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2-dimensional simple factors of $J_0(N)$

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In this paper we investigate abelian variety $\mathcal{A}_f$ which is derived from a newform $f \in S_2(\Gamma_0(N))$ an is $\mathbb{Q}$-simple factors of $Jac(X_0(N))$. We will develop algorithms for computing the period matrix of $\mathcal{A}_f$ and for determing when $\mathcal{A}_f$ is principally polarized. If $\mathcal{A}_f$ is 2-dimensional principally polarized, we give an algorithm for computing the associated hyperelliptic curve C with $Jac(C) \cong \mathcal{A}_f$.

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1 Introduction

For a positive integer N, let $S_2(N)$ denote the space of cusp forms of weight 2 for the group $\Gamma_0(N)$. Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(N)$ be a newform and O_f be the the ring generated over $\mathbb{Z}$ by the complex numbers a_n for all n. Then O_f is an order in a totally real algebraic number field K_f of finite degree d, say. Shimura defined in [Shi] a simple abelian variety $\mathcal{A}_f$ of dimension d, along with an action by O_f, both defined over $\mathbb{Q}$. $\mathcal{A}_f$ is a simple factor of the Jacobian variety $J_0(N)$ of the modular curve $X_0(N)$, and is dual to a subvariety of $J_0(N)$. This factor $\mathcal{A}_f$ is very important due to the generalized Shimura-Taniyama conjecture which asserts that any abelian variety A with real multiplication, both defined over $\mathbb{Q}$, is isogenous to a factor of $J_0(N)$ for a suitable N.

We investigate in this paper the explicit structure of $\mathcal{A}_f$, develop an algorithm for computing the period matrix of $\mathcal{A}_f$ and for determining when $\mathcal{A}_f$ is principally polarized. If $\mathcal{A}_f$ is a two dimensional principally polarized abelian variety, we give also an algorithm to determine the associated hyperelliptic curve C with $Jac(C) \cong \mathcal{A}_f$.

We start our treatment in §2 by showing that $\mathcal{A}_f$ can be obtained as a complex torus $\mathbb{C}^d/\Lambda_f$. Then we derive a formula for the the period matrix of $\mathcal{A}_f$ using certain intersection numbers $e_1, ..., e_d$. We will see that $\mathcal{A}_f$ is principally polarized if $e_1 = \cdots = e_d$.

In §3 we derive firstly an explicit formula for computing the intersection numbers on $H_1(X_0(N),\mathbb{Z})$. Then we show how to compute the period matrix of $\mathcal{A}_f$ explicitly. If $\mathcal{A}_f$ is two dimensional principally polarized, we give also an algorithm for computing the invariants of $\mathcal{A}_f$ and the associated hyperelliptic curve C with $Jac(C) = \mathcal{A}_f$. Our algorithms generalize some results of [Cre2] on modular elliptic curves. To illustrate the method we also give some examples and tables at the end of §3.

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2 The abelian variety $\mathcal{A}_f$

We summarize firstly some results of [Shi].

Let $N > 2$ be a integer, g be the genus of the modular curve $X_0(N)$; we consider the structure of $X_0(N)$ as a real 2-manifold whose first homology group $H_1(X_0(N),\mathbb{Z})$ has rank $2g$. Let $f(z)$ be a cusp form of weight 2 for $\Gamma_0(N)$, and let $\omega(f) = 2\pi i f(z)dz = f(z)\frac{dq}{q}$ be the associated differential. Then ω induces a holomorphic 1-form on $X_0(N)$, which we will also call ω, and the set of such ω forms a $\mathbb{C}$-vector space of dimension g.

Let $f = \sum a_n q^n \in S_2(\Gamma_0(N))$ be a newform, K_f be the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by the complex numbers a_n for all n, I_f be the set of all isomorphisms of K_f into $\mathbb{C}$, $d := \sharp I_f$, $\{f^{\sigma_1}, ..., f^{\sigma_d}\}$ ($\sigma_i \in I_f$) be the complete set of newforms conjugate to f over $\mathbb{Q}$. Shimura showed in [Shi] that there exist an abelian variety $\mathcal{A}_f$ rational over $\mathbb{Q}$ with the properties: $\mathcal{A}_f$ is a simple factor of $J_0(N)$, $dim(\mathcal{A}_f) = d$ and the differential 1-forms $\Omega^1(\mathcal{A}_f) \cong \sum_{\sigma\in I_f} \mathbb{C}\omega(f^\sigma)$.

Set $\mathbf{f} = (f^{\sigma_1}, ..., f^{\sigma_d})^t$ and $\omega(\mathbf{f}) = (\omega(f^{\sigma_1}), ..., \omega(f^{\sigma_d}))^t$, then the image of $H_1(X_0(N),\mathbb{Z})$ under the map

$$H_1(X_0(N),\mathbb{Z}) \longrightarrow \mathbb{C}^d$$

$$c \longmapsto \int_c \omega(\mathbf{f}) := (\int_c \omega(f^{\sigma_1}), ..., \int_c \omega(f^{\sigma_d}))^t$$

is a free $\mathbb{Z}$-module of rank $2d$, in other words it is a lattice Λ_f in $\mathbb{C}^d$. The map

$$P \longmapsto (\int_0^P \omega(f^{\sigma_1}), ..., \int_0^P \omega(f^{\sigma_d}))^t$$

is a morphism of $X_0(N)$ into $\mathbb{C}^d/\Lambda_f$, where 0 is a given point of $X_0(N)$. Shimura showed that $\mathcal{A}_f$ is isomorphic to $\mathbb{C}^d/\Lambda_f$ and $\mathcal{A}_f$ is dual to a subvariety of $J_0(N)$.

We recall now some results of [Hid].

Let $\mathbb{H}$ be the upper half plane. Denote by $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ the extended upper half plane, obtained by including the $\mathbb{Q}$-rational cusps $\mathbb{Q} \cup \{\infty\}$. The quotient space $X_0(N)(\mathbb{C}) = \Gamma_0(N) \setminus \mathbb{H}^*$ may be given the structure of a compact Riemann surface. By utilizing the well-known identity for the smooth compactification $X_0(N)$ of $\Gamma_0(N) \setminus \mathbb{H}$ and Poincaré duality we obtain a non-canonical isomorphism

$$H_1(X_0(N), \mathbb{Z}) \cong H^1(X_0(N), \mathbb{Z}) \cong H^1_p(\Gamma_0(N) \setminus \mathbb{H}, \mathbb{Z}) \cong H^1_p(\Gamma_0(N), \mathbb{Z})$$

where $H^1_p(\Gamma_0(N), \mathbb{Z})$ is the cuspidal cohomology of $\Gamma_0(N)$. Hida defined in [Hid] a pairing $\langle \cdot, \cdot \rangle_N$ on $H^1_p(\Gamma_0(N), \mathbb{Z})$ which comes from the cup product on the cohomology. This pairing corresponds to the intersection numbers on $H_1(X_0(N), \mathbb{Z})$ which is a non-singular skew-symmetric bilinear form on $H_1(X_0(N), \mathbb{Z})$.

Define a subspace $S(f)$ of $S_2(\Gamma_0(N))$ by

$$S(f) := \sum_{\sigma \in I_f} \mathbb{C} f^\sigma \subset S_2(\Gamma_0(N))$$

and let $W_f(\mathbb{R})$ be the isomorphic image of $S(f)$ in $H^1_p(\Gamma_0(N), \mathbb{R})$ under the Eichler-Shimura isomorphism, $dim_{\mathbb{R}}(W_f(\mathbb{R})) = 2d$. Put $L := H^1_p(\Gamma_0(N), \mathbb{Z})$ and $L_f := W_f(\mathbb{R}) \cap L$. Then L_f is a lattice in $W_f(\mathbb{R})$ of rank $2d$ and the $\mathbb{Q}$-linear span $W_f(\mathbb{Q})$ of L_f coincides with $W_f(\mathbb{R}) \cap H^1_p(\Gamma_0(N), \mathbb{Q})$. The binear form $\langle \cdot, \cdot \rangle_N$ is non-degenerate on $W_f(\mathbb{Q})$ and $\langle L_f, L_f \rangle_N \subset \mathbb{Z}$ if $N > 2$. Due to a well-known result in linear algebra we can obtain a symplectic base $\{\mathbf{a}_f, \mathbf{b}_f\} = \{a_1, ..., a_d, b_1, ..., b_d\}$ of L_f with

$$\langle (\mathbf{a}_f, \mathbf{b}_f), (\mathbf{a}_f, \mathbf{b}_f) \rangle_N = \begin{pmatrix} 0 & \Delta_f \\ -\Delta_f & 0 \end{pmatrix}$$

with $\Delta_f = \begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_d \end{pmatrix}$, where $e_1 | e_2 | \cdots | e_d$ are positive integers.

Remark: The number $e_1 e_2 \cdots e_d$ is the so-called cohomologic congrunce number which coincides with a product of special values of Theta functions of a newform f (cf. [Hid]).

Let Y be the orthogonal complement of $W_f(\mathbb{Q})$ in $H^1_p(\Gamma_0(N), \mathbb{Q})$ under $\langle \cdot, \cdot \rangle_N$. We obtain

$$L \subset H^1_p(\Gamma_0(N), \mathbb{Q}) = W_f(\mathbb{Q}) \oplus Y \tag{1}$$

Let M_f be the projection of L to the first direct summand of (1). Then M_f is also a lattice of $W_f(\mathbb{R})$ and $L_f \subset M_f \subset W_f(\mathbb{Q})$. By the definition L_f and M_f are stable under the Hecke operators. Let L_f^* be the dual lattice of L_f under $\langle \cdot, \cdot \rangle_N$ over $\mathbb{Z}$, namely

$$L_f^* := \{x \in W_f(\mathbb{Q}) \mid \langle x, y \rangle_N \in \mathbb{Z}, \forall y \in L_f\}.$$

It can be verified easily that $L_f^* = M_f$. Set $(\mathbf{a}_f^*, \mathbf{b}_f^*) := (\mathbf{a}_f, \mathbf{b}_f)\begin{pmatrix} \Delta_f^{-1} & 0 \\ 0 & \Delta_f^{-1} \end{pmatrix}$. Since $\langle (\mathbf{a}_f^*, \mathbf{b}_f^*), (\mathbf{a}_f, \mathbf{b}_f) \rangle_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, it follows that $\{\mathbf{a}_f^*, \mathbf{b}_f^*\}$ is a base of $L_f^* = M_f$ dual to $\{\mathbf{a}_f, \mathbf{b}_f\}$.

We have seen that $\mathcal{A}_f = \mathbb{C}^d/\Lambda_f$ with $\Lambda_f = \{\int_c \omega(\mathbf{f}) \,|\, c \in L\}$. Because $L \subset W_f(\mathbb{Q}) \oplus Y$, we can write $c = x + y$ with $x \in M_f$, $y \in Y$. Since $W_f(\mathbb{Q}) \perp Y$, we have $\int_y \omega(\mathbf{f}) = 0$ and $\int_c \omega(\mathbf{f}) = \int_x \omega(\mathbf{f})$. This implies that

$$\Lambda_f = \{\int_x \omega(\mathbf{f}) \,|\, x \in M_f\}.$$

Since $\{\mathbf{a}_f^*, \mathbf{b}_f^*\}$ is a base of M_f, the lattice Λ_f is spanned by the matrix $P_f := (\int_{\mathbf{a}_f^*} \omega(\mathbf{f}), \int_{\mathbf{b}_f^*} \omega(\mathbf{f}))$. Define two $d \times d$ matrices

$$A_f := (\int_{\mathbf{a}_f} \omega(\mathbf{f})) = \begin{pmatrix} \int_{a_1} \omega(f^{\sigma_1}) & \cdots & \int_{a_d} \omega(f^{\sigma_1}) \\ \vdots & & \vdots \\ \int_{a_1} \omega(f^{\sigma_d}) & \cdots & \int_{a_d} \omega(f^{\sigma_d}) \end{pmatrix},$$

$$B_f := (\int_{\mathbf{b}_f} \omega(\mathbf{f})) = \begin{pmatrix} \int_{b_1} \omega(f^{\sigma_1}) & \cdots & \int_{b_d} \omega(f^{\sigma_1}) \\ \vdots & & \vdots \\ \int_{b_1} \omega(f^{\sigma_d}) & \cdots & \int_{b_d} \omega(f^{\sigma_d}) \end{pmatrix}.$$

Since $(\mathbf{a}_f^*, \mathbf{b}_f^*) = (\mathbf{a}_f, \mathbf{b}_f)\begin{pmatrix} \Delta_f^{-1} & 0 \\ 0 & \Delta_f^{-1} \end{pmatrix}$, we obtain

$$P_f = (A_f, B_f)\begin{pmatrix} \Delta_f^{-1} & 0 \\ 0 & \Delta_f^{-1} \end{pmatrix} = (A_f\Delta_f^{-1}, B_f\Delta_f^{-1}).$$

Lemma 1

1. A_f is invertible.

2. $\Omega_1 := A_f^{-1}B_f\Delta_f^{-1}$ is symmetric and its imaginary part is positive definite, i.e., $\Omega_1 \in \mathbb{H}_d$, the d-dimensional Siegel half space.

In order to prove this lemma, we need Poincaré Duality:

Lemma 2 *For each cycle $\sigma \in H_1(X_0(N),\mathbb{Z})$ there exists a holomorphic differential $\phi \in \Omega^1(X_0(N))$ such that*

$$\int_\tau \phi = \sharp(\sigma,\tau),\ \forall \tau \in H_1(X_0(N),\mathbb{Z})$$

where $\sharp(\sigma,\tau)$ denotes the intersection number of σ and τ. Moreover, if $\psi \in \Omega^1(X_0(N))$ corresponds to τ, then

$$\int_{X_0(N)} \phi\wedge\psi = \sharp(\sigma,\tau).$$

See [GH] p59 for the proof. The following corallary can also be verified easily.

Corollary 1 *Let $\{u_1,...,u_{2g}\} \subset H_1(X_0(N),\mathbb{Z})$ be a base of $H_1(X_0(N),\mathbb{Q})$, $A = (a_{ij})$ be a $2g\times 2g$ matrix with $a_{ij} := \sharp(u_i,u_j)$. Then*

$$\int_{X_0(N)} \phi\wedge\psi = (\int_{u_1}\phi,...,\int_{u_{2g}}\phi)(-A^{-1})\begin{pmatrix}\int_{u_1}\psi\\ \vdots \\ \int_{u_{2g}}\psi\end{pmatrix}.$$

Proof of Lemma 1: Let $\mathbf{u}_Y \subset Y\cap L$ be a $\mathbb{Q}$-base of Y. Then $\{\mathbf{a}_f,\mathbf{b}_f,\mathbf{u}_Y\}$ is a $\mathbb{Q}$-base of L. Since $(\mathbf{a}_f,\mathbf{b}_f)\perp \mathbf{u}_Y$, we can write

$$\langle(\mathbf{a}_f,\mathbf{b}_f,\mathbf{u}_Y),(\mathbf{a}_f,\mathbf{b}_f,\mathbf{u}_Y)\rangle_N = \begin{pmatrix} 0 & \Delta_f & 0\\ -\Delta_f & 0 & 0\\ 0 & 0 & U_Y\end{pmatrix}.$$

By using the above corollary we obtain

$$\int_{X_0(N)} \phi\wedge\psi = (\int_{\mathbf{a}_f}\phi)\Delta_f^{-1}(\int_{\mathbf{b}_f}\psi)^t - (\int_{\mathbf{b}_f}\phi)\Delta_f^{-1}(\int_{\mathbf{a}_f}\psi)^t - (\int_{\mathbf{u}_Y}\phi)U_Y^{-1}(\int_{\mathbf{u}_Y}\psi)^t.$$

If ϕ or ψ is a linear combination of $\omega(f^{\sigma_1}),...,\omega(f^{\sigma_d})$, then $\int_{\mathbf{u}_Y}\phi = 0$ or $\int_{\mathbf{u}_Y}\psi = 0$. Therefore we obtain

$$\int_{X_0(N)} \phi\wedge\psi = (\int_{\mathbf{a}_f}\phi)\Delta_f^{-1}(\int_{\mathbf{b}_f}\psi)^t - (\int_{\mathbf{b}_f}\phi)\Delta_f^{-1}(\int_{\mathbf{a}_f}\psi)^t. \tag{2}$$

Let $\mathbf{c} = (c_1,...,c_d) \in \mathbb{C}^d$ be such that

$$\mathbf{c}A_f = \sum c_i \int_{\mathbf{a}_f}\omega(f^{\sigma_i}) = 0.$$

Consider the holomorphic differential form $\omega = \sum c_i\omega(f^{\sigma_i}) \in \Omega^1(X_0(N))$, we have $\int_{\mathbf{a}_f} \omega = 0$. This implies also that $\int_{\mathbf{a}_f} \bar{\omega} = 0$ and due to the formula (2) we obtain

$$\int_{X_0(N)} \omega \wedge \bar{\omega} = 0.$$

Therefore $\omega = 0$ and $c_1 = \cdots = c_d = 0$. This proves the assertion 1.

Moreover, since $\Omega^2(X_0(N)) = 0$, we have $\omega(f^{\sigma_i}) \wedge \omega(f^{\sigma_j}) = 0$ for $1 \le i, j \le d$ and the matrix

$$\begin{aligned}(0) \quad &= (\textstyle\int_{X_0(N)} \omega(f^{\sigma_i}) \wedge \omega(f^{\sigma_j}))\\ &= (\textstyle\int_{\mathbf{a}_f} \omega(\mathbf{f}))\Delta_f^{-1}(\int_{\mathbf{b}_f} \omega(\mathbf{f}))^t - (\int_{\mathbf{b}_f} \omega(\mathbf{f}))\Delta_f^{-1}(\int_{\mathbf{a}_f} \omega(\mathbf{f}))^t\\ &= A_f\Delta_f^{-1}B_f^t - B_f\Delta_f^{-1}A_f^t.\end{aligned}$$

This implies that $\Omega_1 := A_f^{-1}B_f\Delta_f^{-1}$ is symmetric.

We apply again the formula (2) to $\phi = \sum c_i\omega(f^{\sigma_i}) = \mathbf{c}\omega(\mathbf{f})$ and $\psi = \bar{\phi}$ with $\mathbf{c} \neq 0$:

$$\int_{X_0(N)} \phi \wedge \bar{\phi} = -2i(\mathbf{c}A_f)Im(\Omega_1)(\overline{\mathbf{c}A_f})^t$$

Since $\frac{i}{2}\int_{X_0(N)} \phi \wedge \bar{\phi} > 0$, $Im(\Omega_1)$ is positive definite. □

We have seen that Λ_f is spanned by the matrix

$$P_f = (A_f\Delta_f^{-1}, B_f\Delta_f^{-1}) = A_f(\Delta_f^{-1}, \Omega_1) = e_d^{-1}A_f(e_d\Delta_f^{-1}, e_d\Omega_1).$$

Set

$$\Omega_f := e_d\Omega_1, \; D_f := e_d\Delta_f^{-1} = \begin{pmatrix} e_d/e_1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Theorem 1 *There exists a base $(v_1, ..., v_d)$ of $\mathbb{C}^d$ (as a complex vector space) and a base $(\lambda_1, ..., \lambda_{2d})$ of Λ_f such that the matrix of $(\lambda_1, ..., \lambda_{2d})$ with respect to $(v_1, ..., v_d)$ takes the form (D_f, Ω_f) where Ω_f is symmetric and $Im(\Omega_f)$ is positive definite. The matrix Ω_f is called the* **period matrix** *of the abelian variety $\mathcal{A}_f$.*

Corollary 2 *If $e_1 = e_2 = \cdots = e_d$, then $\mathcal{A}_f$ is principally polarized.*

In that case $D_f = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ and $\Omega_f = A_f^{-1}B_f$. In next paragraph we will show how to compute the numbers $e_1, ..., e_d$ and the period matrix of $\mathcal{A}_f$ explicitly.

3 Algorithms for computing intersection numbers, period matrix and invariants of $\mathcal{A}_f$

3.1 Intersection numbers

We discuss firstly how to compute the intersection numbers.

Let $\Gamma = SL_2(\mathbb{Z})$, $E_N = \{(c,d) \in \mathbb{Z} \mid gcd(c,d,N) = 1\}$ the set of all elements of order N in $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ and $\sim$ the relation

$$(\lambda c, \lambda d) \sim (c,d),\ \lambda \in (\mathbb{Z}/N\mathbb{Z})^*.$$

Put $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) := E_N/\sim$. It is obvious that the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (c,d)$$

induces an identification between $\Gamma_0(N) \setminus \Gamma$ and $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.

Let W be the co-induced module of $\mathbb{Z}$ on Γ:

$$W := Coind^{\Gamma}_{\Gamma_0(N)}(\mathbb{Z}) \cong \mathbb{Z} \otimes_{\Gamma_0(N)} \Gamma \cong \mathbb{Z} \otimes (\Gamma_0(N) \setminus \Gamma) \cong \mathbb{Z}[\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})] \cong \mathbb{Z}^{\mu}$$

where $\mu = [\Gamma : \Gamma_0(N)]$. The operation of Γ on $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ (resp. W) is given by

$$(u,v).\begin{pmatrix} a & b \\ c & d \end{pmatrix} := (ua + vc, ub + vd).$$

By the Shapiro lemma there is a canonical isomorphism

$$H^1(\Gamma, W) \cong H^1(\Gamma_0(N), \mathbb{Z}).$$

Let $H^1_c(\Gamma, W)$ be the cohomology of Γ with compact support, and $\Gamma_\infty =< T >$ where $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We have then a long exact sequence:

$$\cdots \longrightarrow H^1_c(\Gamma, W) \xrightarrow{i^*} H^1(\Gamma, W) \longrightarrow H^1(\Gamma_\infty, W) \longrightarrow \cdots$$

The cohomology group $H^1_c(\Gamma, W)$ can be described as the group of cocycles which are trivial on Γ_∞:

$$H^1_c(\Gamma, W) = \{\phi : \Gamma \longrightarrow W \mid \phi(ab) = \phi(a).b + \phi(b), \forall a, b \in \Gamma; \phi(T) = 0\}.$$

Put $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $R = TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. It is well-known that S, R generate the group Γ. Since $S^2 = R^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, we have for $\phi \in H^1_c(\Gamma, W)$, $\phi(S).(1+S) =$

0 and $\phi(R).(1+R+R^2)=0$. Moveover, $0=\phi(T)=\phi(R).S+\phi(S)$, i.e., $\phi(R)=-\phi(S).S=\phi(S)$. Therefore

$$H_c^1(\Gamma,W)\cong Ker(1+S)\cap Ker(1+R+R^2).$$

We define now a bilinear form on W:

$$(\cdot,\cdot)_W : W\otimes W \longrightarrow \mathbb{Z}$$

$$(u,v)_W := \sum_{i=1}^{\mu} u_i v_i$$

for $u=(u_1,\dots,u_\mu)$, $v=(v_1,\dots,v_\mu)\in W\cong\mathbb{Z}^\mu$. This bilinear form has the following properties:

1. $(\cdot,\cdot)_W$ is well-defined and non-degenerate.
2. $(\cdot,\cdot)_W$ is symmetric.
3. $(u.r,v.r)_W=(u,v)_W$, $\forall r\in\Gamma$.

The composition of the cup product and the pairing $(\cdot,\cdot)_W$ gives us a bilinear form on $H_c^1(\Gamma,W)$:

$$\langle\cdot,\cdot\rangle_N : H_c^1(\Gamma,W)\times H_c^1(\Gamma,W)\xrightarrow{\cup} H_c^2(\Gamma,W\otimes W)\xrightarrow{(\cdot,\cdot)_W} H_c^2(\Gamma,\mathbb{Z})\xrightarrow{\epsilon}\mathbb{Z}$$

where the cup product of two 1-cocycles ϕ, ψ is defined as a 2-cocycle

$$(\phi\cup\psi)(a,b):=\phi(a).b\otimes\psi(b).$$

The isomorphism ϵ is given by (cf. [Hab] p278)

$$\epsilon(\rho)=\rho(R,S)-\frac{1}{2}\rho(S,S)-\frac{1}{3}(\rho(R,R)+\rho(R^2,R)),\ \forall\rho\in H_c^2(\Gamma,\mathbb{Z}).$$

Recall that $\phi(R)=\phi(S)$ for $\phi\in H_c^1(\Gamma,W)$, we get an explicit formula of $\langle\cdot,\cdot\rangle_N$ on $H_c^1(\Gamma,W)$:

Lemma 3

$$\langle\phi,\psi\rangle_N=\frac{1}{6}((\phi(S).T,\psi(S))_W-(\phi(S),\psi(S).T)_W)$$

This is a generalization to arbitrary level N of a formula of [Hab] p278, which applies only to level $N=1$.

The cuspidal cohomology $H_p^1(\Gamma,W)$ is defined as the image of $H_c^1(\Gamma,W)$ in $H^1(\Gamma,W)$. For classes ϕ, $\psi\in H_p^1(\Gamma,W)$ there exist ϕ', $\psi'\in H_c^1(\Gamma,W)$ with $i^*(\phi')=\phi$, $i^*(\psi')=\psi$. Then the pairing on $H_p^1(\Gamma,W)$ which we denote as $\langle\cdot,\cdot\rangle_N$ again is defined by

$$\langle\phi,\psi\rangle_N:=\langle\phi',\psi'\rangle_N.$$

3.2 The period matrix of $\mathcal{A}_f$

We have shown that the period matrix of $\mathcal{A}_f$ is $\Omega_f = e_d A_f^{-1} B_f \Delta_f^{-1}$. In order to determine the matrices A_f, B_f, we have to calculate the period integrals $\int_{a_i} \omega(f^{\sigma_j})$, $\int_{b_i} \omega(f^{\sigma_j})$.

Lemma 4 *Let* $g = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N)$ *with* $c > 0$. *Set* $y_0 = \frac{1}{Nc}$, $x_1 = -dy_0$, $x_2 = ay_0$, *and let* $f = \sum a_n q^n \in S_2(\Gamma_0(N))$, $\gamma = \{0, g0\} \in H_1(X_0(N), \mathbb{Z})$. *Then*

$$\int_\gamma \omega(f) = \sum_{n=1}^{\infty} \frac{a_n}{n} e^{-2\pi n y_0}(e^{2\pi i n x_2} - e^{2\pi i n x_1}).$$

See [Cre2] p25 for proof.

Because of the complicated structure of $\mathbf{a}_f, \mathbf{b}_f$, it is very time consuming to compute the period integrals directly. To speed up the computations, we consider the modular symbol $\gamma_g := \{0, g0\} \in H_1(X_0(N), \mathbb{Z}) \subset W_f(\mathbb{Q}) \oplus Y$ where $g \in \Gamma_0(N)$. We can write $\gamma_g = (\mathbf{a}_f, \mathbf{b}_f)\mathbf{c}_g + y_g$. Since $W_f(\mathbb{Q}) \perp Y$, the coefficient $\mathbf{c}_g$ can be determined by the following equation

$$\langle(\mathbf{a}_f, \mathbf{b}_f), \gamma_g\rangle_N = \langle(\mathbf{a}_f, \mathbf{b}_f), (\mathbf{a}_f, \mathbf{b}_f)\rangle_N \mathbf{c}_g = \begin{pmatrix} 0 & \Delta_f \\ -\Delta_f & 0 \end{pmatrix} \mathbf{c}_g.$$

Moreover we have

$$\int_{\gamma_g} \omega(\mathbf{f}) = \int_{(\mathbf{a}_f, \mathbf{b}_f)} \omega(\mathbf{f})\mathbf{c}_g + \int_{y_g} \omega(\mathbf{f}) = (A_f, B_f)\mathbf{c}_g.$$

Since the modular symbols $\{0, g0\}$ ($g \in \Gamma_0(N)$) generate the homology $H_1(X_0(N), \mathbb{Z})$, we can find $g_1, \ldots, g_{2d} \in \Gamma_0(N)$ such that the matrix

$$C := (\mathbf{c}_{g_1}, \ldots, \mathbf{c}_{g_{2d}}) = \begin{pmatrix} 0 & -\Delta_f^{-1} \\ \Delta_f^{-1} & 0 \end{pmatrix} \langle(\mathbf{a}_f, \mathbf{b}_f), (\gamma_{g_1}, \ldots, \gamma_{g_{2d}})\rangle_N \in M_{2d\times 2d}(\mathbb{Q})$$

is invertible. We obtain

$$(A_f, B_f) = \int_{(\gamma_{g_1}, \ldots, \gamma_{g_{2d}})} \omega(\mathbf{f}) C^{-1}.$$

The matrix $\int_{(\gamma_{g_1}, \ldots, \gamma_{g_{2d}})} \omega(\mathbf{f})$ can be computed using the formula of Lemma 4. Set

$$(X_1, X_2) := \int_{(\gamma_{g_1}, \ldots, \gamma_{g_{2d}})} \omega(\mathbf{f}) \cdot \langle(\mathbf{a}_f, \mathbf{b}_f), (\gamma_{g_1}, \ldots, \gamma_{g_{2d}})\rangle_N^{-1},$$

then $\Omega_f = -D_f X_2^{-1} X_1$.

3.3 The invariants of $\mathcal{A}_f$ for $d = 2$

We assume throughout this section $d = 2$ and that $\mathcal{A}_f$ is principally polarized.

We summerize some results of [Igu] and [Spa] that we need, and give references to suitable texts.

Lemma 5 *A principally polarized abelian variety of dimension 2 (defined over $\mathbb{C}$) is either the Jacobian of a smooth curve of genus 2 or the canonically polarized product of two elliptic curves.*

See [LB] p348 for proof.

Since $\mathcal{A}_f$ is a simple abelian variety, $\mathcal{A}_f$ is the Jacobian of a hyperelliptic curve C. The invariants of $\mathcal{A}_f$ are defined as the invariants of the associated hyperelliptic curve. We begin with the definition of invariants of a hyperelliptic curve of genus 2.

Let k be a field with $char(k) \neq 2$,

$$f(x) = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in k[x]$$

with $a_6 \neq 0$ or $a_5 \neq 0$. If $disc(f) \neq 0$, then

$$C : y^2 = f(x)$$

is a hyperelliptic curve of genus 2 defined over k. Let $x_1, ..., x_6$ be roots of $f(x)$. Then an expression of the form

$$a_6^m \sum (x_i - x_j)(x_k - x_l) \cdots$$

in which every x_i appears m times in each product and which is symmetric in $x_1, ..., x_6$, can be considered as an irrational form of a homogeneous **integral invariant** of degree m. Write (ij) for $(x_i - x_j)$. Igusa defined in [Igu] four integral invariants

$$I_2 := a_6^2 \textstyle\sum_{15}(12)^2(34)^2(56)^2$$

$$I_4 := a_6^4 \textstyle\sum_{10}(12)^2(23)^2(31)^2(45)^2(56)^2(64)^2$$

$$I_6 := a_6^6 \textstyle\sum_{60}(12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2$$

$$I_{10} := a_6^{10} \textstyle\prod_{i<j}(ij)^2.$$

The calculations of their rational forms are elementary but long and tedious. For the convienient for the reader we give the explicit rational forms, which might be useful to those wishing to write their own programs:

$$
\begin{aligned}
I_2 =\ & -240a_0a_6 + 40a_1a_5 - 16a_2a_4 + 6a_3^2 \\
I_4 =\ & 48a_0a_4^3 + 48a_2^3a_6 + 4a_2^2a_4^2 + 1620a_0^2a_6^2 + 36a_1a_3^2a_5 - 12a_1a_3a_4^2 - 12a_2^2a_3a_5 + 300a_1^2a_4a_6 + \\
& 300a_0a_5^2a_2 + 324a_0a_6a_3^2 - 504a_0a_4a_2a_6 - 180a_0a_4a_3a_5 - 180a_1a_3a_2a_6 + 4a_1a_4a_2a_5 - \\
& 540a_0a_5a_1a_6 - 80a_1^2a_5^2 \\
I_6 =\ & 176a_1^2a_5^2a_3^2 + 64a_1^2a_5^2a_4a_2 + 1600a_1^3a_5a_4a_6 + 1600a_1a_5^3a_0a_2 - 2240a_1^2a_5^2a_0a_6 - 160a_0a_4^4a_2 - \\
& 96a_0^2a_4^3a_6 + 60a_0a_4^3a_3^2 + 72a_1a_3^4a_5 - 24a_1a_3^3a_4^2 + 2250a_1^3a_3a_6^2 - 160a_2^4a_4a_6 - 96a_2^3a_0a_6^2 + \\
& 60a_2^3a_3^2a_6 - 24a_2^2a_3^3a_5 + 8a_2^2a_3^2a_4^2 - 900a_2^2a_1^2a_6^2 - 24a_2^3a_4^3 - 36a_2^4a_5^2 - 36a_1^2a_4^4 + \\
& 424a_0a_4^2a_2^2a_6 + 492a_0a_4^2a_2a_3a_5 + 20664a_0^2a_4a_6^2a_2 + 3060a_0^2a_4a_6a_3a_5 - 468a_0a_4a_3^2a_2a_6 - \\
& 198a_0a_4a_3^3a_5 - 640a_0a_4a_2^2a_5^2 + 3472a_0a_4a_2a_5a_1a_6 - 18600a_0a_4a_1^2a_6^2 - 876a_0a_4^2a_1a_6a_3 + \\
& 492a_1a_3a_2^2a_4a_6 - 238a_1a_3^2a_2a_4a_5 + 76a_1a_3a_2a_4^3 + 3060a_1a_3a_0a_6^2a_2 + 1818a_1a_3^2a_0a_6a_5 - \\
& 198a_1a_3^3a_2a_6 + 26a_1a_3a_2^2a_5^2 - 1860a_1^2a_3a_2a_5a_6 + 330a_1^2a_3^2a_6a_4 + 76a_2^3a_4a_3a_5 - \\
& 876a_2^2a_0a_6a_3a_5 + 616a_2^3a_5a_1a_6 + 2250a_0^2a_5^3a_3 - 900a_0^2a_5^2a_4^2 - 10044a_0^2a_6^2a_3^2 + 162a_0a_6a_3^4 + \\
& 28a_1a_4^2a_2^2a_5 - 640a_1^2a_4^2a_2a_6 + 26a_1^2a_4^2a_3a_5 - 1860a_1a_4a_0a_5^2a_3 + 616a_1a_4^3a_0a_5 - \\
& 18600a_0^2a_5^2a_6a_2 + 59940a_0^2a_5a_6^2a_1 + 330a_0a_5^2a_3^2a_2 - 119880a_0^3a_6^3 - 320a_1^3a_5^3
\end{aligned}
$$

I_{10} is the well-known discriminant of $f(x)$ and can be computed with the following command in MAPLE

$discrim(a6 * x \wedge 6 + a5 * x \wedge 5 + a4 * x \wedge 4 + a3 * x \wedge 3 + a2 * x \wedge 2 + a1 * x + a0, x);$

The quotient of two integral invariants of the same degree is called the **absolute invariant**. Since $I_{10} \neq 0$, we can define three absolute invariants

$$
i_1 := \frac{I_2^5}{I_{10}},\ i_2 := \frac{I_2^3 I_4}{I_{10}},\ i_3 := \frac{I_2^2 I_6}{I_{10}}. \tag{3}
$$

The invariants i_1, i_2, i_3 are very important owing to the following lemma:

Lemma 6 *Let C, C' be two hyperelliptic curves of genus 2. If $I_2 \neq 0$, $I_2' \neq 0$, then C and C' are isomorphic over $\mathbb{C}$ if and only if*

$$
i_1 = i_1',\ i_2 = i_2',\ i_3 = i_3'.
$$

see [Igu] p632 for proof.

On the other hand, the invariants of C can also be determined by Theta functions. We introduce now the definition of Theta functions.

In general, let $\mathbb{H}_d$ be the d-dimensional Siegel half space, it consists of symmetric $d \times d$ complex matrix Ω whose imaginary part is positive definite. We define the Theta function with characteristics $a, b \in \frac{1}{2}\mathbb{Z}^d$ as

$$
\theta \begin{bmatrix} a \\ b \end{bmatrix} : \mathbb{C}^d \times \mathbb{H}_d \longrightarrow \mathbb{C}
$$

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\Omega) := \sum_{n\in\mathbf{Z}^d} exp(\pi i(n+a)^t\Omega(n+a) + 2\pi i(n+a)^t(z+b)).$$

The Theta function has the following properties (cf. [Mum] Chap. II):

1. $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ defines a holomorphic function on $\mathbb{C}^d \times \mathbb{H}_d$.

2. $\theta\left[\begin{smallmatrix} a+p \\ b+q \end{smallmatrix}\right](z,\Omega) = e^{2\pi i a^t q}\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\Omega)$, $\forall a, b \in \frac{1}{2}\mathbb{Z}^d$, $p, q \in \mathbb{Z}^d$.

3. $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](-z,\Omega) = e^{4\pi i a^t b}\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\Omega)$. If $4a^t b \not\equiv 0 \pmod 2$, then $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ is an odd function.

Now let $d = 2$ and $\Omega \in \mathbb{H}_2$. Because of the quasi-periodicity of $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, the Theta functions can be parametrized by the characteristics $a, b \in \{\frac{1}{2}, 0\}^2$. Therefore we have 16 Theta functions (6 of them are odd and 10 functions are even). Define the Theta null-value of a Theta function as

$$\theta\begin{bmatrix} a \\ b \end{bmatrix} := \theta\begin{bmatrix} a \\ b \end{bmatrix}(0,\Omega).$$

Because 6 Theta functions are odd, their Theta null-values vanish. The Theta null-values of the 10 even functions are (we use the notations of [Spa]):

$$v_0 := \theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad v_1 := \theta\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \quad v_3 := \theta\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad v_5 := \theta\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$v_{21} := \theta\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \quad v_{23} := \theta\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \quad v_{25} := \theta\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$v_{41} := \theta\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad v_{43} := \theta\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad v_{45} := \theta\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Let Ω_C be the period matrix of the Jacobi variety J_C of C. We define four modular forms of weight 4,10,12,16:

$$\begin{aligned}
h_4 = \; & v_0^8 + v_1^8 + v_{21}^8 + v_{23}^8 + v_{25}^8 + v_3^8 + v_{41}^8 + v_{43}^8 + v_{45}^8 + v_5^8 \\
h_{10} = \; & (v_0 v_{21} v_{43} v_5 v_1 v_{25} v_{45} v_3 v_{41} v_{23})^2 \\
h_{12} = \; & (v_0 v_1 v_{21} v_{23} v_{25} v_{41})^4 + (v_0 v_{21} v_{23} v_{25} v_3 v_{43})^4 + (v_1 v_{23} v_{25} v_3 v_{41} v_{45})^4 + (v_1 v_{21} v_{25} v_3 v_{43} v_{45})^4 + \\
& (v_0 v_1 v_{21} v_{41} v_{43} v_{45})^4 + (v_0 v_{23} v_3 v_{41} v_{43} v_{45})^4 + (v_0 v_1 v_{21} v_3 v_{41} v_5)^4 + (v_0 v_1 v_{23} v_3 v_{43} v_5)^4 + \\
& (v_1 v_{23} v_{25} v_{41} v_{43} v_5)^4 + (v_{21} v_{25} v_3 v_{41} v_{43} v_5)^4 + (v_0 v_{21} v_{23} v_{25} v_{45} v_5)^4 + (v_0 v_1 v_{25} v_3 v_{45} v_5)^4 + \\
& (v_{21} v_{23} v_3 v_{41} v_{45} v_5)^4 + (v_1 v_{21} v_{23} v_{43} v_{45} v_5)^4 + (v_0 v_{25} v_{41} v_{43} v_{45} v_5)^4 \\
h_{16} = \; & v_0^4 v_1^4 v_{21}^4 v_{23}^4 v_{25}^4 v_3^8 v_{41}^4 + v_0^4 v_1^8 v_{21}^4 v_{23}^4 v_{25}^4 v_3^4 v_{43}^4 + v_0^4 v_{21}^4 v_{23}^4 v_{25}^4 v_3^4 v_{41}^8 v_{43}^4 + v_0^4 v_1^4 v_{21}^4 v_{23}^4 v_{25}^4 v_{41}^4 v_{43}^8 + \\
& v_0^8 v_1^4 v_{23}^4 v_{25}^4 v_3^4 v_{41}^4 v_{45}^4 + v_1^4 v_{21}^8 v_{23}^4 v_{25}^4 v_3^4 v_{41}^4 v_{45}^4 + v_0^8 v_1^4 v_{21}^4 v_{25}^4 v_3^4 v_{43}^4 v_{45}^4 + v_1^4 v_{21}^4 v_{23}^8 v_{25}^4 v_3^4 v_{43}^4 v_{45}^4 + \\
& v_0^4 v_1^4 v_{21}^4 v_{23}^8 v_{41}^4 v_{43}^4 v_{45}^4 + v_0^4 v_1^4 v_{21}^4 v_{25}^8 v_{41}^4 v_{43}^4 v_{45}^4 + v_0^4 v_1^8 v_{23}^4 v_3^4 v_{41}^4 v_{43}^4 v_{45}^4 + v_0^4 v_{21}^8 v_{23}^4 v_3^4 v_{41}^4 v_{43}^4 v_{45}^4 + \\
& v_0^4 v_{23}^4 v_{25}^8 v_3^4 v_{41}^4 v_{43}^4 v_{45}^4 + v_0^4 v_1^4 v_{21}^4 v_3^8 v_{41}^4 v_{43}^4 v_{45}^4 + v_1^4 v_{21}^4 v_{25}^4 v_3^4 v_{41}^8 v_{43}^4 v_{45}^4 + v_1^4 v_{23}^4 v_{25}^4 v_3^4 v_{41}^4 v_{43}^8 v_{45}^4 +
\end{aligned}$$

$$v_0^4v_1^4v_{21}^4v_{23}^4v_{25}^4v_{41}^4v_{45}^8 + v_0^4v_{21}^4v_{23}^4v_{25}^4v_3^4v_{43}^4v_{45}^8 + v_0^4v_1^4v_{21}^4v_{23}^8v_3^4v_{41}^4v_5^4 + v_0^4v_1^4v_{21}^4v_{25}^8v_3^4v_{41}^4v_5^4 +$$
$$v_0^4v_1^4v_{21}^8v_{23}^4v_3^4v_{43}^4v_5^4 + v_0^4v_1^4v_{23}^4v_{25}^8v_3^4v_{43}^4v_5^4 + v_0^8v_1^4v_{23}^4v_{25}^4v_{41}^4v_{43}^4v_5^4 + v_1^4v_{21}^8v_{23}^4v_{25}^4v_{41}^4v_{43}^4v_5^4 +$$
$$v_0^8v_{21}^4v_{25}^4v_3^4v_{41}^4v_{43}^4v_5^4 + v_1^8v_{21}^4v_{25}^4v_3^4v_{41}^4v_{43}^4v_5^4 + v_{21}^4v_{23}^8v_{25}^4v_3^4v_{41}^4v_{43}^4v_5^4 + v_1^4v_{23}^4v_{25}^4v_3^8v_{41}^4v_{43}^4v_5^4 +$$
$$v_0^4v_1^4v_{23}^4v_3^4v_{41}^8v_{43}^4v_5^4 + v_0^4v_1^4v_{21}^4v_3^4v_{41}^4v_{43}^8v_5^4 + v_0^4v_1^8v_{21}^4v_{23}^4v_{25}^4v_{45}^4v_5^4 + v_0^4v_1^4v_{21}^8v_{25}^4v_3^4v_{45}^4v_5^4 +$$
$$v_0^4v_1^4v_{23}^8v_{25}^4v_3^4v_{45}^4v_5^4 + v_0^4v_{21}^4v_{23}^4v_{25}^4v_3^8v_{45}^4v_5^4 + v_0^8v_{21}^4v_{23}^4v_3^4v_{41}^4v_{45}^4v_5^4 + v_1^8v_{21}^4v_{23}^4v_3^4v_{41}^4v_{45}^4v_5^4 +$$
$$v_{21}^4v_{23}^4v_{25}^8v_3^4v_{41}^4v_{45}^4v_5^4 + v_0^4v_{21}^4v_{23}^4v_{25}^4v_{41}^8v_{45}^4v_5^4 + v_0^4v_1^4v_{25}^4v_3^4v_{41}^8v_{45}^4v_5^4 + v_0^8v_1^4v_{21}^4v_{23}^4v_{43}^4v_{45}^4v_5^4 +$$
$$v_1^4v_{21}^4v_{23}^4v_{25}^8v_{43}^4v_{45}^4v_5^4 + v_1^4v_{21}^4v_{23}^4v_3^8v_{43}^4v_{45}^4v_5^4 + v_0^4v_1^8v_{25}^4v_{41}^4v_{43}^4v_{45}^4v_5^4 + v_0^4v_{21}^8v_{25}^4v_{41}^4v_{43}^4v_{45}^4v_5^4 +$$
$$v_0^4v_{23}^8v_{25}^4v_{41}^4v_{43}^4v_{45}^4v_5^4 + v_0^4v_{25}^4v_3^8v_{41}^4v_{43}^4v_{45}^4v_5^4 + v_1^4v_{21}^4v_{23}^4v_{41}^8v_{43}^4v_{45}^4v_5^4 + v_0^4v_{21}^4v_{23}^4v_{25}^4v_{43}^8v_{45}^4v_5^4 +$$
$$v_0^4v_1^4v_{25}^4v_3^4v_{43}^8v_{45}^4v_5^4 + v_{21}^4v_{23}^4v_3^4v_{41}^4v_{43}^8v_{45}^4v_5^4 + v_0^4v_1^4v_{21}^4v_3^4v_{41}^4v_{45}^8v_5^4 + v_0^4v_1^4v_{23}^4v_3^4v_{43}^4v_{45}^8v_5^4 +$$
$$v_1^4v_{23}^4v_{25}^4v_{41}^4v_{43}^4v_{45}^8v_5^4 + v_{21}^4v_{25}^4v_3^4v_{41}^4v_{43}^4v_{45}^8v_5^4 + v_0^4v_1^4v_{21}^4v_{23}^4v_{25}^4v_{41}^4v_5^8 + v_0^4v_{21}^4v_{23}^4v_{25}^4v_3^4v_{43}^4v_5^8 +$$
$$v_1^4v_{23}^4v_{25}^4v_3^4v_{41}^4v_{45}^4v_5^8 + v_1^4v_{21}^4v_{25}^4v_3^4v_{43}^4v_{45}^4v_5^8 + v_0^4v_1^4v_{21}^4v_{41}^4v_{43}^4v_{45}^4v_5^8 + v_0^4v_{23}^4v_3^4v_{41}^4v_{43}^4v_{45}^4v_5^8$$

We obtain by using the tables in [Bol] p483 that

$$i_1 = \frac{h_{12}^5}{h_{10}^6},\ i_2 = \frac{h_{12}^3 h_4}{h_{10}^4},\ i_3 = \frac{h_{12}^2 h_{16}}{h_{10}^4} \tag{4}$$

We have shown how to compute the period matrix of $\mathcal{A}_f$. Hence we can compute the Theta null-values $v_0, v_1, v_3, v_5, v_{21}, v_{23}, v_{25}, v_{41}, v_{43}, v_{45}$ and the absolute invariants i_1, i_2, i_3 of the abelian variety $\mathcal{A}_f$ explicitly. The formula (3)=(4) gives then a system of non-linear equations of the coefficients $a_0, a_1, ..., a_6 \in k$ of the associated hyperelliptic curve C.

In general, this system of equations is extremely complicated. But if the hyperelliptic curve C has a rational point in k (i.e., $C(k) \neq \emptyset$) and $char(k) \neq 5$, then the equation of the curve can be written as

$$C: y^2 = x^5 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{5}$$

with $a_0, a_1, a_2, a_3 \in k$ (cf. [Cas] p41). By using the Buchberger algorithm and Gröbner base (cf. [Map] p473) we have solved this system of equations in several special cases.

Mestre gives a criterion over which field the hyperelliptic curve C is defined.

Assume $I_2 \neq 0$, set

$$x := \frac{8}{225}\frac{20i_2 + i_1}{i_1}$$

$$y := \frac{16}{3375}\frac{-600i_3 + i_1 + 80i_2}{i_1}$$

$$z := \frac{-64}{253125}\frac{-10800000i_1 - 9i_1^2 - 700i_2i_1 + 3600i_3i_1 + 12400i_2^2 - 48000i_2i_3}{i_1^2}.$$

Define

$$L : \sum_{1\leq i,j\leq 3} C_{ij}x_iy_j = 0 \tag{6}$$

the equation of a conic L where C_{ij} is the coefficient of the matrix

$$\begin{pmatrix} x+6y & 6x^2+2y & 2z \\ 6x^2+2y & 2z & 9x^3+4xy+6y^2 \\ 2z & 9x^3+4xy+6y^2 & 6x^2y+2y^2+3xz \end{pmatrix},$$

then the hyperelliptic curve C is defined over a number field k if $L(k) \neq \emptyset$ (cf. [Mes] p332).

3.4 Example and table

Example: Let $N = 63$, $\xi = \sqrt{3}$. $f = \sum a_n q^n \in S_2(\Gamma_0(N))$ with the Fourier coefficients

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
a_p	ξ	0	-2ξ	1	2ξ	2	2ξ	-4	-2ξ	0	-4	2	6ξ	-4	4ξ

is a newform and $K_f = \mathbb{Q}(\sqrt{3})$. Then we get the matrix $\Delta_f = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Therefore $\mathcal{A}_f$ is principally polarized. Denote the period matrix Ω_f by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\begin{aligned} a = \ & -0.72644444909281749715068326927588420603317213842591423092 6\ldots \\ & +i \cdot 0.89755879791552721487836790016613986125849593892686521 41\ldots \end{aligned}$$

$$\begin{aligned} b = \ & -0.36322222454640874854907542105698134213092178722897312799 5\ldots \\ & +i \cdot 0.44877939895776360751290963149702590806944587017468411 70\ldots \end{aligned}$$

$$\begin{aligned} c = \ & -0.36322222454640874862202255001016732932582316087252874325 6\ldots \\ & +i \cdot 0.44877939895776360742367250011281960577756343608577226 84\ldots \end{aligned}$$

$$\begin{aligned} d = \ & 0.27355555090718250279734207741119281169657666084322428409 18\ldots \\ & +i \cdot 0.89755879791552721489131718863651192469523244453325224 22\ldots \end{aligned}$$

The absolute invariants of $\mathcal{A}_f$ are

$$i_1 = 539117.2555879448199220 8602\ldots = \frac{2^3 37^5}{3 \cdot 7^3}$$

$$i_2 = -22816.0014577257878514160438\ldots = -\frac{3 \cdot 37^3 103}{2 \cdot 7^3}$$

$$i_3 = -2197.6840379008591905 8405\ldots = -\frac{5 \cdot 37^2 881}{2^3 \cdot 7^3}.$$

The equation (3)=(4) of the hyperelliptic curve

$$C\ :\ y^2 = x^5 + a_3x^3 + a_2x^2 + a_1x + a_0$$

has a solution:

$$\begin{aligned} a_3 &= -290t^2 \\ a_2 &= 15660t^3 \\ a_1 &= 1541385t^4 \\ a_0 &= 4475628t^5 \end{aligned}$$

with the discriminant $disc(C) = 2^{32}3^{6}5^{20}7^{8}t^{20}$. Taking $t = 1/5$:

$$y^2 = x^5 - \frac{58}{5}x^3 + \frac{3132}{25}x^2 + \frac{308277}{125}x + \frac{4475628}{3125}.$$

Substituting $x \to x - 328/5$ we obtain a hyperelliptic curve

$$C\ :\ y^2 = x^5 - 328x^4 + 43022x^3 - 2820596x^2 + 92430809x - 1211186860$$

with the discriminant $disc(C) = 2^{32}3^{6}7^{8}$.

By using a programe developed by Liu (cf. [Liu]) we obtain a minimal equation of C over $\mathbb{Z}[1/2]$:

$$C\ :\ y^2 = x^5 - 288x^4 + 33166x^3 - 1908900x^2 + 54910233x - 631541988$$

The prime to 2 part of the conductor of C is $3087 = 3^2 7^3$.

Table: 2-dimensional principally polarized factors of $J_0(N)$ for $N \leq 200$.

We explain the notations in the following table. In the first column we give the level N, in the second we give D with $K_f = \mathbb{Q}(\sqrt{D})$, in the third and fourth are the intersection numbers e_1, e_2 and the absolute invariants i_1, i_2, i_3.

N	D	e_1	e_2	i_1	i_2	i_3
23	5	1	1	$\frac{-2^3 11^5 13^5}{23^6}$	$\frac{-11^3 13^3 409^2}{2\cdot 23^6}$	$\frac{-3\cdot 11^2 13^2 713149}{2^3 23^6}$
29	2	1	1	$\frac{2^3 5^5 7^{10}}{29^5}$	$\frac{-5^3 7^6 103\cdot 593}{2\cdot 29^5}$	$\frac{-5^2 7^4 13\cdot 1987\cdot 3229}{2^3 29^5}$
31	5	1	1	$\frac{2^3 5^{10} 41^5}{31^4}$	$\frac{5^6 41^3 281^2}{2\cdot 31^4}$	$\frac{5^4 41^2 94151689}{2^3 31^4}$
63	3	2	2	$\frac{2^3 37^5}{3\cdot 7^3}$	$\frac{-3\cdot 37^3 103}{2\cdot 7^3}$	$\frac{-5\cdot 37^2 881}{2^3 7^3}$
65	3	2	2	$\frac{2^3 19^5 43^5}{3^7 5^5 13^2}$	$\frac{-19^3 43^3 101\cdot 139}{2\cdot 3^5 5^5 13^2}$	$\frac{11\cdot 19^2 43^2 949961}{2^3 3^6 5^5 13^2}$

N	D	e_1	e_2	i_1	i_2	i_3
65	2	2	2	$\frac{-2^{6}5\cdot 313^{5}}{13^{3}}$	$\frac{139\cdot 313^{3}701}{5\cdot 13^{3}}$	$\frac{7\cdot 313^{2}59104229}{2^{3}5^{2}13^{3}}$
67	5	2	2	$\frac{-2^{3}71^{5}}{67^{2}}$	$\frac{-37^{2}71^{3}}{2\cdot 67^{2}}$	$\frac{-71^{2}193\cdot 659}{2^{3}67^{2}}$
73	5	2	2	$\frac{-2^{3}47^{5}}{73^{2}}$	$\frac{-47^{3}61^{2}}{2\cdot 73^{2}}$	$\frac{-11\cdot 47^{2}37633}{2^{3}73^{2}}$
87	5	2	2	$\frac{2^{3}73^{5}}{3^{4}29^{2}}$	$\frac{73^{3}}{2\cdot 3^{4}29^{2}}$	$\frac{31\cdot 67\cdot 73^{2}181}{2^{3}3^{4}29^{2}}$
93	5	4	4	$\frac{-2^{3}23^{5}}{3^{4}31^{2}}$	$\frac{-7^{2}19^{2}23^{3}}{2\cdot 3^{4}31^{2}}$	$\frac{-23^{2}37\cdot 5437}{2^{3}3^{3}31^{2}}$
103	5	2	2	$\frac{-2^{3}47^{5}}{103^{2}}$	$\frac{-47^{3}61^{2}}{2\cdot 103^{2}}$	$\frac{-11\cdot 47^{2}14593}{2^{3}103^{2}}$
107	5	2	2	$\frac{2^{3}73^{5}}{107^{2}}$	$\frac{73^{3}}{2\cdot 107^{2}}$	$\frac{53\cdot 73^{2}2909}{2^{3}107^{2}}$
115	5	4	4	$\frac{2^{3}5^{6}}{23^{2}}$	$\frac{5^{2}11^{2}}{2\cdot 23^{2}}$	$\frac{53\cdot 3209}{2^{3}23^{2}}$
117	3	4	4	$\frac{2^{3}5^{5}59^{5}}{3^{7}13^{3}}$	$\frac{5^{4}7\cdot 59^{3}}{2\cdot 3^{2}13^{3}}$	$\frac{5^{3}43\cdot 59^{2}421}{2^{3}3^{3}13^{3}}$
117	2	8	8	$\frac{2^{3}7^{5}53^{5}}{3^{5}13^{2}}$	$\frac{7^{3}53^{3}34537}{2\cdot 3^{5}13^{2}}$	$\frac{5\cdot 7^{2}19\cdot 53^{2}67\cdot 2311}{2^{3}3^{5}13^{2}}$
125	5	2	2	$2^{3}5^{4}$	$\frac{5^{4}}{2}$	$\frac{5^{2}313}{2^{3}}$
125	5	10	10	$2^{3}5^{4}$	$\frac{5^{4}}{2}$	$\frac{5^{2}313}{2^{3}}$
133	5	4	4	$\frac{2^{3}5^{5}4133^{5}}{7^{4}19^{2}}$	$\frac{5^{3}4133^{3}}{2\cdot 7^{4}19^{2}}$	$\frac{-5^{2}389\cdot 4133^{2}57139}{2^{3}7^{4}19^{2}}$
133	5	4	4	$\frac{-2^{3}23^{5}}{7^{2}19^{2}}$	$\frac{-5^{2}17^{2}23^{3}}{2\cdot 7^{2}19^{2}}$	$\frac{-11^{2}23^{2}1307}{2^{3}7^{2}19^{2}}$
135	13	6	6	$\frac{2^{3}3^{3}11^{5}}{5^{4}}$	$\frac{3^{5}11^{3}41}{2\cdot 5^{4}}$	$\frac{3^{2}11^{2}34613}{2^{3}5^{4}}$
147	2	4	4	$\frac{2^{3}29^{5}}{3^{3}}$	$\frac{13\cdot 29^{3}61}{2\cdot 3^{3}}$	$\frac{19\cdot 29^{2}151}{2^{3}\cdot 3}$
161	5	4	4	$\frac{-2^{3}191^{5}}{7^{4}23^{2}}$	$\frac{-13^{4}191^{3}}{2\cdot 7^{4}23^{2}}$	$\frac{-11\cdot 191^{2}600949}{2^{3}7^{4}23^{2}}$
167	5	2	2	$\frac{2^{3}5^{5}13^{5}}{167^{2}}$	$\frac{5^{3}13^{3}89^{2}}{2\cdot 167^{2}}$	$\frac{3\cdot 5^{2}13^{2}294787}{2^{3}167^{2}}$
175	5	6	6	$\frac{2^{3}113^{5}}{5^{4}7^{4}}$	$\frac{11^{2}113^{3}}{2\cdot 5^{2}7^{4}}$	$\frac{17\cdot 19\cdot 107\cdot 113^{2}}{2^{3}5^{2}7^{4}}$
177	5	4	4	$\frac{-2^{3}23^{5}}{3^{2}59^{2}}$	$\frac{-5^{4}23^{3}}{2\cdot 3^{2}59^{2}}$	$\frac{23^{2}73\cdot 587}{2^{3}\cdot 3\cdot 59^{2}}$
177	5	4	4	$\frac{-2^{3}71^{5}73^{5}}{3^{4}5^{12}59^{2}}$	$\frac{-11^{4}71^{3}73^{3}}{2\cdot 3^{4}5^{8}59^{2}}$	$\frac{-71^{2}73^{2}227\cdot 797077}{2^{3}3^{4}5^{8}59^{2}}$

N	D	e_1	e_2	i_1	i_2	i_3
188	5	6	6	$\frac{-2^5 29^5}{47^2}$	$\frac{-2^9 29^3}{47^2}$	$\frac{-2^3 29^2 281}{47^2}$
189	3	6	6	$\frac{2^3 271^5}{3^7 7^3}$	$\frac{131 \cdot 271^3}{2 \cdot 3^2 7^3}$	$\frac{271^2 133187}{2^3 3^3 7^3}$
191	5	2	2	$\frac{-2^3 47^5}{191^2}$	$\frac{-47^3 97^2}{2 \cdot 191^2}$	$\frac{-47^2 508799}{2^3 191^2}$

Remarks:

1. There are totally 93 2-dimensional factors of $J_0(N)$ for $N \leq 200$. About 30 percent of them are principally polarized.

2. We computed also the equation (6) of the conic L for each factor in the above table and noted that $L(\mathbb{Q}) \neq \emptyset$. This means that the associated hyperelliptic curves are all defined over $\mathbb{Q}$. In the major part of the cases we could only find a solution of the hyperelliptic curve (5) over a number field (of degree 2, 3 or 6). In three favorable cases we were even able to determine the curve equations over $\mathbb{Z}$.

N	Minimal equation of the curve C over $\mathbb{Z}[1/2]$
63	$y^2 = x^5 - 288x^4 + 33166x^3 - 1908900x^2 + 54910233x - 631541988$ prime to 2 part of conductor: $3087 = 3^2 7^3$
117	$y^2 = 3x^5 + 150x^4 + 3250x^3 + 39000x^2 + 264147x + 799370$ prime to 2 part of conductor: $1779573 = 3^4 13^3$
189	$y^2 = 3x^5 + 96x^4 + 1498x^3 + 13356x^2 + 64827x + 132300$ prime to 2 part of conductor: $83349 = 3^5 7^3$

Cremona investigated in [Cre1] certain simple factors of the Jacobian variety $J_0(N)$ with extra twist by the character asssociated to a quadratic number field. There are 12 explicit examples for $N < 300$ of 2-dimensional factors $\mathcal{A}_f$ of $J_0(N)$ with extra twist. In most cases these $\mathcal{A}_f$ split as a product of 2 elliptic curves over a quadratic number field. For example he showed the Jacobian varieties of above 3 curves are isogenous to $E \times E'$ over $\mathbb{Q}(\sqrt{-3})$ where E is an elliptic curve over $\mathbb{Q}(\sqrt{-3})$ and E' is the conjugate curve.

3. We computed even the splitting of $J_0(N)$ and all 2-dimensional factors of $J_0(N)$ for $N \leq 2000$. There are totally 2850 2-dimensional factors of $J_0(N)$.

4. We know only that the hyperelliptic curve C and the abelian variety A_f are both defined over $\mathbb{Q}$ and $A_f \cong Jac(C)$ over $\mathbb{C}$. We expect that A_f would be isogenous to $Jac(C)$ over $\mathbb{Q}$.

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