq \xi \cap \tilde{h}(S^+) = \xi \cap (x^0 + \varepsilon S^+). \tag{12}$$

Comparing (11) and (12), we find that

$$c_{H,k} \geqq \beta > 0.$$

On the other hand it is clear that $c_{H,k+1} \geqq c_{H,k}$ by the monotonicity of the index. Further let us note that if the constant a occurring in the definition of H satisfies $a \in (j\pi, (j+1)\pi)$ (see (9)), then

$$\sup \Phi_H(E^- \oplus E^0 \oplus X_j) < \infty$$

for $X_j = \{x \in E^+ \mid x_k = 0 \text{ for } k > j\}$.

This shows that

$$0<\beta\leqq c_{H,1}\leqq c_{H,2}\cdots\leqq c_{H,nj}<+\infty.$$

It was proved in [3] that the Palais-Smale condition holds for Φ_H. A variant of the proof of [3], Proposition 2 gives the following:

Lemma 1. *If $H\in\mathscr{H}$ and U is an open S^1-invariant neighborhood of the critical set of Φ_H on some level $c>0$, then there exists $\varepsilon>0$ such that for some $h\in\Gamma$*

$$h(\Phi_H^{c+\varepsilon}\setminus U)\subset\Phi_H^{c-\varepsilon}. \qquad \square$$

Here $\Phi_H^d:=\Phi_H^{-1}((-\infty,d])$.

As a consequence of Lemma 1 the numbers $c_{H,j}$ are critical levels provided $c_{H,j}<+\infty$. Next we observe that for $H_1, H_2\in\mathscr{H}$ with $H_1\geqq H_2$ we have

$$c_{H_2,j}\leqq c_{H_1,j}.$$

Given a bounded set S of $\mathbb{C}^n$ we denote by $\mathscr{H}(S)$ the subset of $\mathscr{H}$ consisting of all $H\in\mathscr{H}$ such that

(13) $\qquad H\equiv 0$ on some open neighborhood U of $cl(S)$.

Here cl denotes the closure. Finally we put

(14) $$c_j(S)=\inf_{H\in\mathscr{H}(S)} c_{H,j}$$

For an unbounded set S we define

(15) $$c_j(S)=\sup\{c_j(T)\mid T\subset S,\ T\text{ bounded}\}.$$

As in [3] one verifies easily that (M1)–(M3) hold and that a representation result holds; namely

Proposition 2. *Let Δ be a connected smooth compact hypersurface of restricted contact type. Let B_Δ be the bounded component of $\mathbb{C}^n\setminus\Delta$. Then there exists for given j a closed characteristic P_j on Δ and a positive integer k_j such that*

$$c_j(\Delta)=c_j(B_\Delta)=k_j\left|\int\lambda P_j\right|$$

where $d\lambda=\omega$ on $\mathbb{C}^n$. $\square$

As a Corollary we find that

Corollary 1. $c_j(B^{2n}(1))\geqq\pi$.

Proof. The "smallest" closed characteristic on $S^{2n-1}(1)$ has $\left|\int\lambda P\right|=\pi$. $\square$

This proves $(N1)$. In order to obtain $(N2)$ take a smooth map $\varphi: \mathbb{R} \to [0, +\infty)$ such that $\varphi(s)=0$ for $|s| \leqq 2$ and $\varphi(s)=a|s|$ for $|s|$ large where $a \in (j\pi, (j+1)\pi)$ for some positive integer j. Define $\sigma: E \to \mathbb{R}$ by

$$\sigma(x) = \int_0^1 \varphi(|\pi_1 x|^2)\, dt$$

where $\pi_1: \mathbb{C}^n \to \mathbb{C}$ is the projection onto the first factor. If now S is a bounded subset of $B^2(1) \times \mathbb{C}^{n-1}$ we obtain

$$c_j(S) \leqq \inf_{\operatorname{ind}(\xi) \geqq j} \sup \Phi_\sigma(\xi)$$

where $\Phi_\sigma(x) = A(x) - \sigma(x)$.

Define $\tilde{\xi} \subset E$ by

$$\tilde{\xi} = E^- \oplus E^0 \oplus \hat{X}_j$$

where $\hat{X}_j = \{x \in E^+ \mid x_k = 0 \text{ for } k > j \text{ and } x_k \in \mathbb{C} = \mathbb{C} \times \{0\}^{n-1} \subset \mathbb{C}^n \text{ for } 1 \leqq k \leqq j\}$.

By Proposition 1

$$\operatorname{ind}(\tilde{\xi}) \geqq j$$

Hence

$$c_j(S) \leqq \sup \Phi_\sigma(\tilde{\xi})$$

It follows from the definition of φ that, for some constant $\gamma > 0$, we have

$$\Phi_\sigma(x) \leqq A(x) - \int_0^1 a|\pi_1 x|^2 + \gamma$$

since $j\pi < a < (j+1)\pi$, we must have $A(x) \leqq \int_0^1 a|\pi_1 x|^2$ on $\hat{X}_j$. So, for $x \in \tilde{\xi}$,

$$\Phi_\sigma(x) \leqq 0 + \gamma = \gamma < +\infty.$$

This implies

$$c_j(S) \leqq c < +\infty$$

for every bounded subset $S \subset B^2(1) \times \mathbb{C}^{n-1}$. □

This proves

Corollary 2. $c_j(B^2(1) \times \mathbb{C}^{n-1}) < \infty$ *for every* $j \in \mathbb{N}$.

Finally we note a nice continuity property of symplectic capacities. Consider the space defined by

$$\mathscr{S} = \{B_\Delta \mid \Delta \text{ is a smooth connected compact hypersurface of restricted contact type in } \mathbb{C}^n\}.$$

We introduce the Hausdorff metric on $\mathscr{S}$ by

$$d(B_{\Delta_1}, B_{\Delta_2}) = \sup_{\substack{x\in\Delta_1\\ y\in\Delta_2}} \{\operatorname{dist}(x, \Delta_2) + \operatorname{dist}(\Delta_1, y)\}.$$

Proposition 3. *A symplectic capacity c induces a continuous map* $(\mathscr{S}, d) \to \mathbb{R}$.

Proof. Let $B_\Delta \in \mathscr{S}$. We take a 1-form λ such that λ is nondegenerate on $\ker(\omega|\Delta) \to \Delta$ and $d\lambda = \omega$ on $\mathbb{C}^n$ as used in the definition of restricted contact type. We may of course assume that

$$\lambda = \tfrac{1}{2} \sum_{k=1}^{n} (p_k\, dq_k - q_k\, dp_k)$$

for $z = q + ip$ large. We define a vectorfield η with linear growth by

$$\lambda = i_\eta \omega = \omega(\eta, \cdot).$$

Then

$$\begin{aligned} L_\eta \omega &= d\, i_\eta + i_\eta\, d\omega \\ &= d\lambda \\ &= \omega. \end{aligned}$$

Now assume $B_{\Delta_k} \to B_\Delta$ in $\mathscr{S}$. We use the vectorfield η to obtain a flow $\mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n$, $(z, t) \to z \cdot t$. We observe that η is transversal to Δ. Hence for $\varepsilon > 0$ given and k large enough we have

$$B_\Delta \cdot (-\varepsilon) \subset B_{\Delta_k} \subset B_\Delta \cdot (\varepsilon).$$

Now, since the maps $\mathscr{L}_t: z \to z \cdot t$ are conformally symplectic, in fact $\mathscr{L}_t^* \omega = e^t \omega$, we obtain using axiom (M)

$$e^{-\varepsilon} c(B_\Delta) \leqq c(B_{\Delta_k}) \leqq e^{\varepsilon} c(B_\Delta).$$

Consequently,

$$c(B_{\Delta_k}) \to c(B_\Delta).$$ □

III. Some Examples

Let $0 < r_1 \leqq r_2 \leqq \ldots \leqq r_n < +\infty$ and write $r = (r_1, \ldots, r_n)$. Consider the ellipsoid

$$E(r) = \left\{ z \in \mathbb{C}^n \,\middle|\, \sum \left| \frac{z_i}{r_i} \right|^2 < 1 \right\}.$$

We associate to $r = (r_1, \ldots, r_n)$ a sequence (d_i) as follows. Consider all numbers of the form $k\pi r_j^2$ with $k \in \{1, 2, \ldots\}$ and $j \in \{1, \ldots, n\}$. If the same number is obtained for different choices of k and j we call the number of different choices the multiplicity of the number. Order those numbers and repeat them according to their multiplicity to obtain the sequence (d_i), $d_i = d_i(r)$. For example, if $r = (1, 1, \ldots, 1)$ we have

$$d_1 = \ldots = d_n = \pi$$
$$d_{n+1} = \ldots = d_{2n} = 2\pi$$
$$d_{2n+1} = \ldots = d_{3n} = 3\pi$$
$$\vdots$$

Proposition 4. *$c_j(E(r)) = d_j(r)$ for every j.*

Proof. Let us first assume that the numbers r_j^2 are linearly independent over $\mathbb{Z}$. Then the sequence $d_j(r)$ is strictly monotonic. Define $H(x) := \sum_{i=1}^{n} \left|\frac{z_i}{r_i}\right|^2$ and denote by $\mathscr{F}$ the class of all f: $[0, +\infty) \to [0, +\infty)$, f monotonic, such that $f \circ H \in \mathscr{H}(E(r))$.

Clearly,

$$c_j(E(r)) = \inf\{c_{f \circ H, j} \mid f \in \mathscr{F}\}.$$

Consider $\Phi_{f \circ H}$ defined by

$$\Phi_{f \circ H}(x) = A(x) - \int_0^1 f \circ H(x)\, dt.$$

The critical points of $\Phi_{f \circ H}$ are the solutions x of the problem

$$\dot{x} = f'(H(x))\, i\, H'(x)$$
$$x(0) = x(1).$$

This implies immediately that $y(t) := x(t/f'(H(x(0))))$ is a solution of

$$\dot{y} = i\, H'(y)$$
$$y(0) = y(T),$$

with $T = f'(H(x(0)))$. Hence, with $H(x(0)) =: \tau$

$$\Phi_{f \circ H}(x) = f'(\tau)\tau - f(\tau)$$
$$f'(\tau) = 2\pi p r_i^2$$

for some integer $p \geqq 0$. Now consider the $c_j(f \circ H)$. They are all distinct because otherwise $\Phi_{f \circ H}$ would have uncountably many S^1-orbits on the level $c_j = c_{j+1}$, say. (In fact, as in [1], formula 4.2, one shows that if $c_j = c_{j+1}$ for some j, then the critical set on level c_j has Fadell-Rabinowitz index at least 2.) This would imply uncountably many solutions of $\dot{x} = i H'(x)$ on $\partial E(r)$, which contradicts the nonresonance assumption on the r_i. So

$$c_{f \circ H, j} \geqq \lambda_{f \circ H, j} \tag{1}$$

where $\lambda_{f\circ H,j}$ is the j-th critical value of $\Phi_{f\circ H}$ (starting from the bottom, $\lambda_1=\min$). Let $X_j\subset E^+$ be spanned by the eigenvectors belonging to the first j positive eigenvalues of

$$-i\dot{h}=\lambda H'(h)$$

$$h(0)=h(1).$$

Then, since the eigenspaces are S^1-invariant,

$$E^-\oplus E^0\oplus X_j$$

has index j by Proposition 1. Moreover,

$$\sup \Phi_{f\circ H}((E^-\oplus E^0\oplus X_j)=\lambda_{f\circ H,j}.$$

Hence,

$$c_{f\circ H,j}\leqq \lambda_{f\circ H,j} \tag{2}$$

which proves Proposition 3 in the case that the $\{\pi r_i^2\}$ are independent over $\mathbb{Z}$. If r is arbitrary the result follows from the continuity property established in Proposition 3 observing that $E(r)\in\mathscr{S}$. □

Now define for $r=(r_1,\dots,r_n)$, $0<r_1\leqq r_2\dots\leqq r_n<+\infty$

$$D(r):=B^2(r_1)\times\dots\times B^2(r_n).$$

Proposition 5. $c_j(D(r))=\pi j r_1^2$.

Corollary 3. *If there exists a symplectic embedding $\Psi\colon D(r)\hookrightarrow B^{2n}(r')$, then*

$$\sqrt{n}\, r_1\leqq r'.$$

Proof of Corollary 3. Upon replacing $D(r)$ by $D(\delta r)$ for a given $\delta\in(0,1)$, in order to avoid problems on the boundary, we can extend $\Psi|D(\delta r)$ to a symplectomorphism $\tilde{\Psi}$ defined on all of $\mathbb{C}^n$ (cf. [B]) such that

$$\tilde{\Psi}(D(\delta r))=\Psi(D(\delta r)).$$

Hence

$$\tilde{\Psi}(D(\delta r))\subset B^{2n}(r').$$

Taking the n-capacity we obtain

$$\begin{aligned} n\pi\delta^2 r_1^2 &= c_n(D(\delta r))\\ &= c_n(\tilde{\Psi}(D(\delta r)))\\ &\leqq c_n(B^{2n}(r'))\\ &= \pi(r')^2. \end{aligned}$$

This is true for every $\delta \in (0, 1)$. Hence, taking the square root

$$\sqrt{n}\, r_1 \leqq r'$$

as claimed. □

Remark. Note that for $r=(r_1, \ldots, r_1)$ this result is optimal since we have the obvious embedding $D(r) \hookrightarrow B^{2n}(\sqrt{n}\, r_1)$. If $r_1 < r_2$, say, one might suspect that there are better symplectic invariants, which however have not been found yet.

Proof of Proposition 5. The proof of Proposition 5 consists of two parts. First we derive the estimate (assuming $r=(1, r_2, \ldots, r_n)$

$$c_j(D(r)) \leqq \pi j.$$

Then we show that $c_j(D(r)) \in \mathbb{Z}\pi$. Studying the α-index of the set of trajectories representing c_j we derive that $c_j < c_{j+1}$, implying the desired result. Without loss of generality assume $r_1 = 1$.

Take for $\varepsilon > 0$ a smooth map $f: \mathbb{R} \to \mathbb{R}$ such that

(4)
- $f(s)=0 \qquad s \leqq 1+\varepsilon$
- $f''(s)>0 \qquad s>1+\varepsilon$
- $f(s)=(j+\frac{1}{2})\pi |s|^2 \qquad s$ large
- $f'(s_0)=2j\pi s_0$ for $s_0 > 1+\varepsilon$ implies $f(s_0) \leqq \varepsilon$ and $s_0 \leqq 1+2\varepsilon$.

Define $\sigma: E \to \mathbb{R}$ by

$$\sigma(x) = \int_0^1 f(|\pi_1 x|)\, dt$$

where $\pi_1: \mathbb{C}^n \to \mathbb{C}$ is the projection onto the first factor. We define $\Phi_0(x) = A(x) - \sigma(x)$. From the definition of $c_j(D(r))$ it follows immediately that for $1 \leqq r_2 \leqq r_3 \ldots \leqq r_n$

$$c_j(D(1, r_2, r_3, \ldots, r_n) \leqq \sup \Phi_\sigma(\xi_j)$$

where

$$\xi_j = E^- \oplus E^0 \oplus X_j$$

and

$$X_j = \{x \in E^+ \mid x_k \in \mathbb{C} = \mathbb{C} x \{0\}^{n-1} \subset \mathbb{C}^n \text{ for } 1 \leqq k \leqq j \text{ and } x_k = 0 \text{ for } k > j\}.$$

$\sup \Phi_\sigma(\xi_j)$ is attained at some point $x \in X_j$ satisfying

$$-i\dot{x} = f'(|\pi_1 x|) \frac{\pi_1 x}{|\pi_1 x|}$$
$$x(0) = x(1).$$

Hence with $\tau=|(\pi_1 x)(0)|$

$$\begin{aligned}\sup \Phi_\sigma(x) &= \tfrac{1}{2} f'(\tau)\tau - f(\tau)\\ &\leqq \tfrac{1}{2} 2j\pi|\tau|^2 - f(\tau)\\ &\leqq j\pi(1+2\varepsilon)^2.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary we infer

$$c_j(D(1, r_2, r_3, \ldots r_n)) \leqq \pi j.$$

In particular,

$$c_j(B^2(1)\times\mathbb{C}^{n-1}) \leqq \pi j. \tag{5}$$

The other direction is more delicate. Using monotonicity it is enough to study $D:=D(1,1,1,\ldots,1)$.

Taking the right sequence of Hamiltonians in $\mathscr{H}(D)$ it is quite easy to show that $c_j(D)$ can be represented by a linear combination of loops on $S^1\times\ldots\times S^1$ as follows. Given a sequence x^k of critical points of Φ_{H_k} on level $c_{H_{k},j}$ there exist numbers $\delta_\ell\in\mathbb{C}$ and $j_\ell\in\{1,2,\ldots\}$ for $\ell\in\{1,2,\ldots,n\}$ such that

$$\sum j_\ell =: j' > 0$$

$$|\delta_\ell| = 1 \quad \text{or} \quad \delta_\ell = 0$$

and a subsequence of (x^k) converges to

$$(\delta_1 e^{2\pi i j_1 t}, \ldots, \delta_n e^{2\pi i j_n t})$$

Take a sequence (f_k) of smooth maps satisfying

- $f_k: \mathbb{R}\to\mathbb{R}$
- $f_k(s)=0 \qquad s\leqq 1+\frac{1}{k}$
- $f_k''(s)>0 \qquad s>1+\frac{1}{k}$
- $f_k(s)=(k+\frac{1}{2})\pi|s|^2 \qquad s>1+\frac{2}{k}$
- $f_k'(s_0)=2k\pi s_0$ for $s_0>1+\frac{1}{k}$ implies $f_k(s_0)\leqq\frac{1}{k}$.

It is clear that the Hamiltonians $H_k\in\mathscr{H}(D)$ defined by

$$H_k(x)=\sum_{i=1}^{n} f_k(|x_i|), \qquad x=(x_n,\ldots,x_n)\in\mathbb{C}^n$$

have the property

$$c_{H_k,j} \to c_j(D) \quad \text{as} \quad k \to \infty .$$

Our aim is of course to show that $j'=j$. So we know so far that $c_j(D) \in \{\pi, 2\pi, 3\pi, \ldots\}$. Consider the first $j \in \{1, 2, \ldots\}$ such that

$$c_j(D) = c_{j+1}(D).$$

If no such j exists we have $c_j(D) = \pi j$ by formula (5) and we are done. Hence we must have $c_j(D) = \pi j$ and $c_{j+1}(D) = \pi j$. Our functionals Φ_{H_k} decompose into a sum of functionals on the loop spaces of $\mathbb{C}$. In fact, $\Phi_{H_k}(x) = \sum_{i=1}^{n} \tilde{\Phi}_{f_k(|\cdot|)}(x_i)$. The positive critical set of $\tilde{\Phi}_{f_k}$ is of the form

$$\pi \leqq c_1^k < c_2^k < \ldots \quad \text{with} \quad \lim_k (c_{j+1}^k - c_j^k) = \pi. \tag{6}$$

Now consider the set $\sum$ defined by

$$\sum = \{(\delta_2 e^{2\pi i j_1 t}, \ldots, \delta_n e^{2\pi i j_n t}) \,|\, |\delta_\ell| = 1 \quad \text{or} \quad \delta_\ell = 0$$
$$j_\ell \in \{1, 2, \ldots\}, \sum j_\ell = j\}.$$

We note that $\alpha(\sum) = 1$. Hence we find an open neighborhood U of $\sum$ with

$$\alpha(U) = \alpha(\sum) = 1.$$

By our previous arguments we find that for k large enough the critical sets of Φ_{H_k} on levels $c_{H_k,j}$ and $c_{H_k,j+1}$ are subsets of U. Since Φ_{H_k} has the direct sum form, its gradient flow will have a product form. Since the critical values of Φ_{H_k} are sums of critical values of the $\tilde{\Phi}_{f_k}$ we see that if k is large enough in view of (5), that for given $\varepsilon > 0$ and $\delta \in (0, \pi)$ for a suitable $h \in \Gamma$

$$h(\Phi_{H_k}^{c_{H_k,j+1}+\varepsilon} \setminus U) \subset \Phi_{H_k}^{c_{H_k,j+1}+\varepsilon-\delta}. \tag{7}$$

Since by definition

$$\operatorname{ind}(\Phi_{H_k}^{c_{H_k,j+1}+\varepsilon}) \geqq j+1$$

we obtain as a consequence of $\alpha(U) = 1$ and the subadditivity of the α-index, see [5], that

$$\operatorname{ind}(\Phi_{H_k}^{c_{H_k,j+1}+\varepsilon} \setminus U) \geqq j.$$

Since

$$\begin{aligned} &\operatorname{ind}(h(\Phi_{H_k}^{c_{H_k,j+1}+\varepsilon} \setminus U)) \\ &\quad \geqq \operatorname{ind}(\Phi_{H_k}^{c_{H_k,j+1}+\varepsilon-\delta}) \\ &\geqq j, \end{aligned}$$

we obtain from (7)

$$\operatorname{ind}(\Phi_{H_k}^{c_{H_k,j+1}+\varepsilon-\delta}) \geqq j. \tag{8}$$

This is true for every large k for a given $\varepsilon>0$ and $\delta\in(0,\pi)$. So we obtain from (8) and the definition of $c_j(D)$

$$\pi j=c_j(D)\leqq c_{H_k,j+1}+\varepsilon-\delta$$

for every $\varepsilon>0$ and $\delta\in(0,\pi)$ and all large k. This implies

$$\pi j\leqq \pi j-\pi,$$

giving a contradiction. Consequently $c_j(D)$ is strictly increasing. This implies with our previous discussion

$$c_j(D(1,1,\ldots,1))=\pi j.$$

Using (M) we obtain therefore

$$c_j(D(r))=\pi j$$

as required. □

References

1. Benci, V.: On the critical point theory for indefinite functionals in the presence of symmetries. Trans. Am. Math. Soc. **274**, 533–572 (1982)
2. Benci, V., Rabinowitz, P.H.: Critical point theory for indefinite functionals. Invent. Math. **52**, 336–353 (1979)
3. Ekeland, I., Hofer, H.: Symplectic topology and Hamiltonian dynamics I. Math. Z. **200**, 355–378 see also: C.R. Acad Sci., Paris, Ser. I, **307**, 37–40 (1988)
4. Ekeland, I., Hofer, H.: Convex Hamiltonian energy surfaces and their periodic trajectories. Commun. Math. Phys. **113**, 419–469 (1987)
5. Fadell, E., Rabinowitz, P.H.: Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. Invent. Math. **45**, 139–173 (1978)
6. Fadell, E., Husseini, S., Rabinowitz, P.: Borsuk Ulam Theorems for arbitrary S^1-actions and applications. Trans. Am. Math. Soc. **274**, 345–360 (1982)
7. Floer, A., Hofer, H.: in preparation
8. Gromov, M.: Pseudo-holomorphic curves in symplectic manifolds. Invent. Math. **82**, 307–347 (1985)
9. Hofer, H.: On strongly indefinite functionals and application. Trans. Am. Math. Soc. **275**, 185–214 (1983)
10. Hofer, H., Zehnder, E.: Periodic solutions on hypersurfaces and a result by C. Viterbo, Invent. Math. **90**, 1–7 (1987)
11. Hofer, H., Zehnder, E.: A new capacity for symplectic manifolds. Proceedings of a Conference on the occasion of J. Moser's 60th Birthday. (to appear)
12. Viterbo, C.: A proof of the Weinstein Conjecture in $\mathbb{R}^{2n}$. Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **4**, 337–356 (1987)

Received March 14, 1989; in final form July 4, 1989