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ROTATIONAL HYPERSURFACES OF SPACE FORMS WITH CONSTANT SCALAR CURVATURE

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Let M be a complete rotational hypersurface of a space form with constant scalar curvature S. In this paper we classify these hypersurfaces in the cases of $\mathbf{R}^n$ and $\mathbf{H}^n$, determine the admissible values of S in each of the three spaces and give a geometrical description of the hypersurfaces according to the values of S. In the case of $\mathbf{S}^n$ we find examples of embedded hypersurfaces with constant $S \in (\frac{n-2}{n-1}, 1)$, which are not isometric to product of spheres.

The scalar curvature S of a riemannian manifold is an important geometric invariant, thus the interest in those manifolds with constant S and in particular, in the hypersurfaces of space forms.

One important result is the theorem of A. Ros [7] according to which the only embedded compact hypersurfaces of $\mathbf{R}^n$ with constant S are round spheres. For the non-compact ones there is a theorem of Cheng-Yau [3] stating that the only complete examples with sectional curvatures $K \geq 0$ are $\mathbf{S}^{k-1} \times \mathbf{R}^{n-k}$, $1 \leq k < n$.

In [4] and [5] Hsiang analysed rotational hypersurfaces of space forms with a symmetric function σ_j of the principal curvatures constant, which includes constant scalar curvature when $j = 2$. There he obtains a collection of complete

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hypersurfaces of $\mathbf{R}^n$ and $\mathbf{H}^n$ with $S > 0$ and of $\mathbf{S}^n$ with $S > 1$, but no classification theorem is presented.

In this paper we classify all complete rotational hypersurfaces of $\mathbf{R}^n$ and $\mathbf{H}^n$ with constant scalar curvature (Theorems 3.4 and 3.5). Partial results for $\mathbf{S}^n$ are presented in Theorem 3.6. We also prove that S is precisely greater than or equal to the space form curvature, except in the case of $\mathbf{S}^n$ where any value greater than $(n-3)/(n-1)$ is admissible. In particular we exhibit a collection of new complete hypersurfaces of $\mathbf{H}^n$ with S ranging in $[-1, 0]$, of $\mathbf{R}^n$ with $S = 0$ and of $\mathbf{S}^n$ with S in the interval $(\frac{n-3}{n-1}, 1)$. Surprising examples of embedded hypersurfaces of $\mathbf{S}^n$ with $S < 1$ are presented. We point out that Theorem 3.4 has been announced earlier (see [2]).

Our results suggest interesting problems in Global Differential Geometry. We state below three of these: the first one carries a flavor of Hilbert theorem for surfaces and the second a flavor of Bernstein theorem for minimal surfaces.

- Is there a complete hypersurface of $\mathbf{R}^n$ with constant $S < 0$?
- Is there a nonflat complete graph in $\mathbf{R}^n$ with constant $S = 0$?
- Are there embedded hypersurfaces of $\mathbf{S}^n$ with constant $S \geq 1$ other than product of spheres?

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1. UNIFIED EXPRESSION FOR THE SCALAR CURVATURE.

We denote by $\mathbf{N_c}$ the simply connected n-dimensional space form of constant curvature $c = 0, 1$ or -1. We will take as models for $\mathbf{N_c}$ the euclidean space $\mathbf{R}^n$, the unitary round sphere $\mathbf{S}^n$ and the hyperbolic upper space $\mathbf{H}^n = \{y \in \mathbf{R}^n : y_n > 0\}$.

Let M be a rotational hypersurface of $\mathbf{N_c}$, that is, invariant by the orthogonal group $\mathbf{O}(n-1)$ considered as a subgroup of isometries of the ambient space. To study the geometry of M, we generalize the method used by Spivak ([8], page 173) to compute the intrinsic Gaussian curvature of a rotational surface of $\mathbf{H}^3$.

There, an element of $\mathbf{O}(2)$ fixes all points of a given geodesic γ, which is the axis of revolution, and rotates the initial tangent vector of a geodesic ray starting orthogonally from γ. The orbit of a point p, at a distance $r > 0$ from γ, under the action of $\mathbf{O}(2)$ is a geodesic circle of radius r and center in γ. He chooses (s, θ) for coordinates of the surface, where s is the arc length of its profile curve α and θ is the angle of rotation. The curve α lies in the orbit space $\{(y_1, 0, y_3) \in \mathbf{H}^3 : y_1 \geq 0\}$. The first fundamental form is given by

$$I = \sinh^2 r(s) d\theta \otimes d\theta + ds \otimes ds,$$

so the curvature is intrinsically computed.

In the general case, the orbit of a point in $\mathbf{N_c}$ under the action of $\mathbf{O}(n-1)$ is an $(n-2)$-dimensional geodesic sphere of radius equal to the distance r from the point to γ. We also choose coordinates (s, Θ), where s is as before and $\Theta = (\theta_1, \cdots, \theta_{n-2})$ parametrizes the unitary euclidean $(n-2)$-sphere whose points are in correspondence with the initial velocities of all geodesic rays with length $r(s)$ which are perpendicular to γ. The axis of rotation γ is the vertical geodesic defined by $y_1 = y_2 = \cdots = y_{n-1} = 0$, so the profile curve α lies in the orbit space given by $y_2 = \cdots = y_{n-1} = 0$ and $y_1 \geq 0$.

In this case, the first fundamental form is given by

$$I = f^2(r(s)) \Sigma g_{ij}(\Theta) d\theta_i \otimes d\theta_j + ds \otimes ds,$$

where g_{ij} is the metric of constant sectional curvature 1 in an $(n-2)$-dimensional sphere and $f(r) = r, \sin r$ or $\sinh r$, depending on whether $c = 0, 1$ or -1.

PROPOSITION 1.1. *The scalar curvature S of a rotational hypersurface of $\mathbf{N_c}$ is constant along the orbits and is given by*

$$S = -\frac{2\ddot{F}}{(n-1)F} + \frac{(n-3)(1-\dot{F}^2)}{(n-1)F^2},$$

where $F(s) = f(r(s))$ and s is the arc length of the profile curve.

PROOF. We recall that S is the normalized average of the sectional curvatures on a basis of orthogonal two-dimensional subspaces of the tangent space at a given point. In coordinates (s, Θ) we compute intrinsically from the first fundamental form that $K(\partial/\partial\theta_i, \partial/\partial\theta_j) = (1-\dot{F}^2)/F^2$ and $K(\partial/\partial\theta_i, \partial/\partial s) = -\ddot{F}/F$, $1 \leq i < j \leq n-2$, hence

$$\binom{n-1}{2} S = \binom{n-2}{2} \frac{(1-\dot{F}^2)}{F^2} - (n-2)\frac{\ddot{F}}{F}.$$

REMARK. In terms of extrinsic geometry, we may compute the $n-1$ principal curvatures of the hypersurface and use the Gauss formula $K_{ij} = k_i k_j + c$ to obtain the above expression for S. For the sake of curiosity we write down the principal curvatures in coordinates (s, Θ):

$$k_1 = \cdots = k_{n-2} = -\frac{\sqrt{1 - \dot{F}^2 - cF^2}}{F}, \quad k_{n-1} = \frac{\ddot{F} + cF}{\sqrt{1 - \dot{F}^2 - cF^2}},$$

where the normal points outwards the orbit. The geodesic curvature of α in the orbit space is $-k_{n-1}$.

2. EQUATIONS FOR S CONSTANT. REDUCTION TO A FIRST ORDER SYSTEM.

The proposition 1.1 yields that an $\mathbf{O}(n-1)$ hypersurface of $\mathbf{N_c}$ with constant scalar curvature S is generated by a unit speed curve $\alpha(s)$, to which corresponds a solution $F(s)$ of the 2^{nd} order ODE

$$2F\ddot{F} - (n-3)(1 - \dot{F}^2) + (n-1)SF^2 = 0. \tag{2.1}$$

In order to determine α from a solution of (2.1), we make use of the orbit space geometry. Since $F = f(r)$ is injective for $r > 0$, it suffices to solve the 1^{st} order ODE

$$\dot{r}^2 + \left(\frac{df}{dr}\right)^2 \dot{h}^2 = 1, \tag{2.2}$$

where $h(s)$ measures the riemannian height, with respect to a fixed origin in γ, of the point where the geodesic ray starting from α will meet γ. The equation (2.2) states that α is parametrized by arc length. The factor $(df/dr)^2$ is intrinsic to the geometry of $\mathbf{N_c}$, as it can be seen by standard computation. We fix as origins the points $(0, ..., 0) \in \mathbf{R}^n$, $(0, ..., 0, 1) \in \mathbf{S}^n \subset \mathbf{R}^{n+1}$ or $(0, ..., 0, 1) \in \mathbf{H}^n$, according to $c = 0, 1$ or -1, respectively. It is well known from hyperbolic geometry that $\sinh r = \tan\phi$ and $\exp(h) = \rho$, where ρ and ϕ are polar coordinates in the euclidean plane.

Therefore our problem of classification is reduced to the determination of all complete integral curves of the system given by equations (2.1) and (2.2). Of course we include in this set those curves meeting orthogonally the orbit space boundary, i.e., those solutions where $r(s) \to 0$ together with $\dot{r}^2(s) \to 1$, as $s \to s_0$.

REMARK. From now on, we assume that $n \geq 4$. The case $n = 3$ is of a different nature due to the vanishing of the factor $(n-3)(1-\dot{F}^2)$ in equation (2.1).

PROPOSITION 2.2. *Equation (2.1) is equivalent to its first order integral*

$$F^{n-3}(1-\dot{F}^2) - SF^{n-1} = K, \qquad (2.3)$$

where K is a constant; moreover, for a constant solution equals to F_0, one has that $S > 0$ and $F_0^2 = (n-3)/(n-1)S$, so

$$K_0 = \frac{2}{(n-1)}\left[\frac{(n-3)}{(n-1)S}\right]^{\frac{(n-3)}{2}}.$$

PROOF. The left hand side of equation (2.1) multiplyed by $-\dot{F}F^{n-4}$ is precisely the derivative of $F^{n-3}(1-\dot{F}^2) - SF^{n-1}$, which is the left hand side of equation (2.3); let $F(s) = F_0$ in (2.1), so that $(n-1)SF_0{}^2 = (n-3)$ and the corresponding value of the constant K is K_0.

COROLLARY 2.3. *The constant solutions of the system formed by (2.1) and (2.2) correspond to embedded cylinders equidistant from γ, that is, to hypersurfaces isometric to $\gamma \times \mathbb{S}^{n-2}{}_{(r_0)}$, with $F_0 = f(r_0)$ as in Proposition 2.2. Moreover, for $c = 0$ or -1, S ranges in $(0, \infty)$, while for $c = 1$ it ranges in $(\frac{n-3}{n-1}, \infty)$. Figure 1 illustrates these curves.*

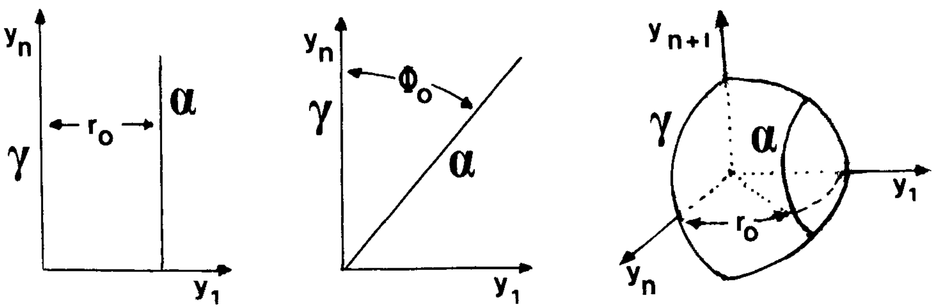

Figure 1: Cylinders in space forms

PROOF. We have that $r(s)$ is constant, so $h(s) = as$, with $a = 1$, $1/\cosh r_0$, or $1/\cos r_0$, depending on $c = 0, -1$ or 1, respectively. In all cases, the orbit is isometric to a round sphere of radius r_0, where $F_0 = f(r_0)$.

Clearly, the rotational hypersurfaces are riemannian products of the axis γ by the constant orbit. As ${F_0}^2 = \frac{(n-3)}{(n-1)S}$, we have that for $c = 0$ or -1 the only restriction to S is to be positive, while for $c = 1$ the value ${F_0}^2 = \sin^2 r_0$ must be smaller than 1.

3. THE THEOREMS OF CLASSIFICATION.

Equation (2.3) tells us that a local solution F of (2.1) paired with its first derivative is a subset, denoted by $(F, \dot{F})$, of a level curve for the function H defined by

$$H(u,v) = u^{n-3}(1 - v^2 - Su^2),$$

with $u > 0$.

DEFINITION 3.1. *We say that a solution $F > 0$ of (2.3) is* **complete** *if either F is defined for all s or if $(F, \dot{F})$ admits only (0,1) and (0,-1) as limit values.*

LEMMA 3.2. *All solutions of equation (2.3) can be extended to complete solutions. The sets $(F, \dot{F})$ are connected components of the level curves indicated in Figures 2,3 and 4.*

PROOF. Let us map the open half plane $\{(u, v) : u > 0\}$ by level curves $H = K$. Each curve is a smooth union of two graphs

$$(\pm v)^2 = 1 - Su^2 - \frac{K}{u^{n-3}},$$

except for the level K_0 given by proposition 2.2, when $S > 0$. The level curve $H = K_0$ consists of the unique critical point of H, which is on the horizontal axis, as it can be seen from

$$\nabla H = u^{n-4}((n-3)(1-v^2) - (n-1)Su^2, -2uv).$$

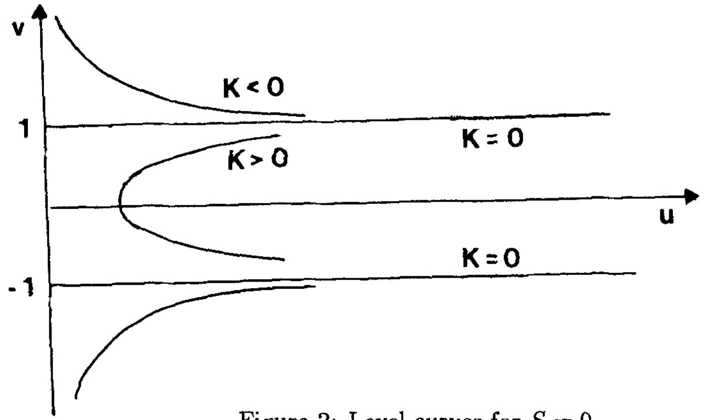

Figure 2: Level curves for $S = 0$

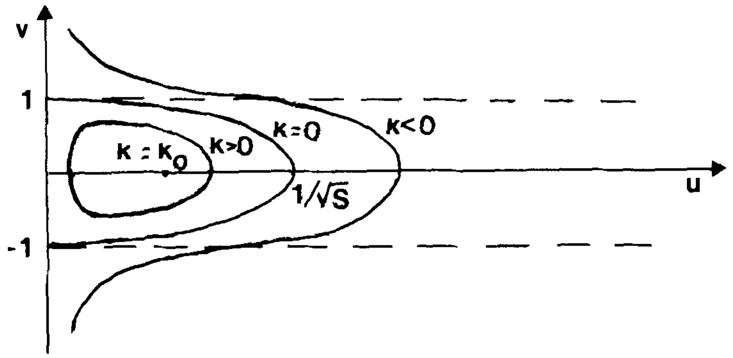

Figure 3: Level curves for $S > 0$

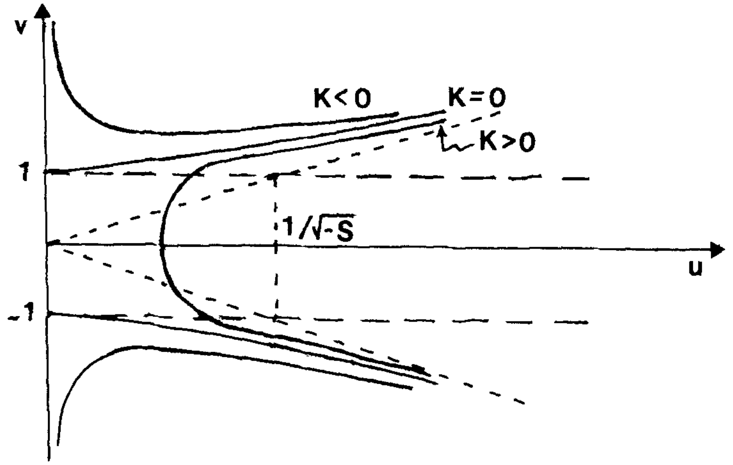

Figure 4: Level curves for $S < 0$

For $K = 0$, the level curve $v^2 + Su^2 = 1$ is a conic whose boundary in the closed half plane consists of $(0,1)$ and $(0,-1)$.

For $K \neq 0$, we claim that the level curve is closed in the open half plane, hence it is a complete subset. Indeed, if $u \to 0^+$ one gets that $v^2 \to \infty$, so either the level curve asymptotes the vertical axis or it is at a positive distance from it.

We consider the foliation of the open half plane by level curves $H = K$ for $S = 0,\ S > 0$ and $S < 0$:

CASE $S = 0$ (figure 2). Here K takes all values and H has no critical points. The conic at $K = 0$ is formed by two half lines. For $K > 0$ one has that $u^{n-3} \geq K$.

CASE $S > 0$ (figure 3). Here H has a maximum at K_0, so K takes all values $\leq K_0$. The conic at $K = 0$ is a half ellipse. For $0 < K \leq K_0$, the level curve is compact and for $K < 0$, it asymptotes the vertical axis.

CASE $S < 0$ (figure 4). Here K takes all values and H has no critical points. The conic $K = 0$ is formed by two half branches of an hyperbola . For $K > 0$, the level curve asymptotes the conic from its inner side, and for $K < 0$, from its outer side.

It follows from the theory of ODE that a local solution $F(s)$ of (2.3) can be extended through values of s for which $(F, \dot{F})$ is interior to the half space. We look at the level curves maps and conclude that the solutions F corresponding to $K \neq 0$ are complete. Furthemore, the solutions corresponding to $K = 0$ may be extended up to a value where $(F, \dot{F}) \to (0, \pm 1)$, hence they are complete in the sense of definition 3.1.

LEMMA 3.3. *Given a solution (r, h) of the system given by equations (2.1) and (2.2), it determines a solution F of (2.3) such that*

$$\dot{F}^2 \leq 1\ , \quad \frac{\dot{F}^2}{1 + F^2} \leq 1 \quad \text{or} \quad \frac{\dot{F}^2}{1 - F^2} \leq 1\ ,$$

depending on $c = 0, -1$ or 1, respectively.

PROOF. It follows from equation (2.2) that $\dot{r}^2 \leq 1$. This inequality gives the desired result, for $F(s) = r(s),\ \sinh r(s)$ or $\sin r(s)$, respectively.

THEOREM 3.4 (CLASSIFICATION in $\mathbf{R}^n$).

i) Up to vertical translation, there is precisely a one parameter family of complete rotational hypersurfaces of constant scalar curvature $S = 0$, converging to a hyperplane $\mathbf{R}^{n-1}$. In $\mathbf{R}^4$ the profile curve is a parabola, in $\mathbf{R}^5$ it is a catenary and in $\mathbf{R}^n$, $n \geq 6$, it asymptotes two horizontal lines. In all cases, the hypersurfaces are embedded.

ii) For any $S > 0$, there is a one parameter family of complete embedded hypersurfaces of constant scalar curvature S, all periodic and cylindrically bounded, which converges to a sequence of spheres, two by two vertically tangent (see figure 5).

iii) There is no complete rotational hypersurface with constant scalar curvature $S < 0$.

PROOF. By lemma 3.2, all solutions of equation (2.3) are complete. But those corresponding to complete hypersurfaces of $\mathbf{R}^n$ must also satisfy inequality $\dot{F}^2 \leq 1$, as asserted by Lemma 3.3. Thus, only level curves contained in the region $v^2 \leq 1$ will be taken into consideration. The admissible values of K are directly indicated in figures 2, 3 and 4, depending on $S = 0$, $S > 0$ or $S < 0$.

In any of the three cases we have that $u = r$ and $v = \dot{r}$, so the intersection of a level curve with the horizontal axis $v = 0$ corresponds precisely to the points where the distance r from the profile curve α to the axis of rotation γ is critical; clearly the symmetry of all level curves allows critical distances only of maximal and minimal type.

Without loss of generality, we will take $h(0) = 0$ for initial height always.

For $S = 0$, figure 2 yields that K takes values in $[0, \infty)$. The value $K = 0$ gives us the trivial solution $r(s) = s$ and $h(s) = 0$, $s \geq 0$, corresponding to the hyperplane generated by the horizontal line $h = 0$.

For a fixed $K > 0$, r attains a unique minimum $r_1 > 0$ which we take as initial distance $r(0)$.

It follows from equations (2.2) and (2.3) that

$$\dot{h}^2 = 1 - \dot{r}^2 = \frac{K}{r^{n-3}} ,$$

where $K = {r_1}^{n-3}$. Clearly r has no upper bound, so no hypersurface is cylindrically bounded. Away from r_1, we may divide $\dot{h}^2$ by $\dot{r}^2$ to get

$$\left(\frac{dh}{dr}\right)^2 = \frac{{r_1}^{n-3}}{r^{n-3} - {r_1}^{n-3}},$$

hence the profile curve is formed by two symmetric graphs as given below

$$\pm h = \sqrt{{r_1}^{n-3}} \times \int_{r_1}^{r} \frac{dr}{\sqrt{r^{n-3} - {r_1}^{n-3}}}.$$

We can solve these convergent integrals when $n = 4$ and $n = 5$, and get that $\pm h = 2\sqrt{r_1}\sqrt{r - r_1}$ and $\pm h = r_1 \log(\frac{\sqrt{r^2 - r_1^2} + r}{r_1})$, respectively. We observe that the profile curves are parabolas * $r - r_1 = h^2/4\,r_1$ in the case $n = 4$, and catenaries $r/r_1 = \cosh(h/r_1)$ in the case $n = 5$.

When $n = 6$, we compare the integrand with $1/\sqrt{r^{n-3}/2}$, for $r \geq 2r_1$, and get that the integral $\int_{2r_1}^{r} \frac{dr}{\sqrt{r^{n-3} - {r_1}^{n-3}}}$ is uniformly bounded by the constant $\int_{2r_1}^{\infty} \frac{\sqrt{2}dr}{\sqrt{r^{n-3}}}$, so the profile curve is asymptotic to two horizontal lines $\pm h = constant$. Geometrically , this means that except for dimensions $n = 4$ and $n = 5$, the distance of a rotational hypersurface of $\mathbf{R}^n$ with 0 scalar curvature from the axis reaches infinity in a finite interval of height. This property also holds for rotational hypersurfaces of $\mathbf{R}^n$ with 0 mean curvature, except for $n = 3$ (see [1]).

The embeddedness is clear, since the two graphs glue smoothly at $(r_1, 0)$ with tangent line parallel to γ. This completes the proof of i).

For $S < 0$, figure 4 yields that for $K > 0$ there are local solutions, although none of these can be completed due to the fact that $\dot{F}^2$ reaches 1 in a finite interval. For $K \leq 0$ not even local solutions do exist. This proves iii).

For $S > 0$, it is immediate from figure 3 that the admissible set for K is $[0, K_0]$ and that all level curves are compact. Also the level curves for negative values of K correspond to non-complete hypersurfaces, for they reach the region $v^2 > 1$.

The value $K = K_0$ gives us the solution $r(s) = r_0$ and $h(s) = s$, corresponding to the cylinder of Corollary 2.3.

The value $K = 0$ gives us the solution

* I am obliged to Robert Bartnik for pointing out the longtime knowledge of these parabolas by physicists (see [6]).

$$r(s) = \frac{1}{\sqrt{S}} \sin(\sqrt{S}\, s), \quad h(s) = \frac{1}{\sqrt{S}}(1 - \cos(\sqrt{S}\, s))$$

and its translations, corresponding to a sequence of (n-1)-spheres of radius $1/\sqrt{S}$, two by two tangent at points intersecting the axis of revolution.

For a fixed $K \in (0, K_0)$, it follows from the compacteness of the level curve that the function $r(s)$ is periodic and varies monotonically from a minimum $r_1 > 0$ to a maximum $r_2 < 1/\sqrt{S}$, while its square derivative $\dot{r}^2(s)$ is bounded away from 1. Therefore, $\dot{h}^2 = 1 - \dot{r}^2$ is everywhere positive, so h is monotonic and the profile curve is embedded.

Figure 5 below pictures geometrically how the periodic hypersurfaces converge on one side to the cylinder and on the other side to the spheres.

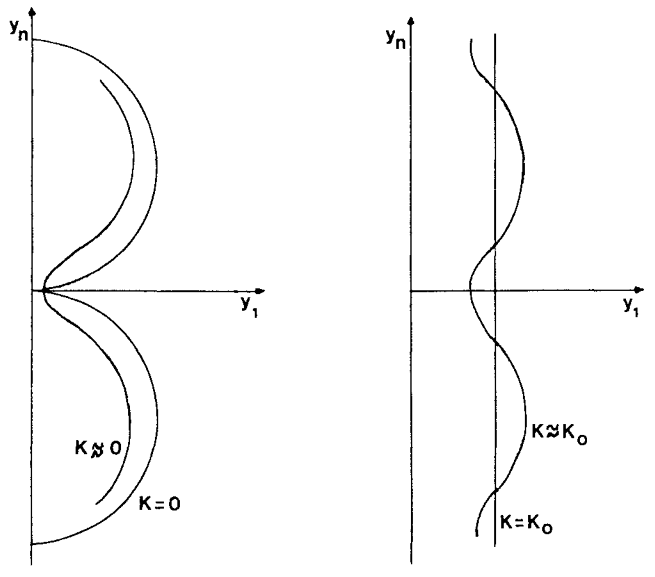

Figure 5: Hypersurfaces of $\mathbf{R}^n$ with $S > 0$

REMARK. Any profile curve reaching the axis γ does it orthogonally. This behaviour is in contrast with dimension 3, for there are surfaces of revolution in $\mathbf{R}^3$ with constant Gaussian curvature $K = 0$ generated by curves meeting the axis at an acute angle, e.g., flat cones. Analogously for $K > 0$.

THEOREM 3.5 (CLASSIFICATION in $\mathbf{H}^n$).

i) Up to translation, the complete rotational hypersurfaces of constant scalar curvature $S \in [-1, 0]$ form a one-parameter family of examples which converge to a totally geodesic hyperbolic plane $\mathbf{H}^{n-1}$, to a hypersphere or to a horosphere, depending on $S = -1$, $S \in (-1, 0)$ or $S = 0$, respectively. The profile curves are asymptotic to two geodesic lines, except when $S = 0$. In all cases, they are embedded and not cylindrically bounded (see figure 6).

ii) For any $S > 0$, there is a one-parameter family of complete embedded hypersurfaces of constant scalar curvature S, all periodic and cylindrically bounded, converging to a sequence of geodesic spheres, two by two vertically tangent (see figure 7).

iii) There is no complete rotational hypersurface with constant scalar curvature $S < -1$.

PROOF. The proof is analogous to the euclidean one. In hyperbolic space, $F = \sinh r$ and one has that $\dot{F}^2 \leq 1 + F^2$, so only level curves contained in the region $v^2 \leq 1 + u^2$ will be taken into consideration. The admissible values of K are indicated in figures 2, 3 and 4. Again we take $h(0) = 0$ as initial value. Furthermore, using that $F(s) = \sinh r(s)$, it follows from equation (2.2) that

$$\dot{h}^2 = \frac{1 - \dot{r}^2}{\cosh^2 r} = \frac{1 - (\dot{F}^2 / \cosh^2 r)}{\cosh^2 r} = \frac{1 + F^2 - \dot{F}^2}{(1 + F^2)^2};$$

we call the far right hand side *hyperbolic expression* of $\dot{h}^2$.

Let us recall that $\sinh r = \tan\phi$ and $\exp h = \rho$, where ρ and $\pi/2 - \phi$ are the usual polar coordinates in the euclidean plane. We observe that for a given solution (r, h), its symmetric $(r, -h)$ will correspond in hyperbolic plane to the reflection of the original solution around the geodesic $h = 0$, that is, to the curve obtained by euclidean inversion with respect to the circle $\rho = 1$.

For $S \in [-1, 0]$, figures 2 and 4 yield that the admissible set for K is $[0, \infty)$, since the hyperbola $v^2 + Su^2 = 1$ is inside the region $v^2 - u^2 \leq 1$, when $S \in [-1, 0)$.

The value $K = 0$ in equation (2.3) gives us that $\dot{F}^2 = 1 - SF^2$. Putting $F(0) = 0$ one gets $F(s) = s$ or $F(s) = (\sinh\sqrt{-S}\, s)/\sqrt{-S}$, depending on $S = 0$

or $S \in [-1, 0)$, respectively. These solutions put into the hyperbolic expression of $\dot{h}^2$ yield

$$\dot{h}^2 = \left(\frac{s}{1+s^2}\right)^2$$

or

$$\dot{h}^2 = \frac{(-S)(1+S)(\sinh^2\sqrt{-S}s)}{(\sinh^2\sqrt{-S}s - S)^2}.$$

When $S = -1$, the profile curve has equations $r(s) = s$ and $h(s) = 0$, corresponding to the hyperbolic $(n-1)$-space generated by the geodesic $\rho = 1$.

When $S \in (-1, 0)$, there are two solutions for different signs of $\dot{h}$: if $\dot{h} < 0$, direct integration gives us the solution

$$\frac{1}{\rho(s)} = \frac{\cosh\sqrt{-S}s - \sqrt{1+S}}{1-\sqrt{1+S}}\ , \ \ \tan\phi(s) = \frac{\sinh\sqrt{-S}s}{\sqrt{-S}},$$

which parametrizes an euclidean half-circle of radius $R = \frac{1-\sqrt{1+S}}{-S}$ and center $(0, R\sqrt{1+S})$; if $\dot{h} > 0$, one gets ρ instead of $\frac{1}{\rho}$, hence the inversion of the previous circle, now with center in the negative y_n-axis. The corresponding hypersurfaces are called hyperspheres.

When $S = 0$, it follows by integration that $\pm h(s) = \log\sqrt{1+s^2}$; besides, $\tan\phi(s) = F(s) = s$, so elementary trigonometry implies that $\cos^2\phi = 1/(1+s^2)$. Again we obtain two solutions for different signs of $\dot{h}$: if $\dot{h} > 0$, then $\rho\cos\phi = 1$ and the profile curve is half of the euclidean horizontal line $y_n = 1$; if $\dot{h} < 0$, then $\rho = \cos\phi$ and the profile curve is half of the euclidean circle tangent to y_1-axis. The corresponding hypersurfaces are called horospheres.

For a fixed $K > 0$, r attains a unique minimum $r_1 > 0$, which we take as initial value $r(0)$. Clearly $F_1 = \sinh r_1$ is determined by the equality $F_1^{n-3}(1-SF_1^2) = K$, since the left hand side is an increasing function of F_1 as long as $S \leq 0$. Also F has no upper bound, so no hypersurface is bounded by a hyperbolic cylinder.

Substituting $\dot{F}^2 = 1 - SF^2 - \frac{K}{F^{n-3}}$ in the hyperbolic expression of $\dot{h}^2$, one gets

$$\dot{h}^2 = \frac{(1+S)F^2 + \frac{K}{F^{n-3}}}{(1+F^2)^2}.$$

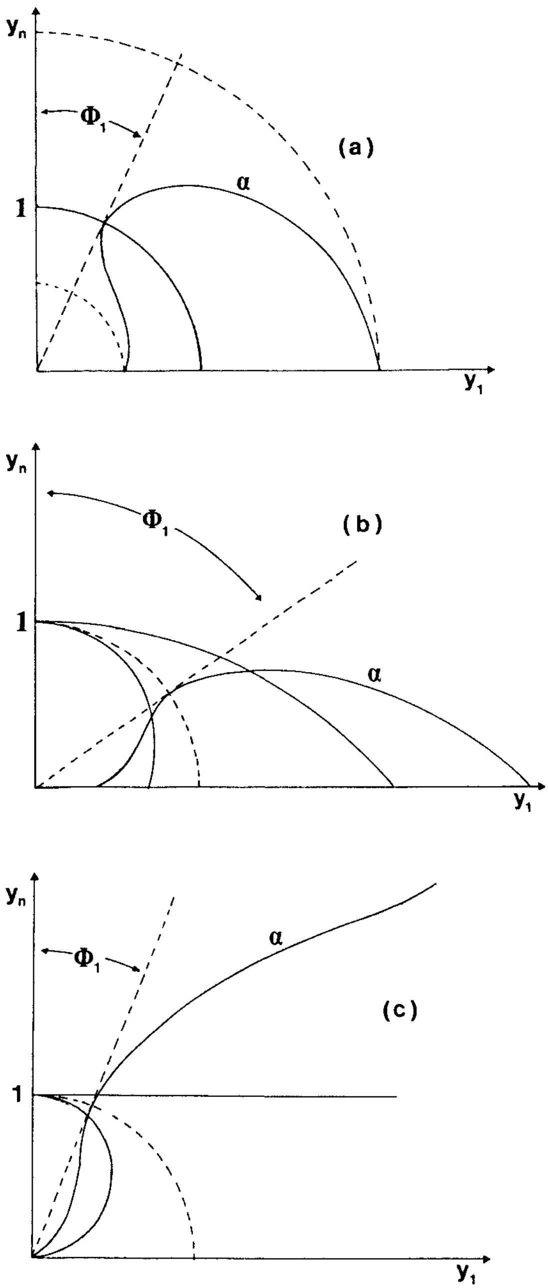

Figure 6: Hypersurfaces of $\mathbf{H}^n$ with (a) S=-1, (b) $S \in (-1,0)$ and (c) $S = 0$

Away from F_1, we may divide $\dot{h}^2$ by $\dot{F}^2$ to arrive at

$$\left(\frac{dh}{dF}\right)^2 = \frac{(1+S)F^{n-1}+K}{(1-SF^2)F^{n-3}-K} \times \frac{1}{(1+F^2)^2},$$

hence the profile curve is formed by two graphs, symmetric with respect to inversion around $\rho = 1$.

When $S = 0$, $\left|\frac{dh}{dF}\right| \geq \frac{F}{1+F^2}$ on (F_1, ∞), so $h(F)$ is unbounded. On the contrary, when $S \in [-1, 0)$, $\left|\frac{dh}{dF}\right|$ is bounded at infinity by $\left(\frac{1+S}{-S}\right) \times \left(\frac{1}{1+F^2}\right)$, so the integral $h(F)$ is uniformly bounded. That means the profile curve asymptotes two geodesics $h = \pm constant$. The embeddedness is clear, since the two graphs glue smoothly at $r = r_1$ with tangent orthogonal to $\rho = 1$. This completes the proof of i). Figure 6 illustrates the geometry.

The proof of iii) is immediate from figure 4, for $\frac{\dot{F}^2}{1+F^2}$ always reaches 1 in a finite interval when $S < -1$. In other words, all level curves asymptot the hyperbola $v^2 + Su^2 = 1$, which is exterior to the region $v^2 - u^2 \leq 1$ precisely when $S < -1$.

The proof of ii) goes as in the previous theorem, so K varies in $[0, K_0]$. The profile curve oscillates periodically around the cylinder of Corollary 2.3 and converges to it as $K \to K_0$. Moreover, as $K \to 0$, we claim that the profile curves converge to a sequence of geodesic circles. Indeed, the limit solution satisfies $1 - \dot{F}^2 = SF^2$, hence $F(s) = \frac{\sin\sqrt{S}s}{\sqrt{S}}$ which yields $\dot{h}^2 = ((1+S)\sin^2\sqrt{S}\,s)/(S + \sin^2\sqrt{S}\,s)$: integration for $\dot{h} < 0$ gives us

$$\rho(s) = \frac{\sqrt{1+S}-1}{\sqrt{1+S}-\cos\sqrt{S}\,s},$$

so we have an euclidean half-circle of radius $R = \frac{1}{\sqrt{1+S}+1}$ and center $R\sqrt{1+S}$, which is well known to be a geodesic circle in the hyperbolic plane; integration for $\dot{h} > 0$ gives $1/\rho$, hence the inverted circle. Subsequent inversions of the hypersurface produce a sequence of geodesic spheres.

As in the previous theorem, given a fixed $K \in (0, K_0)$, it follows from the compacteness of the level curve that the function F, hence also r, is periodic and varies monotonically from a minimum F_1 to a maximum $F_2 < 1/\sqrt{S}$, while its derivative satisfies $\dot{F}^2 < 1 - SF^2$. Thus, $\dot{r}^2 = \frac{\dot{F}2}{1+F^2} < \frac{1-SF_1^2}{1+F_1^2}$ is bounded away

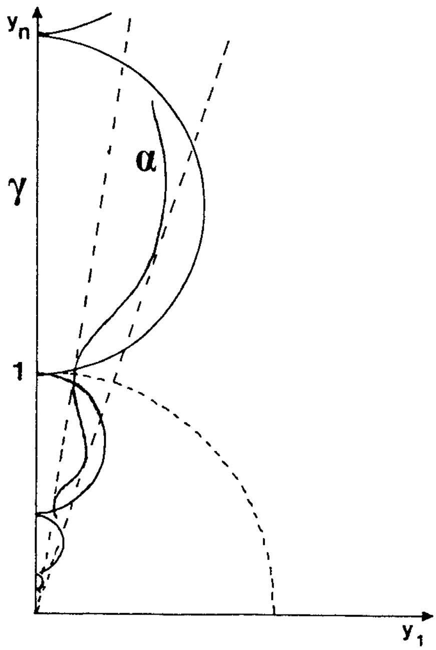

Figure 7: Hypersurfaces of $\mathbf{H}^n$ with $S > 0$

from 1, $\dot{h}^2$ is everywhere positive and consequently, h is monotonic and the profile curve is embedded. Figure 7 pictures the geometry.

THEOREM 3.6. *i) For $S \in (\frac{n-3}{n-1}, 1)$, there exists an infinitely countable family of complete immersed rotational hypersurfaces of $\mathbf{S}^n$ with constant scalar curvature S, converging to the embedded cylinder of Corollary 2.3. When $S \in (\frac{n-2}{n-1}, 1)$, one of these hypersurfaces is embedded.*

ii) For $S \geq 1$, there exists a countable family of complete immersed rotational hypersurfaces of $\mathbf{S}^n$ with constant scalar curvature S, converging on one side to the cylinder of Corollary 2.3 and on the other side to a sequence of isometrically embedded spheres of radius $1/\sqrt{S}$. Possibly except for finite values of S, this family is infinite.

iii)There is no complete rotational hypersurface of $\mathbf{S}^n$ *with constant scalar curvature* $S \leq \frac{n-3}{n-1}$.

PROOF. In spherical space, one has that $F = \sin r$ and $\dot{F}^2 \leq 1 - F^2$, so only level curves contained in the region $u^2 + v^2 \leq 1$ will be taken into consideration.

Clearly complete solutions do not exist for $S = 0$ or $S < 0$, since all level curves reach the exterior of the circle $u^2 + v^2 = 1$. Also, when $S \in (0, \frac{n-3}{n-1}]$, it follows from Proposition 2.2 that $F_0^2 = \frac{n-3}{(n-1)S} \geq 1$, so any curve at a level $0 < K < K_0$ encloses the point $(F_0, 0)$, therefore escaping the unitary circle. The value $K = K_0$ provides an inner level curve only when $S = \frac{n-3}{n-1}$, in which case the solution is $F = 1$ and the curve is reduced to a point.

For $S \in (\frac{n-3}{n-1}, 1)$, figure 3 yields that the admissible set for K is $[1-S, K_0]$, for the ellipse $v^2 + Su^2 = 1$ is exterior to $u^2 + v^2 \leq 1$ while the interior curve passing trough $(1,0)$ has level $K = 1 - S$, as it can be seen from $v^2 = 1 - Su^2 - \frac{K}{u^{n-3}}$. And for $S \geq 1$, the admissible set for K is $[0, K_0]$. Clearly any curve at an intermediate level K is compact and the associated solution $r(s)$ attains a unique minimum $r_1 > 0$ taken as initial value $r(0)$. The value $K = K_0$ gives us the solution $r(s) = r_0$, corresponding to the spherical cylinder of Corollary 2.3.

Using that $F(s) = \sin r(s)$, it follows from equation (2.2) that

$$\dot{h}^2 = \frac{1 - \dot{r}^2}{\cos^2 r} = \frac{1 - F^2 - \dot{F}^2}{(1 - F^2)^2};$$

we call the far right hand side the *spherical expression* of $\dot{h}^2$.

Substituting $\dot{F}^2 = 1 - SF^2 - \frac{K}{F^{n-3}}$ into the spherical expression of $\dot{h}^2$, one further gets

$$\dot{h}^2 = \frac{\frac{K}{F^{n-3}} + (S-1)F^2}{(1 - F^2)^2}.$$

Away from $F_1 = \sin r_1$, we may divide it by $\dot{F}^2$ to arrive at

$$\left(\frac{dh}{dF}\right)^2 = \frac{K + (S-1)F^{n-1}}{(1 - SF^2)F^{n-3} - K} \times \frac{1}{(1 - F^2)^2}.$$

When $S \in (\frac{n-3}{n-1}, 1)$, we claim that the value $K = 1 - S$ gives us a curve which starts at a distance r_1 from γ and then spirals indefinitely around the point

$(1,0,...,0)$, which is at a distance $r = \pi/2$ from the axis γ. Indeed, the latest equation becomes

$$\left(\frac{dh}{dF}\right)^2 = \frac{(1-S)(1-F^{n-1})}{S(1-F^{n-1})-(1-F^{n-3})} \times \frac{1}{(1-F^2)^2} ,$$

hence

$$\left|\frac{dh}{dF}\right| \geq \left[\frac{1-S}{S-\frac{n-3}{n-1}}\right]^{1/2} \times \frac{1}{1-F^2} ,$$

for as F increases to 1 one has that $\frac{1-F^{n-3}}{1-F^{n-1}}$ converges to $\frac{n-3}{n-1}$. Now we use that

$$\int_{F_1}^{F} \frac{dF}{1-F^2} = \frac{1}{2}\left(\log\frac{1+F}{1-F} - \log\frac{1+F_1}{1-F_1}\right) \to \infty,$$

as $F \to 1$. Of course this curve does not generate a hypersurface in the induced topology of the sphere.

For $S \geq 1$, we claim that $K = 0$ gives us half of an euclidean circle of radius $1/\sqrt{S}$ starting orthogonally from γ, which can be continually reflected. Indeed, integration of $\dot{F}^2 = 1 - SF^2$, with $F(0) = 0$, yields $F(s) = \frac{\sin\sqrt{S}\,s}{\sqrt{S}}$, hence

$$\dot{h}^2 = \frac{S(S-1)\sin^2\sqrt{S}\,s}{(S-\sin^2\sqrt{S}\,s)^2} .$$

When $S = 1$, we have that $r(s) = s$ and $h(s)$ is constant, so the profile curve is a great circle which generates a totally geodesic $(n-1)$-sphere. When $S > 1$, elementary integration gives us that

$$h(s) = -\arctan\frac{\cos(\sqrt{S}\,s)}{\sqrt{S-1}} ,$$

once we take spherical coordinates in the orbit space with $h(0) = -\arctan\frac{1}{\sqrt{S-1}}$; the solution satisfies $\cos r(s)\cos h(s) = \sqrt{1-\frac{1}{S}}$, which is the equation of a half-circle of radius $1/\sqrt{S}$ cut in the orbit space from a plane whose distance to the point $(0,0,0)$ is $\sqrt{1-\frac{1}{S}}$.

The corresponding hypersurface is an isometrically embedded $(n-1)$-sphere of radius $1/\sqrt{S}$ entirely contained in an open hemisphere of $\mathbf{S}^n$.

An intermediate value of K gives us a periodic solution $r(s)$ which oscillates around $r = r_0$, from a minimum $r_1 > 0$ to a maximum $r_2 < 1$. As $\dot{h}$ never vanishes, for otherwise $F^2 + \dot{F}^2 = 1$, one may consider h increasing, so the period P of the curve with respect to the variable h is obtained by integration of $\frac{dh}{dF}$

$$\frac{P(K)}{2} = h(F_2) - h(F_1) = \int_{F_1}^{F_2} \frac{\sqrt{K + (S-1)F^{n-1}}}{\sqrt{\phi(F) - K}} \times \frac{dF}{1 - F^2} \ ,$$

where F_1 and F_2 are the solutions of $\phi(F) = K$, with $\phi(F) = (1 - SF^2)F^{n-3}$. Of course the curve determines a complete hypersurface of $\mathbf{S}^n$ precisely when P is an integer divisor of $2r\pi$, where r is the number of turns in the orbit space given by the profile curve before it closes. Embeddedness occurs when $r = 1$.

We already have that the limit values of $P(K)$ for $K = 1 - S$ or $K = 0$ in the situations $S < 1$ or $S \geq 1$ are ∞ and $2 \arctan \frac{1}{\sqrt{S-1}}$, respectively.

We claim that for $S > \frac{n-3}{n-1}$, the period $P(K)$ converges to $\frac{2\pi}{\sqrt{(n-1)S-(n-3)}}$ as $K \to K_0$. Indeed, both F_1 and F_2 converge to F_0, where ϕ attains its maximal value K_0 and $\frac{d^2\phi}{dF^2}(F_0) = -2(n-3)F_0^{n-5} < 0$ (we recall from Proposition 2.2 that $(n-1)S{F_0}^2 = (n-3)$). Also, the factor $\sqrt{K + (S-1)F^{n-1}}/1 - F^2$ converges to $\sqrt{F_0}^{n-3}/\sqrt{1 - {F_0}^2} > 0$. Taylor's approximation of ϕ around F_0 up to second order implies that the desired limit is the same as

$$\frac{\sqrt{F_0}^{n-3}}{\sqrt{1 - {F_0}^2}} \times \lim_{\epsilon \to 0} 2 \int_{F_0 - \sqrt{\epsilon/A}}^{F_0 + \sqrt{\epsilon/A}} \frac{dF}{\sqrt{\epsilon - A(F - F_0)^2}} \ ,$$

where $A = (n-3)F_0^{n-5}$. Since the integral converges to $\pi/\sqrt{A}$, straightforward computation proves the claim.

When $S \in (\frac{n-3}{n-1}, 1)$, it follows from continuity of $P(K)$ that for each rational $\frac{r}{s} > \frac{1}{\sqrt{(n-1)S-(n-3)}}$, there exists $K \in (1 - S, K_0)$ such that $P(K) = 2\pi\frac{r}{s}$ is the period of a closed curve which generates an immersed complete hypersurface of $\mathbf{S}^n$ with constant scalar curvature S. At least for values of S in the subinterval $(\frac{n-2}{n-1}, 1)$ one has that $(n-1)S-(n-3) > 1$, hence the closed curve corresponding to $\frac{r}{s} = 1$ determines an embedded hypersurface of $\mathbf{S}^n$ with constant scalar curvature which is neither a sphere nor a cylinder.

When $S \geq 1$, continuity gives us that there exists $K \in (0, K_0)$ such that $P(K)$ is equal to $\frac{2\pi r}{s}$, for a given rational $\frac{r}{s}$ in the open interval of extremals $1/\sqrt{(n-1)S - (n-3)}$ and $\frac{1}{\pi}$ arctan$(1/\sqrt{S-1})$. It is easy to see that this open interval is non-empty, except possibly for two values of $S > 1$. We observe that this interval is not sufficiently large to contain rationals of type $\frac{1}{s}$, so the profile curves determine non-embedded hypersurfaces.

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