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Some remarks on the study of good contractions

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Let X be a complex projective variety with log terminal singularities admitting an extremal contraction in terms of Minimal Model Theory, i.e. a projective morphism $\varphi : X \longrightarrow Z$ onto a normal variety Z with connected fibers which is given by a (high multiple of a) divisor of the type $K_X + rL$, where r is a positive rational number and L is an ample Cartier divisor. We first prove that the dimension of any fiber F of φ is bigger or equal to $(r-1)$ and, if φ is birational, that $dimF \geq r$, with the equalities if and only if F is the projective space and L the hyperplane bundle (this is a sort of "relative" version of a theorem of Kobayashi-Ochiai). Then we describe the structure of the morphism φ itself in the case in which all fibers have minimal dimension with the respect to r. If φ is a birational divisorial contraction and X has terminal singularities we prove that φ is actually a "blow-up".

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Introduction.

Let X be a complex Fano manifold of dimension n and of index m, that is m is the largest integer for which $-K_X = mL$ with L an ample divisor; then $m \leq (n+1)$ and the equality holds if and only if $(X, L) = (\mathbf{P}^n, \mathcal{O}(1))$; this is a famous theorem of Kobayashi-Ochiai (see [K-O]).

We consider here a "relative" version of this theorem which holds also in the singular case. We will set up the problem properly in the section 1, we now briefly summarize saying that we consider a complex projective variety X with log terminal singularities admitting an extremal contraction in terms of Minimal Model Theory. That is we have a projective morphism $\varphi : X \to Z$ onto a normal variety Z with connected fibers which is given by a (high multiple of a) divisor of the type $K_X + rL$, where r is a positive rational number and L is a Cartier ample divisor.

We first prove that the dimension of any fiber F of φ is bigger or equal to $(r-1)$ and, if φ is birational, that $dimF \geq r$. Moreover if the equality holds then $F \simeq P^{(r-1)}$, respectively $F \simeq P^r$ in the birational case, and $L_{|F} \simeq \mathcal{O}(1)$ (see the theorem (2.2)).

Then we describe the structure of the morphism φ itself in the case in which all fibers have minimal dimension with the respect to r (section 3). If φ is of fiber type and $cod_X Sing(X) > dimZ$ then φ makes X a projective bundle over Z (actually a scroll relative to L); this is a result of Fujita (see [Fu1] and also [B-S]). If φ is a birational divisorial contraction and X has terminal singularities we prove that φ is actually a "blow-up" (see the theorems (3.1) and (3.2)). This implies in particular the existence of the *first reduction* (see the definition in [B-S]) for a polarized pair (X, L) where X is a variety with at most terminal singularities.

All these results were known in the smooth case (see section 4 in [A-W1]); the theorems in section 3 were proved in a previous paper (see [An]) under the strong assumption that X has Gorenstein singularities.

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1. Notations.

(1.0) We work with schemes defined over complex numbers: a variety X is a reduced separated scheme of finite type over $\mathbf{C}$.

By K_X we will denote its canonical divisor, if K_X is Cartier the associated line bundle will be denoted by the same name. More generally: we will confuse Cartier divisors and line bundles whenever it makes sense.

We will assume that X has at worst log terminal singularities; in particular K_X is a $\mathbf{Q}$-Cartier divisor and X has rational (thus Cohen-Macaulay) singularities. For the definition of log terminal singularities we refer to [K-M-M] as well for most of our definitions and notations. We will give however some basic definitions in order to state our main objects.

(1.1) A *contraction* is a proper map $\varphi : X \to Z$ of normal irreducible varieties with connected fibers. The map φ is birational or otherwise $dimZ < dimX$, in the latter case we say that φ is of fiber type. The exceptional locus $E(\varphi)$ of a birational contraction φ is equal to the smallest subset of X such that φ is an isomorphism on $X \setminus E(\varphi)$. We denote by F a non trivial fiber of φ.

The contraction φ is called *good* if the anti-canonical $\mathbf{Q}$-divisor $-K_X$ is φ-ample. We say that φ is *elementary* if $PicX/\varphi^*(PicZ) \simeq \mathbf{Z}$.

In order to understand the structure of φ "locally", i.e. in a neighborhood of a chosen fiber F, we assume that the target Z is affine.

For a good contraction φ we will consider a φ-ample line bundle L such that $K_X + rL$ is a pullback of a line bundle from Z for some rational number r; if Z is affine then $K_X + rL = \varphi^*(\mathcal{O}_Z)$. Such an L always exists for good contractions. If X is smooth then r can be chosen ≥ 1. We sometime say that $K_X + rL$ is a *good supporting divisor* for the contraction φ.

(1.2) Another equivalent point of view is the following: let X be a log terminal variety which is not minimal; that is K_X is not nef. Let L be an ample line bundle on X and define

$$\tau(X, L) = \min\{t \in \mathbf{R} : K_X + tL \text{ is nef}\};$$

τ is called the *nef value* (or the threshold value) of the pair (X, L) and it is a positive number. By the Kawamata's rationality theorem τ is a rational number (see [K-M-M], theorem (4.1.1)) and a high multiple of the divisor $K_X + \tau L$ gives a good contraction φ as above ($\tau = r$). The contraction φ is called in this case the *nef value morphism* (relative to (X, L)).

A fundamental feature of good contractions is the following (relative) vanishing theorem (see [K-M-M], section (1-2), or [E-W], corollary 6.11):

Theorem (1.3). (Kawamata-Viehweg-Kollár Vanishing theorem) *Let $\varphi : X \to Z$ be a good contraction. Assume that L is a φ-ample line bundle and that φ is supported by $K_X + rL$. Then for any integer $t > -r$ we have*

$$R^i\varphi_*(tL) = 0 \text{ for } i > 0.$$

Moreover if r is an integer

$$R^i\varphi_*(-rL) = 0 \text{ for } i > dimX - dimZ.$$

With the vanishing theorem in [A-W1] it was proved the following very useful theorem.

Theorem(1.4). *Let $\varphi : X \to Z$ be a good contraction supported by $K_X + rL$. Let F be a component of a fiber of φ. Assume moreover that*

$$dimF \leq r$$

or, if φ is birational, that

$$dimF \leq r + 1$$

Then $Bs|L| := supp(coker(\varphi^*\varphi_*L \to L))$ does not intersect F.

The next result deals with the case $n = 2$ and it is due to F.Sakai ([Sa]; see also [A-S]).

Theorem (1.5). *Let S be a normal surface and let L be an ample line bundle on S. Let C be an irreducible curve on S with $C^2 < 0$ and $(K_S + L).C = 0$ (C is called a redundant curve). Then C meets at most one singularity y such that*

(i) y is a rational double point of type A_n for some n;

(ii) the strict transform $\overline{C}$ of C through a minimal resolution of singularities, $\pi : \overline{S} \to S$, is a (-1)-curve meeting one of the end components of the chain of (-2)-curves of $\pi^{-1}(y)$. In particular C can be contracted to a smooth point.

We conclude this section with a definition which will be used to extend the last theorem in the case $dimX \geq 3$.

Definition (1.6). Let X be a normal variety; $p \in X$ is called a cA_n singularity if there exist $(n-2)$ hyperplane sections through p, $H_i, i = 1, ..., (n-2)$, intersecting scheme theoretically in a normal surface $\cap H_i := S$ and p is a A_n

singularity of S. Equivalently p is cA_n if it is locally analytically isomorphic to the (hypersurface) singularity given by

$$f + \sum_{i=1}^{(n-2)} t_i g_i$$

where $f \in k[x, y, z]$ is the polynomial $x^2 + y^2 + z^n, n \geq 1$ and $g_i \in k[x, y, z, t_1, ..., t_{(n-2)}]$.

Remark (1.6.1). If H is a hyperplane section of a normal projective variety X then K_X is locally free if and only if K_H is locally free; this follows from the residue isomorphism, or adjunction formula, $(K_X + H)_{|H} = K_H$. In particular it implies that a cA_n singularities is Gorenstein.

2. Relative Kobayashi-Ochiai criterion.

The following is a more general and "relative" version of the theorem of Kobayashi-Ochiai. The main ideas of proof are contained in the work of T. Fujita (see the book [Fu2]); here we essentially add the power of the base point free theorem (1.4). The special case $r = (n + 1)$ of the theorem was proved in [Ma].

Theorem (2.1). *Let X be a projective variety with log terminal singularities and let φ be a good contraction supported by $K_X + rL$; equivalently let φ be the nef value morphism of the pair (X, L), with $\tau(X, L) = r$ (see the definitions in (1.1) or in (1.2)). Let F be a component of a non trivial fiber $F_1 = \varphi^{-1}(z)$, F' be its normalization and let L' be the pull back to F' of the restriction of L to F. Let finally $\lfloor r \rfloor$ be the integral part of r and let $r' = -\lfloor -r \rfloor$.*

Then:

(I,i) $dimF \geq (r - 1)$;

(I,ii) if $dimF < r$ then $F \simeq \mathbf{P}^{(r'-1)}$ and $L_F = \mathcal{O}(1)$;

(I,iii) if $dimF < (r + 1)$ then $\Delta(F', L') = O$.

If moreover $dimF > dimX - dimZ$, for instance if φ is birational, then

(II,i) $dimF \geq r$;

(II,ii) if $dimF = r$ then $F \simeq \mathbf{P}^r$ and $L_F = \mathcal{O}(1)$;

(II,iii) if $dimF \leq (r + 1)$ then $\Delta(F', L') = O$.

If all components of the fiber satisfy $dimF < r$ in the case (I.ii) or $dimF \leq r$ in the case (II.iii), then the fiber is actually irreducible.

In the cases (I.iii) and (II.iii) with φ birational the fiber F is normal; therefore $\Delta(F, L) = 0$.

Proof. Let t be any integer with $t > -r$ or, more generally, $t \geq -r$ if $dimF > dimX - dimZ$; with this assumptions the proof goes similarly for the cases I) and II). The vanishing theorem (1.3) gives $R^s\varphi_*(tL) = 0$ for such integers t and for $s = dimF$.

We claim that we can "shrink" the vanishing of these groups to a component of a non trivial fiber, i.e. we claim that $H^s(F, tL) = 0$. We use the same argument of [Y-Z], lemma 4; for a different one see [Fu1], lemma (11.3). The

Formal Function Theorem (see theorem 11.1, section III in [Ha]), applied to the map $\varphi : X \to Z$, gives, for t as above,

$$0 = R^s\varphi_*(tL)^* = \lim_{\leftarrow} H^s(F_m, tL_{|F_m})$$

where F_m is the subscheme of X defined by the ideal sheaf $\mathcal{I}^m$, with $\mathcal{I}$ the ideal of F_1 has a fiber, and the inverse limit is taken with respect to m. On the other hand, for any pair of integer (k, m) such that $k > l$, we have a surjective map

$$H^s(F_k, tL|F_k) \to H^s(F_l, (tL)_{|F_l});$$

this follows by tensorizing the short exact sequence

$$0 \to \frac{\mathcal{I}^l}{\mathcal{I}^k} \to \mathcal{O}_{F_k} \to \mathcal{O}_{F_l} \to 0$$

with tL and, after taking the long cohomolgy sequence, noticing that $dim(Supp(\frac{\mathcal{I}^l}{\mathcal{I}^k}) \leq s$. In particular this implies that the natural projection map

$$\lim_{\leftarrow} H^s(F_m, tL_{|F_m}) \to H^s(F_k, tL|F_k)$$

is surjective for any positive integer k. Thus, taking $k = 1$, we get

$$H^s(F_1, tL|F_1) = 0.$$

F is a component of $(F_1)_{red}$, thus we can consider the exact sequence

$$0 \to \frac{\mathcal{I}_F}{\mathcal{I}} \to \mathcal{O}_{F_1} \to \mathcal{O}_F \to 0;$$

taking the long cohomology sequence, and since $dim(Supp(\frac{\mathcal{I}_F}{\mathcal{I}}) \leq s$, we finally obtain the claim.

Let $g : W \to F$ be a desingularization of F; by the Leray spectral sequence for g, exactly as in Lemma 2.4 of [Fu1], we get

$$H^s(W, g^*(tL)) = 0.$$

On the other hand, since $g^*(L)$ is nef and big on W, by the Kawamata-Viehweg vanishing theorem we have that $H^i(W, g^*(tL)) = 0$ for $t < 0$ and $i < s$.

Consider now the Hilbert polynomial $\chi(t) := \chi(W, g^*(tL))$; it is a polynomial in t of degree equal to $dimW = dimF$. By what proved above χ is zero for all integers t such that $0 > t > -r$; if $dimF > dimX - dimZ$ and r is an integer then χ is zero also for $t = -r$. The inequalities in (i) follows then immediately since $deg\chi \geq \{$*number of its zeros*$\}$.

Assume now that $dimF < r$ or, if $dimF > dimX - dimZ$, that $dimF \leq r$; from above we have that $H^s(W, g^*(tL)) = 0$ for any integer t such that $0 > t \geq -dimF$. We apply the proposition 2.2 in [Fu3] which implies that $F' = \mathbf{P}^{r'-1}$, resp. $\mathbf{P}^r$ if $dimF > dimX - dimZ$, and $L_F = \mathcal{O}(1)$.

By the theorem (1.4) we may assume that L is φ-spanned by global sections; therefore $F = F' = \mathbf{P}^{(r'-1)}$, resp. $\mathbf{P}^r$, and $L = \mathcal{O}(1)$.

Assume then that $dimF < (r+1)$ or $\leq (r+1)$ if $dimF > dimX - dimZ$; we have
$H^q(W, g^*(tL)) = 0$ for $q > 0$ and $t \geq (1 - dimW)$. We apply in this case the proposition 2.12 in [Fu3] which gives that $\Delta(F', L') = 0$.

We assume now that $dimF < r$, or $dimF \leq r$ if $dimF > dimX - dimZ$, for every component of a fiber; we want to prove that in this case the fiber is irreducible. To do that we apply an "horizontal slicing argument" as described and used in [A-W1] and then in [An]. More precisely, assume by contradiction that the fiber has (at least) two irreducible components intersecting in a subvariety of dimension $t \leq (r-1)$; by the base point freeness of L, (1.4), we can choose $t+1$ sections of L intersecting transversally in a variety with log terminal singularities and meeting the two irreducible components not in their intersection (we say that we "slice horizontally" with section of L). By construction the map φ restricted to this variety has non connected fibers and this is in contradiction with the lemma 2.6.3 in [A-W1].

We finally prove the normality of F in the cases (I,iii) and (II,iii) with φ birational. In the case X is a smooth 4-fold, $r = 1$ and φ is birational this was proved in the paper [A-W2]; we will only notice that this proof holds also in the more general set up. Namely, let C be any curve which is the scheme theoretic intersection of $s-1$ sections of L and the fiber F; we claim that $h^1(C, \mathcal{O}_C) = 0$. This follows from the vanishing theorem (1.3) using the formal function theorem as above and then the long exact sequence associated to the short sequence

$$0 \to -L \to O_F \to O_D \to 0,$$

where D is a section of L.

We can now apply the following proposition which is proved in [A-W2].

Proposition (2.2). *Let F be an irreducible (reduced) variety and let L be an ample and spanned line bundle; assume that for every curve C which is the scheme theoretic intersection of $s-1$ sections of L we have $g(C) =: h^1(\mathcal{O}_{|C}) = 0$. Then F is normal, L is very ample and the pair (F, L) has delta genus zero, $\Delta(F, L) = 0$ (in particular the pairs (F, L) are classified, see [Fu2]).*

Remark (2.3). If X is smooth and a component of the fiber has dimension $r-1$, resp. r in the birational case, then all components have this dimension; in fact, by slicing with sections of L, we can reduce to the case $r = 2$, resp $r = 1$. Then the claim is proved in [A-W2].

3. Divisorial contraction with "long ray"; blow-ups.

In this section we want to describe the structure of the morphism φ itself in the case in which all fibers have minimal dimension with the respect to r.

Assume that φ is of fiber type and $dimF = (r'-1)$ for all fibers F; if $cod_X Sing(X) > dimZ$ then Z is smooth and φ makes X a projective bundle over Z (actually a scroll relative to L); this is a result of Fujita in the smooth case (see [Fu1],lemma (2.12)) generalized to the singular case in [B-S], proposition (1.4) (they actually assume only that X is normal and with Cohen-Macaulay singularities).

If φ is birational we prove the following.

Theorem (3.1). *Let X be a normal complex variety with log terminal singularities. Let $\varphi : X \to Z$ be a good contraction onto an affine normal variety Z; let $K_X + rL = \varphi^*\mathcal{O}_Z$ for some φ-ample line bundle L. Assume that φ is birational and that $r \geq (n-1)$. Then the map φ is a divisorial contraction contracting a prime divisor E to a point, $\varphi_{|X\setminus E}$ is an isomorphism and $E \simeq \mathbf{P}^{(n-1)}$, $L_E = \mathcal{O}(1)$.*

Moreover every point $x \in E \subset X$ is a Gorenstein point for X, actually if it is not smooth it is a cA_n singularity; $E \cap SingX$ is either empty or of pure dimension $(n-2)$.

In particular if X has singularities in codimension 3 (for instance if X has terminal singularities) then X is smooth around E, $\varphi(E) = p$ is a smooth point of Z and φ is the blow up of p.

Proof. Since φ is birational we have that $dimE \leq (n-1)$. We apply the theorem (2.1.1): first we have that $dimE \geq dimF \geq (n-1)$, therefore the equality holds and φ contracts E to a point. Secondly it gives that E is irreducible and isomorphic to $\mathbf{P}^{(n-1)}$ while $L_E = \mathcal{O}(1)$.

We claim that in the linear system $|L|$ we can take $(n-2)$ divisors, $H_1, ..., H_{(n-2)}$, such that each H_i contains the point x and $H_1 \cap ... \cap H_i := X_i$ are normal varieties for $i = 1, ..., n-2$. We prove this by induction: since $L_i := L_{|X_i}$ is base point free and ample the linear system $|L_i - x|$ has finite base point. In particular a general divisor in $|L_i - x|$ has singularities in codimension two. Moreover every elements in $|L_i - x|$ have Cohen-Macaulay (C-M) singularities, since a Cartier divisor on a C-M variety has C-M singularities. Therefore by, Serre criterion, the general member of $|L - x|$ is normal; take it to be X_{i+1}.

Let $S := X_{(n-2)}$ and $\varphi' := \varphi_{|S}$; by adjunction the map φ' is supported by $K_S + L'$ where $L' := L_{|S}$. The map φ' contracts the irreducible curve $C = S \cap E$ to a point; moreover the curve C is rational, C^2 is negative by the Mumford-Grauert criterion and $(K_S + L').C = 0$. Then, by the theorem (1.5), we have that x is at worst an A_n singularities for S. Therefore x is, by definition (see (1.6)), at worst a cA_n singularity for X; in particular the singularities of X along E are Gorenstein.

The last part of the theorem follows now applying the main theorem of [L-S] (note that $-K_{X|E} = \mathcal{O}(1)$). In particular this theorem implies that $SingX \cap E$ is either empty or of pure dimension $(n-2)$.

Remark (1) The theorem extends the main result of Lipmann and Sommese [L-S] to the case of log-terminal non Gorenstein singularities.
(2) Bt the theorem the point $z := \varphi(E)$ is a smooth point of Z if $dimSingX \leq (n-3)$; it can be conjectured that it is always smooth, this is true for $n = 2$, as it follows from (1.5).
(3) The theorem implies the existence of the *first reduction* for pairs (X, L), with X a log terminal variety with $codimSingX \geq 3$ (for instance terminal singularities) and L an ample line bundle on X (see [B-S] and [An]). Namely we have

Corollary (3.1.1). *Let X be a projective variety with terminal singularities and let L be an ample line bundle on X. If the nef value of the pair (X, L)*

is $\tau = (n-1)$ and the nef morphism ϕ is birational then $\phi : X \longrightarrow X'$ is the simultaneous contraction to distinct smooth points of disjoint divisors $E_i \cong \mathbf{P}^{n-1}$ such that $E_i \subset reg(X)$, $\mathcal{O}_{E_i}(E_i) \cong \mathcal{O}_{\mathbf{P}^{n-1}}(-1)$ and $L_{E_i} \cong \mathcal{O}(1)$ for $i = 1, \dots, t$. Furthermore $L' := (\phi_* L)^{**}$ and $(K_{X'} + (n-1)L')$ are ample and $(K_X + (n-1)L) \cong \phi^*(K_{X'} + (n-1)L')$. The pair (X', L') is called the first reduction of the pair (X, L).

The next theorem was proved for $r = 1$ in [K-M] (see (4.9) (4.9.6) and (4.10.1) in [K-M]); we adapt here the argument of [K-M] for $r > 1$.

Theorem (3.2). *Let X be a normal variety with at most terminal singularities and let $\varphi : X \to Z$ be a good contraction. Assume that φ is of divisorial type, that $K_X + rL$ is a good supporting divisor for φ and that all fibers of φ have dimension $\leq r$. Then the exceptional set is a $\mathbf{Q}$-Cartier divisor E, $B := \varphi(E)$ has pure codimension $(n-1-r)$, $\varphi_*(\mathcal{O}_X(-mE)) = I_B^{(m)}$ ($I_B^{(m)}$ denotes symmetric power) and $X \simeq Proj_Y(\sum_{m=0}^{\infty} I_B^{(m)})$.*
In particular if B is a complete intersection then Z is smooth and X has only index 1 points.

Proof. We first claim that the general fiber of φ is contained in $Reg(X)$; to prove it we use a "vertical slicing" argument (see also [A-W1]). More precisely take $n-1-r$ general very ample divisors on Z, call them H_i, and consider the intersection $Z'' := \cap H_i$. The variety $X'' := \varphi^{-1}(Z'')$ has again terminal singularities, by the Bertini theorem, and the restriction of φ to X'' is given by a high multiple of $K_{X''} + rL_{|X''}$ and contracts a *general* fiber F, being now a divisor in X'', to a point. We can then apply the above theorem and obtain the claim.

The above argument gives also that Z'' is smooth; since it is an intersection of Cartier divisors this implies that Z is smooth along Z''. Therefore $SingZ \cap B$ is a strict subvariety of B

Let B' be the closed subset of B which is the image of fibers of φ intersecting $Sing(X)$. Let $S := SingZ \cup B'$; from the first part we have that $dimS \leq (n-3)$ and $\varphi^{-1}(S)$ has codimension at least two in X. On the other hand $Z - S$ and $X - \varphi^{-1}(S)$ are smooth; therefore by the theorem (4.1) in [A-W1] (see also the corollary (4.11) in [A-W1])

$$X - \varphi^{-1}(S) = B_{B-S}(Z - S)$$

($B_C(A)$ denotes the blow-up of $C \subset A$). Then the proof goes on exactly as at pages 577 and 578 of [K-M].

Corollary (3.2.1). *If in the above theorem we assume moreover that Z is factorial (this happens for instance if X is factorial) and that $dimSingX \leq (r-1)$, then B is a complete intersection and Z is smooth.*

Proof. Take a point $q \in B$. We can take r general sections of L, $\mathcal{D}_1, \dots, \mathcal{D}_r$, intersecting transversally in a smooth variety X' and intersecting the fiber $\varphi^{-1}(q)$ in a finite number of points. By the lemma (2.6.3) in [A-W1] we have that the map $\varphi_{|X'}$ has connected fibers, therefore it is an isomorphism with its image $Z' = \varphi(X')$. Therefore $Z' \subset Z$ is smooth; since Z' is an irreducible component of $\varphi(\mathcal{D}_1) \cap \ldots \cap \varphi(\mathcal{D}_r)$ and Z is factorial, Z is smooth in a neighborhood of Z'. Moreover B is a local complete intersection since it is a hypersurface of the smooth variety Z'.

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