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# Partial Regularity for Certain Classes of Polyconvex Functionals Related to Nonlinear Elasticity

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Consider the variational integral $J(u) := \int_\Omega |\nabla u|^p + H(\det \nabla u)\, dx$ where $\Omega \subset \mathbf{R}^n$ and $p \geq n \geq 2$. $H : (0,\infty) \to [0,\infty)$ is a smooth convex function such that $\lim_{t \downarrow 0} H(t) = \infty$. We approximate $J$ by a sequence of regularized functionals $J_\delta$ whose minimizers converge strongly to an $J$–minimizing function and prove partial regularity results for $J_\delta$–minimizers.

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## 0 Introduction

We study a special class of polyconvex variational integrals which are related to nonlinear elasticity. Our main purpose is to illustrate some ideas which might lead to partial regularity of minimizers for stored energies studied in the papers of John Ball (see [3],[4]). To be precise consider a bounded open set $\Omega$ in $\mathbf{R}^n$ and a real number $p \geq n$. We require $n \geq 2$ and $p > 2$ — the case $n = p = 2$

has been treated in the paper [6]. Suppose further that we are given a function $u_0 \in H^{1,p}(\Omega, \mathbf{R}^n)$ such that

$$\tau \le \det \nabla u_0(x) \le 1/\tau \quad \text{a.e. on } \Omega$$

for some $\tau \in (0,1)$. Then we look at the variational problem

$$\begin{cases} J(u) := \int_\Omega |\nabla u|^p + H(\det \nabla u)\, dx \\ \to \min \ \text{in } \mathcal{C} := \{w \in H^{1,p}(\Omega, \mathbf{R}^n) : w = u_0 \text{ on } \partial\Omega\} \end{cases} \tag{0.1}$$

with $H : (0,\infty) \to [0,\infty)$ of class $C^2$, strictly convex and with the property

$$\lim_{t \downarrow 0} H(t) = \infty. \tag{0.2}$$

Integrands of this type occur as stored energy densities for certain models from nonlinear elasticity (see Ball [3],[2] and Ogden [9]) and from the work of Ball [3],[4] or Müller [8] we deduce that problem (0.1) has at least one solution $u \in \mathcal{C}$. Up to now nothing is known about the regularity properties of minimizers $u$ but the results described below give rise to the following

**CONJECTURE:** *There is an open subset $\Omega_0$ of $\Omega$ whose complement has vanishing Lebesgue measure such that $u \in C^{1,\alpha}(\Omega_0)$ for any $0 < \alpha < 1$. Moreover, $x_0 \in \Omega_0$ if and only if the following conditions hold:*

*a)* $x_0$ *is a Lebesgue point for* $\nabla u$

*b)* $\det \nabla u(x_0) > 0$

*c)* $\displaystyle ⨍_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^p\, dx \longrightarrow 0$ *as* $r \downarrow 0$.

Here and in the sequel we use the symbol $(f)_{x_0,r}$ to denote the mean value $⨍_{B_r(x_0)} f\, dx$ of the function $f$.

As a first approach towards this conjecture we consider the case

$$\lim_{t \to \infty} H'(t) = \infty \tag{0.3}$$

and replace (0.1) by a sequence of more regular variational problems

$$J_\delta(v) := \int_\Omega |\nabla v|^p + h_\delta(\det \nabla v)\, dx \longrightarrow \min \ \text{in } \mathcal{C} \tag{0.4}$$

where for $0 < \delta < \tau$

$$h_\delta(t) := \begin{cases} H'(\delta)(t-\delta) + H(\delta) & , \ t \le \delta \\ H(t) & , \ \delta \le t \le \delta^{-1} \\ H'(1/\delta)(t - 1/\delta) + H(1/\delta) & , \ t \ge \delta^{-1} \end{cases}$$

is defined for all $t \in \mathbb{R}$ with linear growth at $\pm\infty$. (In case that (0.3) does not hold we just let $h_\delta(t) = H(t)$ for all $t \geq \delta$.)

It is easy to show

$$h_\varepsilon(t) \leq h_\delta(t) \quad \text{if} \quad \delta < \varepsilon, \quad h_\delta(t) \to H(t) \quad \text{as} \quad \delta \downarrow 0 \tag{0.5}$$

if we define $H(t) = \infty$ for $t \leq 0$.

**THEOREM 1:** *Problem (0.4) admits a solution $u_\delta$. After passing to a subsequence we also have*

$$u_\delta \to: u \quad in \quad H^{1,p}(\Omega, \mathbf{R}^n) \quad strongly$$

*and $u$ is a solution of (0.1).*

We obtained this result in [6] and it is worth remarking that Theorem 1 gives existence of solutions to our original problem without using the elaborate arguments of Ball or Müller.

According to Theorem 1 the sequence $\{u_\delta\}$ converges strongly to a solution of (0.1) and it therefore seems reasonable to analyze the regularity properties of these functions.

**THEOREM 2:** *There exists an open subset $\Omega_\delta$ of $\Omega$ such that $u_\delta \in C^{1,\alpha}(\Omega_\delta)$ for any $0 < \alpha < 1$. We have the estimates*

$$\mathcal{L}^n(\Omega - \Omega_\delta) \leq \min\{H(\delta), H(1/\delta)\}^{-1} J(u_0) \longrightarrow 0 \quad as \quad \delta \downarrow 0$$

*and $\delta < \det \nabla u_\delta(x) < \delta^{-1}$ on $\Omega_\delta$.*

Unfortunately our integrand

$$f(Q) := |Q|^p + h_\delta(\det Q), \quad Q \in \mathbf{R}^{n \times n},$$

does not satisfy the hypotheses which are usually imposed on the data. For example, Anzellotti – Giaquinta [2] require the integrand to be a strictly convex function but $f$ is only quasiconvex. On the other hand Acerbi – Fusco [1] discuss integrands of the form $|Q|^2 + f(Q)$ by the way compensating the degeneracy caused by $|Q|^p$. In the same spirit Evans – Gariepy [5] require the second derivatives of the integrand to grow like $|Q|^{p-2}$ which is false for our function $f$. It should also be noted that $f$ is not of class $C^2$ since $h_\delta$ is only in the space $C^{1,1}(\mathbb{R})$. One might therefore ask if some of the difficulties can be avoided by replacing $h_\delta$ by a sequence $\tilde{h}_\delta$ of smoother functions, for example the second order Taylor approximation. But in this case $\tilde{h}_\delta(\det Q)$ behaves in certain directions like $|Q|^{2n}$, moreover the approximation property (0.5) which is essential for the proof of Theorem 1 will in general be violated.

Despite of these difficulties we are going to prove Theorem 2 in the spirit of Evans – Gariepy where of course the special form of the function $h_\delta$ is very helpfull. The main step towards partial regularity is an energy decay estimate which is shown by contradiction. In this step we consider a sequence of scaled minimizers converging weakly to a solution of a linear elliptic system with constant coefficients. In section 2 we improve this result to strong convergence which immediately leads to the desired contradiction.

# 1 An Energy Decay Estimate

For the rest of this section we fix $\delta \in (0, \tau)$ and assume that all the hypotheses of Theorem 2 are satisfied. We write $u$ and $h$ in place of $u_\delta$ and $h_\delta$.

Suppose that we are given $A_0 \in \mathbf{R}^{n\times n}$ such that

$$a_0 := \det A_0 \in (\delta, 1/\delta).$$

Then we can calculate $\sigma = \sigma(A_0, \delta)$ such that

$$\det A \in \left[\frac{1}{2}(a_0 + \delta), \frac{1}{2}(a_0 + \delta^{-1})\right] \subset (\delta, \delta^{-1})$$

holds for all $A \in \mathbf{R}^{n\times n}$, $|A - A_0| \le \sigma$.

**MAIN LEMMA:** *There is a constant $c_* = c_*\big(A_0, p, H''(a_0)\big)$ with the following property: For each $t \in (0,1)$ there exists $\varepsilon = \varepsilon(A_0, t, \delta)$ such that, for every ball $B_R(x_0) \subset \Omega$, the conditions*

$$|(\nabla u)_{x_0,R} - A_0| \le \sigma,$$

$$E\big(u, B_R(x_0)\big) := \fint_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 + |\nabla u - (\nabla u)_{x_0,R}|^p \, dx < \varepsilon^2$$

*imply*

$$E\big(u, B_{tR}(x_0)\big) \le c_* t^2 E\big(u, B_R(x_0)\big).$$

From this result the statement of Theorem 2 follows in a routine manner: Let $\Omega'_\delta := \{x \in \Omega : \delta < \det \nabla u(x) < \delta^{-1}\}$. Minimality of $u$ implies

$$\int_{\Omega - \Omega'_\delta} h(\det \nabla u)\, dx \le \int_\Omega |\nabla u_0|^p + H(\det \nabla u_0)\, dx$$

and the term on the left is bounded below by the quantity

$$\mathcal{L}^n(\Omega - \Omega'_\delta)\ \min\{H(\delta), H(1/\delta)\}.$$

Next consider $x_0 \in \Omega'_\delta$ such that

$$x_0 \quad \text{is a Lebesgue point for} \quad \nabla u \tag{1.1}$$

$$\delta < \det \nabla u(x_0) < \delta^{-1} \tag{1.2}$$

$$E\big(u, B_r(x_0)\big) \longrightarrow 0 \qquad \text{as} \quad r \downarrow 0. \tag{1.3}$$

Clearly (1.1)–(1.3) are true for $\mathcal{L}^n$ - almost all $x_0 \in \Omega'_\delta$. If we take $A_0 = \lim\limits_{r\downarrow 0}(\nabla u)_{x_0,r}\ \big( = \nabla u(x_0)\big)$ then iteration of the Main Lemma gives $u \in C^1$ in a neighborhood of $x_0$. This shows $u \in C^1(\Omega_\delta)$ with

$$\Omega_\delta := \{x_0 \in \Omega'_\delta :\ x_0 \text{ satisfies } (1.1) - (1.3)\}$$

and the proof of Theorem 2 is complete.

The proof of the Main Lemma proceeds in several steps: We fix $t \in (0,1)$ and define $c_*$ later on. If the lemma were false then we could find a sequence of balls $B_{R_k}(x_k) \subset \Omega$ such that

$$|A_k - A_0| \le \sigma,\ A_k := \fint_{B_{R_k}(x_k)} \nabla u \, dx,$$

$$E\big(u, B_{R_k}(x_k)\big) = \varepsilon_k^2 \longrightarrow 0 \qquad \text{but}$$

$$E\big(u, B_{tR_k}(x_k)\big) > c_* t^2\ E\big(u, B_{R_k}(x_k)\big).$$

We let

$$v_k(z) := \frac{1}{\varepsilon_k R_k}\left[u(x_k + R_k z) - (u)_{x_k,R_k} - R_k\, A_k z\right],\ z \in B_1,$$

hence

$$(\nabla v_k)_{0,1} = 0, \quad (v_k)_{0,1} = 0, \quad \nabla v_k(z) = \varepsilon_k^{-1}\big(\nabla u(x_k + R_k z) - A_k\big),$$

$$\fint_{B_1} |\nabla v_k|^2\, dz \ = \ \varepsilon_k^{-2} \fint_{B_{R_k}(x_k)} |\nabla u - A_k|^2\, dx \le 1,$$

$$\fint_{B_1} |\nabla v_k|^p\, dz \ = \ \varepsilon_k^{-p} \fint_{B_{R_k}(x_k)} |\nabla u - A_k|^p\, dx \le \varepsilon_k^{2-p},$$

that is

$$\int_{B_1} |\varepsilon_k^{1-2/p}\ \nabla v_k|^p\, dz \le 1. \tag{1.4}$$

After passing to subsequences we can arrange

$$\begin{cases} A_k \rightharpoonup: A, \det A \in [\frac{1}{2}(a_0+\delta), \frac{1}{2}(a_0+\delta^{-1})], \\ v_k \rightharpoonup v \quad \text{weakly in} \quad H^{1,2}(B_1, \mathbb{R}^n), \\ v_k \to v \quad \text{strongly in} \quad L^2(B_1, \mathbb{R}^n), \\ \varepsilon_k^{1-2/p} \nabla v_k \rightharpoonup 0 \quad \text{weakly in} \quad L^p(B_1, \mathbb{R}^{n\times n}). \end{cases} \tag{1.5}$$

For the last statement we use (1.4) to deduce

$$\varepsilon_k^{1-2/p} \nabla v_k \rightharpoonup: F \quad \text{weakly in} \quad L^p(B_1, \mathbb{R}^{n\times n}).$$

This implies for any testvector $\psi$

$$\begin{aligned} &|\int_{B_1} F : \psi \, dz| = \lim_{k\to\infty} |\int_{B_1} \varepsilon_k^{1-2/p} \nabla v_k : \psi \, dz| \\ &\le \limsup_{k\to\infty} \varepsilon_k^{1-2/p} \|\nabla v_k\|_{L^2(B_1)} \|\psi\|_{L^2(B_1)} = 0 \end{aligned}$$

since we assume $p > 2$. From (1.5) we also deduce

$$\begin{cases} \varepsilon_k \nabla v_k \longrightarrow 0 \quad \text{strongly in} \quad L^p(B_1, \mathbb{R}^{n\times n}) \\ \qquad\qquad\qquad \text{and almost everywhere on} \quad B_1. \end{cases} \tag{1.6}$$

**LEMMA 1:** *The weak limit $v$ satisfies*

$$\begin{cases} \int_{B_1} & p|A|^{p-2}[\nabla v + (p-2)|A|^{-2}(A : \nabla v)A] : \nabla \varphi \\ & + H''(\det A)(Cof\, A : \nabla v)(Cof\, A : \nabla \varphi)\, dz = 0 \end{cases} \tag{1.7}$$

*for all $\varphi \in C_0^1(B_1, \mathbb{R}^n)$.*

Here Cof $A$ denotes the cofactor matrix of $A$ which by definition satisfies the equation

$$A \circ (\mathrm{Cof}\, A)^T = \det A \, \mathbb{1}.$$

We calculate

$$\det(A+B) - \det A = \int_0^1 \frac{d}{ds} \det(A+sB)\, ds = \int_0^1 \mathrm{Cof}(A+sB) : B\, ds.$$

Since $A \neq 0$ and $H''(\det A) \geq 0$ (1.7) is a linear elliptic system with constant coefficients. This gives $v \in C^\infty(B_1, \mathbb{R}^n)$ and the estimate (see [7])

$$\fint_{B_t} |\nabla v - (\nabla v)_t|^2\, dz \le c_0\, t^2 \fint_{B_1} |\nabla v - (\nabla v)_1|^2\, dz \tag{1.8}$$

with $c_0$ depending only on $A_0$ ,$p$ and $C^2$–norms of $H$ near $\det A_0$. We define $c_* := 2c_0$.

In order to verify (1.7) we rewrite the Euler equation for $u$ in terms of the scaled functions:

$$\begin{aligned} 0 &= \int_{B_1} p\varepsilon_k^{-1}\big(|\varepsilon_k\nabla v_k + A_k|^{p-2}(\varepsilon_k\nabla v_k + A_k) - |A_k|^{p-2}A_k\big) : \nabla\varphi\, dz \\ &+ \int_{B_1} \varepsilon_k^{-1}\big(h'(\det[\varepsilon_k\nabla v_k + A_k]) - h'(\det A_k)\big)\, \mathrm{Cof}\,(\varepsilon_k\nabla v_k + A_k) : \nabla\varphi\, dz \\ &=: I_k + II_k, \qquad \varphi \in C_0^1\,(B_1, \mathbf{R}^n). \end{aligned}$$

Clearly

$$\begin{aligned} I_k &= p\,\varepsilon_k^{-1} \int_{B_1}\int_0^1 |A_k + s\varepsilon_k\nabla v_k|^{p-2}\,\varepsilon_k\nabla v_k : \nabla\varphi\, ds\, dz \\ &+ p(p-2)\varepsilon_k^{-1} \int_{B_1}\int_0^1 |A_k + s\varepsilon_k\nabla v_k|^{p-4}(A_k + s\varepsilon_k\nabla v_k) : \varepsilon_k\nabla v_k \\ &\quad (A_k + s\varepsilon_k\nabla v_k) : \nabla\varphi\, ds\, dz \\ &=: I_k^1 + I_k^2, \\ I_k^1 &= p\int_{B_1} |A_k|^{p-2}\nabla v_k : \nabla\varphi\, dz \\ &+ p\int_{B_1}\Big(\int_0^1 |A_k + s\varepsilon_k\nabla v_k|^{p-2}\, ds - |A_k|^{p-2}\Big)\nabla\varphi : \nabla v_k\, dz \end{aligned}$$

with

$$\lim_{k\to\infty} I_k^1 = p\int_{B_1} |A|^{p-2}\nabla v : \nabla\varphi\, dz.$$

This will follow as soon we can also show that the second term in $I_k^1$ vanishes as $k \to \infty$. To this purpose fix $\varepsilon > 0$ and select $M \subset B_1$ such that (recall (1.6))

$$\mathcal{L}^n(M) < \varepsilon, \quad \varepsilon_k\nabla v_k \to 0 \quad \text{uniformly on} \quad B_1 - M.$$

Then

$$\begin{aligned} &\Big| \int_{B_1}\Big(\int_0^1 |A_k + s\varepsilon_k\nabla v_k|^{p-2}\, ds - |A_k|^{p-2}\Big)\nabla v_k : \nabla\varphi\, dz\,\Big| \\ &\le \int_{B_1 - M} |\nabla v_k|\,|\nabla\varphi|\, dz \quad \sup_{B_1-M}\Big|\int_0^1 |A_k + s\varepsilon_k\nabla v_k|^{p-2}\, ds - |A_k|^{p-2}\Big| \\ &+ \int_M c\ \ \big(1 + \varepsilon_k^{p-2}|\nabla v_k|^{p-2}\big)|\nabla v_k|\,|\nabla\varphi|\, dz \end{aligned}$$

where here and in the sequel $c$ denotes a constant independent of $k$.

The quantities involving $B_1 - M$ go to zero as $k \to \infty$. For the rest we observe

$$\begin{aligned} \int_M \cdots &\le c\|\nabla\varphi\|_{L^\infty(B_1)}\Big(\int_M |\nabla v_k| dz + \varepsilon_k^{p-2}\int_M |\nabla v_k|^{p-1}\, dz\Big) \\ &\le c\|\nabla\varphi\|_{L^\infty(B_1)}\Big(\mathcal{L}^n(M)^{1/2}\|\nabla v_k\|_{L^2(B_1)} + \varepsilon_k^{p-2}\int_M |\nabla v_k|^{p-1}\, dz\Big) \\ &\le c\|\nabla\varphi\|_{L^\infty(B_1)}\Big(\sqrt{\varepsilon} + \varepsilon\int_M \varepsilon_k^{p-2}|\nabla v_k|^p\, dz + \varepsilon^{2-p}\varepsilon_k^{p-2}\Big) \end{aligned}$$

so that $\limsup\limits_{k\to\infty} \int\limits_M \ldots \le c\|\nabla\varphi\|_{L^\infty(B_1)}(\sqrt{\varepsilon}+\varepsilon)$.
Since $\varepsilon$ was arbitrary the claim for $\lim\limits_{k\to\infty} I_k^1$ follows.

In a similar way we obtain

$$\lim_{k\to\infty} I_k^2 = \int\limits_{B_1} p\,(p-2)\,|A|^{p-4}(A:\nabla v)(A:\nabla\varphi)\,dz$$

and it remains to discuss

$$\begin{aligned}
I\!I_k &= \int\limits_{B_1} \varepsilon_k^{-1} \int\limits_0^1 \tfrac{d}{ds} h'\left(\det A_k + s[\det(A_k+\varepsilon_k\nabla v_k) - \det A_k]\right) ds \\
&\quad \cdot\left(\operatorname{Cof}(A_k+\varepsilon_k\nabla v_k):\nabla\varphi\right) dz \\
&= \int\limits_{B_1} \tfrac{1}{\varepsilon_k}\Big\{\int\limits_0^1 h''(\ldots)ds\Big\}\left(\det[A_k+\varepsilon_k\nabla v_k] - \det A_k\right) \\
&\quad \cdot\left(\operatorname{Cof}(A_k+\varepsilon_k\nabla v_k):\nabla\varphi\right) dz \\
&= \int\limits_{B_1} \Big\{\int\limits_0^1 h''(\ldots)ds\Big\}\Big(\int\limits_0^1 \operatorname{Cof}(A_k + r\varepsilon_k\nabla v_k):\nabla v_k\,dr\Big) \\
&\quad \cdot\left(\operatorname{Cof}(A_k+\varepsilon_k\nabla v_k):\nabla\varphi\right) dz.
\end{aligned}$$

By the same reasoning as for $I_k^1$ we arrive at (see also the calculation in the proof of Lemma 2)

$$\lim_{k\to\infty} I\!I_k = \int\limits_{B_1} h''(\det A)(\operatorname{Cof} A:\nabla v)(\operatorname{Cof} A:\nabla\varphi)\,dz$$

and Lemma 1 follows.

By assumption we have

$$\fint\limits_{B_{tR_k}(x_k)} |\nabla u - (\nabla u)_{x_k,tR_k}|^2 + |\nabla u - (\nabla u)_{x_k,tR_k}|^p\,dx > c_* t^2 \varepsilon_k^2$$

so that

$$\fint_{B_t} |\nabla v_k - (\nabla v_k)_t|^2 + \varepsilon_k^{p-2}|\nabla v_k - (\nabla v_k)_t|^p\,dz > c_*\,t^2.$$

Suppose now that we already know

**LEMMA 2:** *The convergence properties stated in (1.5) can be improved to*

$$\begin{aligned}
&\nabla v_k \to \nabla v \quad in\ L^2_{loc}(B_1, \mathbb{R}^{n\times n}) \\
&\varepsilon_k^{1-2/p}\nabla v_k \to 0 \quad in\ L^p_{loc}(B_1, \mathbb{R}^{n\times n})
\end{aligned}$$

*strongly.*

Then the last inequality turns into

$$\fint_{B_t} |\nabla v - (\nabla v)_t|^2\, dz \geq c_* t^2.$$

From weak convergence in $H^{1,2}(B_1, \mathbf{R}^n)$ we infer

$$\fint_{B_1} |\nabla v|^2\, dz \leq \liminf_{k\to\infty} \fint_{B_1} |\nabla v_k|^2\, dz \leq 1,$$

hence

$$\fint_{B_t} |\nabla v - (\nabla v)_t|^2\, dz \geq c_* t^2 \fint_{B_1} |\nabla v - (\nabla v)_1|^2\, dz$$

contradicting (1.8) and our choice of $c_*$.

# 2 Strong Convergence of the Scaled Sequence

It remains to prove Lemma 2. Define

$$\begin{aligned} f_k(Q) := \;& \varepsilon_k^{-2}\big(|A_k + \varepsilon_k Q|^p - |A_k|^p - p|A_k|^{p-2}A_k : \varepsilon_k Q\big) \\ &+ \varepsilon_k^{-2}\big(h(\det[A_k + \varepsilon_k Q]) - h(\det A_k) - h'(\det A_k)\{\det(A_k + \varepsilon_k Q) - \det A_k\}\big) \end{aligned}$$

for $Q \in \mathbf{R}^{n\times n}$ and observe

$$I_k^r(v_k, B_r) := \int_{B_r} f_k(\nabla v_k)\, dz \leq I_k^r(w) \tag{2.1}$$

for any $w \in H^{1,p}_{\text{loc}}(B_1, \mathbf{R}^n)$, spt $(v_k - w) \subset B_r$, $r < 1$.

We claim

$$0 \leq f_k(Q) \leq \lambda(|Q|^2 + \varepsilon_k^{p-2}|Q|^p),\; Q \in \mathbf{R}^{n\times n}, \tag{2.2}$$

for some positive constant $\lambda$ independent of $k$.

Case 1: $\varepsilon_k|Q| \leq \sigma$ Then

$$\begin{aligned} &\varepsilon_k^{-2}\big(|\varepsilon_k Q + A_k|^p - |A_k|^p - \varepsilon_k Q : A_k|A_k|^{p-2}\big) = \\ &p\int_0^1 \big(|A_k + s\varepsilon_k Q|^{p-2}\, Q : Q + (p-2)|A_k + s\varepsilon_k Q|^{p-4}(Q : [A_k + s\varepsilon_k Q])^2\big)(1-s)\, ds \\ &\leq c|Q|^2 \end{aligned}$$

and

$$\begin{aligned}
&\varepsilon_k^{-2}\big(h(\det[A_k+\varepsilon_k Q]) - h(\det A_k) - h'(\det A_k)\{\det(A_k+\varepsilon_k Q) - \det A_k\}\big)\\
&\quad = \varepsilon_k^{-2}\big(\det(A_k+\varepsilon_k Q) - \det A_k\big)^2\\
&\qquad \int_0^1 h''\big(\det A_k + s(\det[A_k+\varepsilon_k Q] - \det A_k)\big)(1-s)\,ds\\
&\quad \le \varepsilon_k^{-2}\big(\sup_{\mathbb{R}} h''\big)\Big\{\int_0^1 \tfrac{d}{ds}\det(A_k+s\varepsilon_k Q)\,ds\Big\}^2\\
&\quad = \varepsilon_k^{-2}\big(\sup_{\mathbb{R}} h''\big)\Big\{\int_0^1 \operatorname{Cof}(A_k+s\varepsilon_k Q):\varepsilon_k Q\,ds\Big\}^2\\
&\quad \le c|Q|^2.
\end{aligned}$$

Case 2: $\varepsilon_k|Q| \ge \sigma$ Then

$$\begin{aligned}
&\varepsilon_k^{-2}\big(|A_k+\varepsilon_k Q|^p - |A_k|^p - \varepsilon_k Q : A_k|A_k|^{p-2}\big) =\\
&\varepsilon_k^{p-2}\,p\int_0^1\Big\{\big|\tfrac{A_k}{\varepsilon_k}+sQ\big|^{p-2}Q:Q + (p-2)\big|\tfrac{A_k}{\varepsilon_k}+Q\big|^{p-4}\Big(Q:\big[\tfrac{A_k}{\varepsilon_k}+sQ\big]\Big)^2\Big\}(1-s)\,ds\\
&\le c\,\varepsilon_k^{p-2}\,|Q|^2\int_0^1 \big|\tfrac{A_k}{\varepsilon_k}+sQ\big|^{p-2}\,ds\\
&\le c\,\varepsilon_k^{p-2}\,|Q|^2\Big(\big|\tfrac{A_k}{\varepsilon_k}\big|^{p-2} + |Q|^{p-2}\Big)\\
&\le c\,\varepsilon_k^{p-2}\,|Q|^2\,(\varepsilon_k^{2-p} + |Q|^{p-2})\\
&\le c\,\varepsilon_k^{p-2}\,|Q|^2\,(\sigma^{-p+2}|Q|^{p-2} + |Q|^{p-2})\\
&= c\,|Q|^p\,\varepsilon_k^{p-2}
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon_k^{-2}\big(h(\det[A_k+\varepsilon_k Q]) - \dots\big)\\
&\le \varepsilon_k^{-2}\big(1 + |\det(A_k+\varepsilon_k Q)| + \sup_{\mathbb{R}}|h'|\,|\det(A_k+\varepsilon_k Q) - \det A_k|\big)\\
&\le c\,\varepsilon_k^{-2}\Big(1 + \varepsilon_k^n|Q|^n + \Big|\int_0^1 \operatorname{Cof}(A_k+s\varepsilon_k Q):\varepsilon_k Q\,ds\Big|\Big)\\
&\le c\,\varepsilon_k^{-2}\big(1+\varepsilon_k^n|Q|^n\big) = c\big(\varepsilon_k^{-2} + \varepsilon_k^{n-2}|Q|^n\big)\\
&\le c\,\big(\sigma^{-2}|Q|^2 + \varepsilon_k^{-2}(\varepsilon_k|Q|)^p(\varepsilon_k|Q|)^{n-p}\big)\\
&\le c\,\big(\sigma^{-2}|Q|^2 + \varepsilon^{p-2}|Q|^p\,\sigma^{n-p}\big)\\
&\le c\,\big(|Q|^2 + \varepsilon_k^{p-2}|Q|^p\big)
\end{aligned}$$

which proves (2.2). Following Evans - Gariepy [5] we define the measures

$$\mu_k(Z) := \int_Z |\nabla v_k|^2 + \varepsilon_k^{p-2}|\nabla v_k|^p\,dz,\quad Z \subset B_1,$$

which are uniformly bounded on account of (1.5). Thus there is a measure $\mu$ such that $\mu_k \rightharpoonup \mu$ at least for a subsequence. We fix $0 < r < 1$ with the property

$\mu(\partial B_r) = 0$. Let $0 < s < r$ and choose $\eta \in C_0^1(B_1, [0,1])$ satisfying $\eta = 1$ on $B_s$, $\eta = 0$ on $B_1 - B_r$. Next recall that $v$ is smooth so that $I_k^r(v)$ makes sense. From (2.1) we then deduce

$$\begin{aligned} I_k^r(v_k) - I_k^r(v) \;&\leq\; I_k^r\big(v_k + \eta(v - v_k)\big) - I_k^r(v) \\ &\underset{(2.2)}{\leq} c \int_{B_r - B_s} \Big(|\nabla v_k|^2 + |\nabla v|^2 + \varepsilon_k^{p-2}\left[|\nabla v_k|^p + |\nabla v|^p\right] \\ &\qquad + |\nabla \eta|^2 |v_k - v|^2 + \varepsilon_k^{p-2} |\nabla \eta|^p |v_k - v|^p\Big)\, dx \end{aligned}$$

which gives

$$\limsup_{k\to\infty} \big(I_k^r(v_k) - I_k^r(v)\big) \leq c \left(\mu(\overline{B_r - B_s}) + \int_{B_r - B_s} |\nabla v|^2\right).$$

Taking the limit $s \nearrow r$ we arrive at

$$\limsup_{k\to\infty} \big(I_k^r(v_k) - I_k^r(v)\big) \leq 0 \tag{2.3}$$

for $\mathcal{L}^1$–almost all $r \in (0,1)$.

On the other hand we have

$$I_k^r(v_k) - I_k^r(v) =: I_k + II_k,$$

where

$$\begin{aligned} I_k &= \varepsilon_k^{-2} \int_{B_r} |A_k + \varepsilon_k \nabla v_k|^p - |A_k + \varepsilon_k \nabla v|^p - p|A_k|^{p-2} \varepsilon_k A_k : (\nabla v_k - \nabla v)\, dx \\ &= \varepsilon_k^{-2} \int_{B_r} |A_k + \varepsilon_k \nabla v_k|^p - |A_k + \varepsilon_k \nabla v|^p \\ &\qquad - p|A_k + \varepsilon_k \nabla v|^{p-2}(A_k + \varepsilon_k \nabla v) : \varepsilon_k(\nabla v_k - \nabla v)\, dx \\ &\quad + \varepsilon_k^{-2} \int_{B_r} p\big(|A_k + \varepsilon_k \nabla v|^{p-2}(A_k + \varepsilon_k \nabla v) - |A_k|^{p-2} A_k\big) : \varepsilon_k(\nabla v_k - \nabla v)\, dx \\ &=: III_k + IV_k, \end{aligned}$$

$$\begin{aligned} III_k &= \varepsilon_k^{-2} \int_{B_r} \int_0^1 \tfrac{d^2}{ds^2} \mid A_k + \varepsilon_k \nabla v + s\varepsilon_k(\nabla_k - \nabla v) \mid^p (1-s)\, ds\, dx \\ &\geq p \int_{B_r} \int_0^1 \mid A_k + \varepsilon_k \nabla v + s\varepsilon_k(\nabla v_k - \nabla v) \mid^{p-2} (1-s)\, ds |\nabla v_k - \nabla v|^2\, dx. \end{aligned}$$

Using

$$\int_0^1 \mid \alpha + s\beta \mid^{p-2} (1-s)\, ds \geq c(p)(|\alpha|^{p-2} + |\beta|^{p-2})$$

we see

$$\begin{aligned} III_k &\geq c(p) \int_{B_r} (|A_k + \varepsilon_k \nabla v|^{p-2} + \varepsilon_k^{p-2} |\nabla v_k - \nabla v|^{p-2})\, |\nabla v_k - \nabla v|^2\, dx \\ &\geq c(p) \int_{B_r} (1 + \varepsilon_k^{p-2} |\nabla v_k - \nabla v|^{p-2}) |\nabla v_k - \nabla v|^2\, dx \end{aligned}$$

being valid for large enough $k$. Here again the local boundedness of $\nabla v$ enters in an essential way. Similar to the discussion of the quantity $I_k^1$ in section 1 we can prove

$$\lim_{k\to\infty} IV_k = 0$$

Finally we have (using convexity of $h$)

$$\begin{aligned} II_k &= \varepsilon_k^{-2} \int_{B_r} h(\det[A_k + \varepsilon_k \nabla v_k]) - h(\det[A_k + \varepsilon_k \nabla v]) \\ &\quad - h'(\det A_k)(\det[A_k + \varepsilon_k \nabla v_k] - \det[A_k + \varepsilon_k \nabla v])\, dx \\ &\geq \varepsilon_k^{-2} \int_{B_r} \big(h'(\det[A_k + \varepsilon_k \nabla v]) - h'(\det A_k]\big) \\ &\quad \cdot (\det[A_k + \varepsilon_k \nabla v_k] - \det[A_k + \varepsilon_k \nabla v])\, dx. \end{aligned}$$

Writing

$$h'(\det[A_k + \varepsilon_k \nabla v]) - h'(\det A_k) =$$

$$\int_0^1 h''\big(\det A_k + s\{\det(A_k + \varepsilon_k \nabla v) - \det A_k\}\big)\, ds\, \{\det(A_k + \varepsilon_k \nabla v) - \det A_k\}$$

and observing

$$\begin{aligned} &\det(A_k + \varepsilon_k \nabla v) - \det A_k = \int_0^1 \operatorname{Cof}(A_k + s\varepsilon_k \nabla v) : \varepsilon_k \nabla v\, ds \\ &\det(A_k + \varepsilon_k \nabla v_k) - \det(A_k + \varepsilon_k \nabla v) = \\ &\int_0^1 \operatorname{Cof}\big(A_k + \varepsilon_k \nabla v + s\varepsilon_k(\nabla v_k - \nabla v)\big) : \varepsilon_k(\nabla v_k - \nabla v)\, ds \end{aligned}$$

we get

$$\begin{aligned} II_k \geq &\int_{B_r} \Big(\int_0^1 h''(\ldots) ds\Big)\Big(\int_0^1 \operatorname{Cof}(A_k + s\varepsilon_k \nabla v) : \nabla v\, ds\Big) \\ &\Big(\int_0^1 \operatorname{Cof}\big(A_k + \varepsilon_k \nabla v + s\varepsilon_k(\nabla v_k - \nabla v)\big) : (\nabla v_k - \nabla v)\, ds\Big)\, dx. \end{aligned}$$

We claim that the right hand side vanishes as $k \to \infty$. To this purpose we consider the case $n \geq 3$ ($n = 2$ follows by simplification) and observe

$$\int_0^1 h''(\ldots)\, ds \overset{k\to\infty}{\longrightarrow} h''(\det A) \qquad \text{uniformly on } B_r$$

and (using $v \in C^\infty(\overline{B_r})$)

$$\int_0^1 \operatorname{Cof}(A_k + s\varepsilon_k \nabla v) : \nabla v\, ds \overset{k\to\infty}{\longrightarrow} \operatorname{Cof} A : \nabla v \qquad \text{uniformly on } B_r.$$

We therefore have to show

$$\int_{B_r}\int_0^1 \operatorname{Cof}\big(A_k + \varepsilon_k \nabla v + s\varepsilon_k(\nabla v_k - \nabla v)\big) : (\nabla v_k - \nabla v)\, ds\, dx \overset{k\to\infty}{\longrightarrow} 0.$$

To this purpose we choose $M \subset B_r$ such that

$$\mathcal{L}^n(M) < \varepsilon, \ \ \varepsilon_k(\nabla v_k - \nabla v) \overset{k\to\infty}{\longrightarrow} 0 \quad \text{uniformly on } B_r - M.$$

Clearly

$$\int_{B_r - M} \int_0^1 \operatorname{Cof}\big(A_k + \varepsilon_k \nabla v + s\varepsilon_k(\nabla v_k - \nabla v)\big) : (\nabla v_k - \nabla v)\, ds\, dx \overset{k\to\infty}{\longrightarrow} 0.$$

Then, recalling (1.5), (1.6)

$$\begin{aligned}
&\Big| \int_M \int_0^1 \operatorname{Cof}(\ldots) : (\nabla v_k - \nabla v)\, ds\, dx \Big| \\
&\le \int_M c\,(1 + \varepsilon_k^{n-1} |\nabla v_k|^{n-1}(|\nabla v_k| + |\nabla v|)\, dx \\
&\le c\Big\{ \int_M |\nabla v_k| + \int_M \varepsilon_k^{n-1} |\nabla v_k|^n \Big\} \\
&\le c\big(\mathcal{L}^n(M)^{1/2} \|\nabla v_k\|_2\big) + c\Big(\varepsilon_k^{\frac{n-1}{n-2}} \int_{B_r} \varepsilon_k^{p-2} |\nabla v_k|^p\Big)^{(1-\lambda)\frac{n}{p}} \\
&\le c\big(\sqrt{\varepsilon} + c\, \varepsilon_k^{\frac{n-1}{n-2}\frac{n}{p}(1-\lambda)}\big) \overset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Here we have used the interpolation inequality

$$\|\nabla v_k\|_n \le \|\nabla v_k\|_2^{\lambda}\, \|\nabla v_k\|_p^{1-\lambda} \qquad \text{with} \quad \frac{1}{n} = \frac{\lambda}{2} + \frac{1-\lambda}{p}.$$

Collecting the various estimates we end up with

$$\limsup_{k\to\infty} \big(I_k^r(v_k) - I_k^r(v)\big) \ge$$

$$c \limsup_{k\to\infty} \int_{B_r} |\nabla v_k - \nabla v|^2 + \varepsilon_k^{p-2} |\nabla v_k - \nabla v|^p\, dx$$

for $\mathcal{L}^1$–almost all $0 < r < 1$ which together with (2.3) completes the proof of Lemma 2 and hence the proof of Theorem 2.

# 3 Remarks

Theorems 1 and 2 easily extend to more general functionals of the type

$$F(u) = \int_\Omega f(\nabla u) + h_\delta(\det \nabla u)\, dx$$

with $f$ of growth order $p$ in $\nabla u$ and being strictly convex. For example we may take

$$f(Q) = \big(A_{\alpha\beta}^{ij}\, Q_\alpha^i\, Q_\beta^j\big)^{p/2}$$

with constant coefficients satisfying the Legendre – Hadamard condition. We leave the details to the reader.

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