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Absolute Continuity of Hamiltonians with von Neumann Wigner Potentials II

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Physicists have known for a long time that there is a close connection between the asymptotics of the eigenfunctions of a linear differential equation and the spectral properties of the corresponding differential operator. According to this eigenfunctions associated with the absolutely continuous spectrum behave almost like plane waves. Here this is extended by showing that values λ, for which all solutions are bounded belong to the absolutely continuous spectrum of the corresponding differential operator. In addition we continue our analysis of [1] and extend all results of this paper. As in [1] we treat only separated Schrödinger- and Dirac operators, and we state conditions which imply the absolute continuity of the spectra of these operators off a finite or countable resonance set in terms of the potential V. These conditions are satisfied for example if V admits a decomposition near infinity of the form

$$V = S + P_1 + P_2 + \dots + P_n + W$$

where W is a Wigner von Neumann potential, $S \in \mathcal{L}^1$ and where P_i is i-times differentiable with $P_i^{(j)} \in \mathcal{L}^{\frac{n}{j}}$ for $j \leq i$ and $P_i^{(i)} \in \mathcal{L}^1$. For $n = 1$ and $W = 0$ these are just the results of Weidmann [16] and Heinz [11]. Thus rougly speaking slower decay can be compensated by better smoothness. Even though subordinacy requires control of the eigenfunctions only for real eigenvalues λ, we analyze eigenfunctions also for complex λ in order to deduce the continuity of the m-function, a limiting absorption principle and the singularity of m near the resonances.

This paper is divided into four sections. In the first we collect all information on methods and results used in the analysis of spectra. Here we also state our first main theorem. The second section is devoted to asymptotic integration, while Schrödinger- and Dirac operators with smooth potentials are studied in the third part. Embedded eigenvalues are investigated in section IV.

I

In this paper we study the separated Schrödinger equation

$$\tau_S y = -y'' + Vy = \lambda y$$

and the separated Dirac equation

$$\tau_D u = \begin{pmatrix} V_1 - m & V_2 \\ V_2 & V_3 + m \end{pmatrix} u + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} u' = \lambda u$$

on (a, ∞). Though in most cases a is finite, we will occasionally also allow $a = -\infty$ or a singular endpoint a. With suitable conditions on the potentials $V, V_1, \dots$, which are always assumed to be real and locally integrable, these differential operators τ_S and τ_D can be extended to selfadjoint operators $\mathcal{H}_S$ and $\mathcal{H}_D$ on the corresponding Hilbert spaces [16]. These operators we call Hamiltonians.

In the system notation τ_D becomes

$$u' = \begin{pmatrix} -V_2 & -V_3 - m + \lambda \\ V_1 - m - \lambda & V_2 \end{pmatrix} u. \tag{1}$$

This system also covers the systems form of τ_S if one replaces $-V_3 - m + \lambda$ by 1 and V_2 and m by 0.

Thus both equations can be treated in a unified manner. In the proofs however, we shall mostly adher to the Dirac case, because it is more complicated.

Definition: Let τ be a Sturm Liouville differential operator on $(a, b), -\infty \leq a < b \leq \infty$, which is singular at b [16]. A solution y of $\tau y = \lambda y$ is subordinate at b, if for any linear independent solution of $\tau z = \lambda z$

$$\lim_{d \to b-} ||y||_d / ||z||_d = 0$$

where

$$||f||_d^2 = \int_c^d |f(s)|^2 \, ds.$$

This definition is clearly independent of $c, a < c < b$.

If τ is singular at a, subordinacy at a can be defined similarly. In the same manner subordinacy can be defined for Dirac systems.

One speaks of sequential subordinacy if the limit above is required for a sequence $d_n \to b_-$ only.

By abusing the notation slightly we shall also speak of subordinacy for Schrödinger– respectively Dirac Hamiltonians.

Subordinate solutions are thus functions which decay faster near the singularity than the other solutions. Eigenfunctions are in general subordinate. Subordinacy generalizes therefore the notion of a principal solution and an eigenfunction. In order to state the main result of Gilbert and Pearson [8,9] and its extension to Dirac operators [1], we need two more definitions.

Definition: Let S be a measurable subset of $\mathbf{R}$ and let μ be a Borel measure on $\mathbf{R}$. Then S is a set of minimal support of μ, if

i) $\mu(\mathbf{R} \backslash S) = 0$

ii) for any measurable subset $T \subset S$ with $\mu(T) = 0$ one has $\lambda(T) = 0$, where λ is the Lebesgue measure.

Minimal supports are clearly not unique. But this lack of nonuniqueness will not concern us here.

Definition: For a Borel measure μ on $\mathbf{R}$ the absolutely continuous (singular continuous, discrete) part of μ will be denoted by $\mu_{ac}(\mu_{sc}, \mu_d)$.

In the same fashion these suffixes will be used to denote the corresponding minimal supports $M, M_{ac}, \ldots$ and if $\mathcal{H}$ is a selfadjoint operator with spectrum $\sigma(\mathcal{H}), \sigma_{ac}(\mathcal{H})$ or $\sigma_d(\mathcal{H})$ will denote the absolutely continuous or discrete spectrum of $\mathcal{H}$.

If μ is a spectral measure minimal supports M_{ac}, M_{sc} and M_d of μ_{ac}, μ_{sc} respectively μ_d can be choosen such that their closure contains the corresponding spectra σ_{ac}, σ_{sc} respectively σ_d [8, Lemma 5].

Lemma 1: a) Let $\mathcal{H}$ be a Hamiltonian which is derived from a Schrödinger– or Dirac differential expression τ on (a, ∞) with two singular limit point endpoints. Then minimal supports M_{ac} and M_{sc} of the absolutely continuous respectively singular continuous part of the spectral measure μ of $\mathcal{H}$ are given by

i) $M_{ac} = \{\lambda \in \mathbf{R} \mid$ There is no solution of $\tau u = \lambda u$ which is subordinate at a or ∞ or both.$\}$

ii) $M_{sc} = \{\lambda \in \mathbf{R} \quad R \mid$ There is a solution of $\tau u = \lambda u$ which is subordinate at a and ∞, but which is not square integrable.$\}$

b) If a is a regular endpoint the minimal supports are given by

i) $M_{ac} = \{\lambda \in \mathbf{R} \mid$ There is no solution of $\tau u = \lambda u$ which is subordinate at ∞.$\}$

ii) $M_{sc} = \{\lambda \in \mathbf{R} \mid$ $\tau u = \lambda u$ has a subordinate solution satisfying the boundary condition at a.$\}$

c) (b) remains valid also if a is a singular limit circle endpoint.

This Lemma has been shown by Gilbert and Pearson [8,9] and has been extended to Dirac equations by Behncke [1].

It is the aim of this paper to derive spectral properties of Schrödinger– or Dirac–Hamiltionians from the asymptotics of their eigenfunctions. The methods to be used are:

i) The technique of Weidmann

ii) Subordinacy

iii) The continuity of the m-function and

iv) The limiting absorption principle

Of these methods the last two rely on the analysis of the eigenfunctions in the complex domain. Because of the appearance of exponentially growing terms, this leads to considerable difficulties and requires among other things a modification of the $1 + Q$-transformation. For the limiting absorption principle it is also necessary to control the resolvent, i.e. the eigenfunctions near 0, though this may be overcome partially by applying the resolvent only to functions which vanish near zero. In contrast to these techniques the first two methods rely on the analysis of the eigenfunctions for real eigenvalues only. Apart from using real and generally bounded eigenfunctions, these methods exhibit a better stability under perturbations. For periodic differential operators this has recently been shown for Weidmann's technique by Stolz [15].

In Weidmann's approach one has to verify the following two conditions

(W1) For $0 < a_1 \leq a_2$ and $\lambda_1 \leq \lambda_2$ there exist constants C, C' and c such that for all $d > c, 0 < b \leq c, (\tau - \lambda)u = 0$ implies $|u(x)| \leq C||u||_{[c,d]} \cdot ||1||^{-1}_{[c,d]}$ for all $x \in [a_1, a_2]$ and $\lambda \in [\lambda_1, \lambda_2]$.

Moreover

(W2) The (generalized) Prüfer angle $\Theta(x, \mu)$ corresponding to τ restricted to $[b, d]$ satisfies $|\Theta(d, \mu_1) - \Theta(d, \mu_2)| \leq C'|d - b|\ |\mu_1 - \mu_2|$ for $\mu_1, \mu_2 \in [\lambda_1, \lambda_2]$

In contrast to this, the non (sequential) subordinacy condition amounts to:

(P) For any two nontrivial solutions u_1, u_2 of $(\tau - \lambda)v = 0$ $\quad \liminf\limits_{d \to \infty} \dfrac{||u_1||_{[c,d]}}{||u_2||_{[c,d]}} > 0$

This condition is closely related to (W1), because if u is a nontrivial subordinate solution and v an arbitrary nontrivial solution of $(\tau - \lambda)w = 0$ the condition (W1) can be written as

$$|u(x)| \leq C \frac{||u||_{[c,d]}}{||v||_{[c,d]}} \frac{||v||_{[c,d]}}{||1||_{[c,d]}} \qquad x \in [a_1, a_2]$$

Since the first quotient on the right hand side converges to 0 for $d \to \infty$, we would immediately get a contradiction if the second quotient was bounded, i.e. if on the average the $\mathcal{L}^2$-norm of v did not grow faster than the $\mathcal{L}^2$-norm of 1 or $\sin(\lambda t + \alpha)$.

These considerations allow us to simplify condition (P) in our situation and at the same time extend Weidmann's methods considerably.

Theorem 1: Let $-\infty \leq a$, $\lambda_1, \lambda_2 \in \mathbf{R}$ and let τ_D be a Dirac differential expression on (a, ∞) such that each solution u of $\tau_D u = \lambda u$ is bounded in $[c, \infty]$. Then $[\lambda_1, \lambda_2]$ belongs to the spectrum of any Dirac Hamiltionian $\mathcal{H}_D$ associated to τ_D and $\sigma(\mathcal{H}_D)$ is absolutely continuous in (λ_1, λ_2). The result is also valid for Schrödinger operators τ_S if the boundedness is required for y and y'.

Proof: We shall show that for each $\lambda \in [\lambda_1, \lambda_2]$ no solution exists which is subordinate at ∞. Contrary to this claim let u be such a (real) subordinate solution with $||u(c)|| = 1$. Moreover let v be a complementary real solution, i.e. the Wronskian $W(u, v)$ of u and v is 1. Then $1 = u_1(x)v_2(x) - v_1(x)u_2(x)$ for $x \in (a, \infty)$. Compute the $\mathcal{L}^2$-norms of both sides and use the Cauchy Schwarz inequality

$$||1||^2_{[c,d]} = |d - c| \leq 2||u||_{[c,d]}||v||_{[c,d]} \leq 2C||u||_{[c,d]}(d - c)^{\frac{1}{2}}$$

to show that u is not subordinate.

In the Schrödinger case one shows first that u and v are oscillatory and deduces from $1 = uv' - vu'$ that there are constants c_1 and c_2 with $0 < c_1 \leq u^2 + u'^2, v^2 + v'^2 \leq c_2$. With this one shows that the distance of successive zeros of u respectively v are bounded and bounded away from zero. This finally easily yields a contradiction to the assumed subordinacy of u.

Corollary: Let $-\infty \leq a$, $\lambda_1, \lambda_2 \in \mathbf{R}$ and let $\tau_S(y) = -y'' + qy$ be a Schrödinger differential expression on (a, ∞) such that

i) $q = q_1 + q_2$ with $q_2 \in \mathcal{L}^1, q_1 \in \mathcal{L}^1_{loc}$ with $q_{1-} = (|q_1| - q_1) \cdot \frac{1}{2}$ bounded.

ii) For all $\lambda \in [\lambda_1, \lambda_2]$ all solutions y of $\tau_S y = \lambda y$ are bounded on $[c, \infty)$ for some $c > a$.

Then $[\lambda_1, \lambda_2] \in \sigma(\mathcal{H}_S)$ and $\sigma(\mathcal{H}_S)$ is absolutely continuous in (λ_1, λ_2).

Proof: Because of the conditions on the eigenfunctions τ_S is clearly limit point at infinity. Assume y is a nontrivial real solution of $\tau_S y = \lambda y$. Then

$$\begin{aligned}\int_c^d y'^2 dt &= yy'|_c^d - \int_c^d yy'' dt = yy'|_c^d - \int_c^d y^2(q - \lambda)\, dt \\ &\leq yy'|_c^d + ||y||_\infty^2 \cdot ||q_2|| + ||(q_1 - \lambda)_-||_\infty ||y||_{[c,d]}^2.\end{aligned}$$

Since the boundary terms can easily shown to be finite, $||y'||_{[c,d]}^2$ can be estimated in terms of $||y||_{[c,d]}^2$. Now the proof can be completed as in the Theorem.

Remark 1: It should be noted that the boundedness of the eigenfunctions is required only at one singular endpoint. The conditions on q_{1-} can certainly be weakened, for example $\sup_{x>c} |\int_x^{x+1} q_{1-} dt| < \infty$ suffices. Another independent proof of this has been given by Stolz (private communication). A little thought also shows that in this case $0 < Im\ m(\lambda) < \infty$ for $\lambda \in (\lambda_1, \lambda_2)$.

The results of this Corollary are clearly stable with respect to $\mathcal{L}^1$-perturbations. This is also true for the results of the Theorem and holds in fact in a much wider context. To see this let

$$u'(t) = A(t)u(t)$$

be a system on n equations on (a, ∞), for which all solutions are known to be bounded. Consider the perturbed system

$$v'(t) = (A(t) + B(t))v(t) \qquad \text{with } B_{ij} \in \mathcal{L}^1 \quad i, j = 1, ..., n.$$

If Y is the fundamental matrix of the unperturbed system, the variation of constants method

$$w = Y^{-1}v$$

leads to

$$w' = Y^{-1}BYw.$$

This equation however has only bounded solutions, because the right hand side is integrable.

Thus $\mathcal{L}^1$-pertubations preserve boundedness and thus in the situation of the theorem the absolutely continuous spectrum. This for example applies to Schrödinger or Dirac operators with periodic or even quasiperiodic potentials [6,13].

II

In order to derive the asymptotics of the solutions of (1), we shall use the method of asymptotic integration. For this we need the basic result of Levinson [7]. Levinson's Theorem states that a system

$$u'(t) = (\text{ diag } (\lambda_1(t), ..., \lambda_n(t)) + R(t))u(t) = A(t)u(t)$$

has an almost diagonal fundamental matrix if the diagonal is not too small - dichotomy conditions - and if R is not too large - $R \in \mathcal{L}^1$.

Here we need a slight refinement of this Theorem, because our matrices also depend on the spectral parameter λ. Better error estimates and conditions, which give a uniform dependence of A on λ, thus allow us to conclude that the fundamental matrix $Y(t, \lambda)$ is almost diagonal and depends continuously on λ, even in the asymptotic regime. Since the precise statement of this is rather technical, we will not elaborate further on this.

Levinson's Theorem cannot be applied directly to (1). Instead the asymptotic integration is achieved by a series of successive transformations until this result becomes applicable. Three types of transformations are used mainly. These are diagonalization, perturbation and the "$1 + Q$-transformation" used by Harris and Lutz [10]. For later use we give a brief outline of these methods. Because of its importance for our work we also state the basics of a complex $1 + Q$-transformation.

Diagonalization

Consider the linear differential system

$$y' = (B + B_1)y \tag{2}$$

on $[a, \infty)$ and assume the leading part $B = B(t)$ can be diagonalized by a smooth matrix function $A = A(t), A^{-1}BA = \text{diag}(\lambda_1(t), ..., \lambda_n(t)) = \Lambda$. Then the transformation

$$z = A^{-1}y$$

leads to

$$z' = [\Lambda + A^{-1}B_1A - A^{-1}A']z \tag{3}$$

This transformation is thus useful if $A^{-1}A'$ is small, i.e. if B is sufficiently smooth.

Perturbation

Levinson's fundamental theorem [7] asserts that $\mathcal{L}^1$-perturbations have at most a multiplicative $(1 + o(1))$ effect. We shall use this result however mostly in the following form.

Let Y be a fundamental matrix for the system

$$y' = By$$

such that Y and Y^{-1} are uniformly bounded . Moreover assume that B_1 is integrable. Then the transformation

$$y = Yz \quad \text{or} \quad z = Y^{-1}y \tag{4}$$

transforms (2) into

$$z' = (Y^{-1}B_1Y)z \tag{5}$$

The latter system has a fundamental matrix Z with

$$Z(t) = 1 + C(t) \quad \text{with} \quad C' \in \mathcal{L}^1 \quad \text{and} \quad \|C(t)\| \leq \quad K \quad \cdot \Sigma \int_t^\infty |B_{1i,j}(s)| ds \tag{6}$$

This follows easily from Levinsons result. Thus in this case (2) has a fundamental matrix $Y(t) \cdot (1 + C(t))$. Of course (6) can also be shown in more general situations.

The $1 + Q$-Transformation

Assume that B_1 in (2) is conditionally integrable and let $Q = \tilde{B}_1$ be a matrix function with

$$Q' = B_1 + C \quad \text{with} \quad C \in \mathcal{L}^1 \quad \text{and} \quad Q(t) \to 0 \tag{7}$$

Then $(1 + Q)^{-1}$ exists for large t and the transformation

$$z = (1 + Q)^{-1}y \tag{8}$$

leads from (2) to

$$z' = (1 + Q)^{-1}[B + BQ + B_1Q - C]z \tag{9}$$

This transformation is thus useful if BQ and B_1Q are smaller than B_1. This applies in particular to rapidly oscillating not necessarily bounded B_1. This method has been used extensively by Harris and Lutz [10].

The complex $1 + Q$-Transformation

Later on we shall frequently encounter systems of the following form

$$y' = \begin{pmatrix} i\mu + q_3 & q_1 \\ q_2 & -i\mu - q_3 \end{pmatrix} y$$

where $\mu = \mu_1 + i\mu_2$ is a smooth complex valued function with $\mu' \in \mathcal{L}^1$ and where the q_i are conditionally integrable. More specifically we assume $q_i = \Sigma f_{ij} \sin g_j$ with $f_{ij} \in \mathcal{L}^2$ and $f'_{ij} \in \mathcal{L}^1$.

Now the transformation (8) with

$$Q = \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix}$$

leads to a system in Levinson form if $Q_i(t) \to 0$ for $t \to \infty, Q_i \in \mathcal{L}^2$ and

$$Q_1' = 2i\mu Q_1 + q_1 \quad \text{and} \quad Q_2' = -2i\mu Q_2 + q_2. \tag{10}$$

With $\sigma(t) = \int_c^t \mu(s)ds$ and $\sigma_i(t) = \int_c^t \mu_i(s)ds$ we write solutions of (10) as

$$\begin{aligned} Q_1(t) &= e^{2i\sigma}\left[\int_c^t e^{-2i\sigma(s)}q_1(s)ds - \int_c^\infty e^{-2i\sigma_1(s)}q_1(s)ds\right] \\ &= e^{2i\sigma}\left[\int_c^t (1-e^{-2\sigma_2})e^{-2i\sigma}q_1 ds - \int_t^\infty e^{-2i\sigma_1}q_1 ds\right] \end{aligned} \tag{11}$$

and

$$Q_2(t) = e^{-2i\sigma(t)} \int_\infty^t e^{2i\sigma(s)}q_2(s)ds. \tag{12}$$

With the aid of partial integration it is now possible to show $Q_i(t) \to 0$ and $Q_i(t) \in \mathcal{L}^2$ uniformly in μ_2, if $|2\mu(t) - g_j(t)| \geq \delta > 0$ near infinity. In a simpler situation this transformation has been used by Ben Artzi and Devinatz [4].

In the remainder we shall employ a series of such transformations. Our guiding principle in this is to handle the smooth or slowly oscillating parts first by diagonalization and then to transform the oscillating terms by repeated $1 + Q$-transformations.

III

Because of Levinson's result it is advantageous to introduce a class of functions, wich is particularly suited for a repeated diagonalization modulo $\mathcal{L}^1$-terms. For a natural number n let $\mathcal{D}_n(\mathbf{R})$ respectively $\mathcal{D}_n(\mathbf{C})$ denote the class of real respectively complex valued functions on $\mathbf{R}_+$ which admit a decomposition

$$f = f_1 + f_2 + \dots + f_n \quad \text{near infinity where} \tag{13.1}$$

$$f_k \text{ is } k\text{-times differentiable, } f_k^{(j)} \in \mathcal{L}^{\frac{n}{j}} \quad \text{for } 1 \leq j \leq k \quad \text{and } f_k^{(k)} \in \mathcal{L}^1 \tag{13.2}$$

$$f_k^{(j)}(t) \to 0 \qquad \text{for } t \to \infty \text{ and } 0 \leq j < k \leq n. \tag{13.3}$$

In the remainder we shall also consider bounded functions where (13.3) is only required for $1 \leq j < k \leq n$. This defines the space $\mathcal{E}_n$.

As mentioned above the definition of $\mathcal{D}_n$ is motivated by the diagonalization procedure and the $\mathcal{L}^1$-perturbation result, where each successive diagonalization introduces higher derivatives. For this reason we define the k-th derivative modulo $\mathcal{L}^1$ of such a function by

$$f^{[k]} = f_{k+1}^{(k)} + \dots + f_n^{(k)} \quad \text{and} \quad f^{[n]} = 0 \tag{14}$$

This extends to products of such functions by

$$(f \cdot g)^{[k]} = \sum \binom{k}{j} f^{[j]} g^{[k-j]}$$

Because of (13.2) we see that $(f \cdot g)^{[k]} \in \mathcal{L}^{\frac{n}{k}}$. By using power series this can also be extended to analytic functions of $\mathcal{D}_n$. We note that the operation $\sim$ defined above (7) is in some sense the antiderivative of this. Likewise it is clear that neither $\tilde{f}$ nor $f^{[k]}$ are unique. But this nonuniqueness will not concern us here. For the same reason we shall denote generic $\mathcal{L}^1$-terms by R or r.

In this paper we shall study the equation (1) for $|Re\ \lambda| > m \geq 0$ because it generalizes the separated Dirac equation. The restriction of the eigenvalue parameter λ to $|Re\lambda| > m \geq 0$ in the Dirac case and $Re\lambda > 0$ in the Schrödinger case implies that we study only the asymptotics of the positive energy continuous eigenfunctions. In the remainder λ may vary freely in a set K which is definded by the following inequalities $0 \leq Im\lambda < \varepsilon, |Re\lambda - \lambda_1| < \varepsilon$ for $\lambda_1 \in [-m,m]^c$ or $\lambda_1 > 0$ respectively and $0 < \varepsilon < \frac{1}{2} \mid |\lambda_1| - m \mid$ or $0 < \varepsilon < \frac{1}{2}\lambda_1$ respectively.

For the potentials in (1) we assume near infinity

$$\begin{aligned} &V_i = S_i + P_i + W_i, \quad S_i \in \mathcal{L}^1, P_i \in \mathcal{D}_n(\mathbf{R}) \quad i = 1,3 \\ &V_2 = S_2 + P_2 + W_2, P_2 \text{ differentiable }, P_2', P_2 \cdot P_i', S_2 \in \mathcal{L}^1 \\ &P_2 \text{ real and } P_2(t) \to 0 \text{ for } t \to \infty \end{aligned} \tag{15}$$

The Wigner von Neumann potentials W_1, W_2 and W_3 will be specified later. (1) can now be written as

$$u' = \left(\begin{pmatrix} 0 & -m+\lambda \\ -m-\lambda & 0 \end{pmatrix} + \mathcal{P} + \mathcal{S} + \mathcal{W} \right) u \tag{16}$$

and according to our general philosophy we begin by diagonalizing the smooth part of the system. For simplicity replace $P_1 - m - \lambda$ by P_1 and $P_3 + m - \lambda$ by P_3. Define now

$$\mu(t) = \left(P_1 \cdot P_3 - P_2^2\right)^{\frac{1}{2}}$$

where the square root is defined such that $Im\mu(t) \geq 0$ if $Im\lambda \geq 0$. Since $P_1(t) \to -m - \lambda, P_3(t) \to m - \lambda$ and $P_2(t) \to 0$ for $t \to \infty$, this can always be achieved for sufficiently large t. $\pm i\mu(t)$ are just the eigenvalues of

$$\begin{pmatrix} -P_2 & -P_3 \\ P_1 & -P_2 \end{pmatrix} = \mathcal{P}$$

The corresponding eigenvectors are $(1, a_\pm)^t$ where

$$a_\pm = \frac{(\mp i\mu - P_2)}{P_3}$$

Thus the matrix

$$A = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_1 a_+ & \varphi_2 a_- \end{pmatrix}$$

satisfies $A^{-1}\mathcal{P}A = \text{diag}(i\mu, -i\mu)$. Because of (3) we also have to compute $A^{-1}A'$.

$$A^{-1}A' = [\varphi_1\varphi_2(a_- - a_+)]^{-1} \begin{pmatrix} \varphi_2[\varphi_1'(a_- - a_+) - \varphi_1 a_+'] & -\varphi_2^2 a_-' \\ \varphi_1^2 a_+' & \varphi_1[\varphi_2'(a_- - a_+) + \varphi_2 a_-'] \end{pmatrix}$$

The diagonal terms of this matrix vanish if

$$\varphi_1' = \frac{a_+'}{(a_- - a_+)}\varphi_1 \quad \text{and} \quad \varphi_2' = \frac{-a_-'}{(a_- - a_+)}\varphi_2$$

Now

$$\frac{\pm a'_\pm}{(a_- - a_+)} = -\frac{\mu'}{2\mu} + \frac{P'_3}{2P_3} \mp \frac{P'_2}{2i\mu} \pm \frac{P'_3 P_2}{P_3 2i\mu}$$

Since by assumption the last two terms are integrable, $A^{-1}A'$ will have approximately, i.e. modulo $\mathcal{L}^1$, a zero diagonal if for $i = 1, 2$

$$\varphi'_i = \left(-\frac{\mu'}{2\mu} + \frac{P'_3}{2P_3}\right)\varphi_i \quad \text{or} \quad \varphi_i = \left[\frac{P_3}{\mu}\right]^{\frac{1}{2}} = \varphi$$

Here the sign of the square root should be chosen such that φ is a continuous function of λ. For sufficiently small $Im\lambda$ there is in fact a power series expansion of μ and φ around $Re\lambda$ for t near infinity.

The transformed system, $z = A^{-1}u$ is now

$$z' = \left[\operatorname{diag}(i\mu, -i\mu) + \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix} + R + A^{-1}\mathcal{S}A + A^{-1}\mathcal{W}A\right] z$$

where $q_1 = \frac{-a'_-}{a_- - a_+}$ and $q_2 = \frac{a'_+}{a_- - a_+}$. This can be rewritten as

$$z' = \left[\operatorname{diag}(i\mu, -i\mu) + \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} + R + A^{-1}\mathcal{W}A\right] z$$

where $q = -\frac{\mu^{[1]}}{2\mu} + \frac{P_3^{[1]}}{2P_3} \in \mathcal{L}^n$.

We shall continue now the diagonalization of the differentiable part, i.e. the first two terms of this equation. Since this transformation will be repeated several times, we rewrite this system in a slightly more general form as

$$z' = \left[\operatorname{diag}(i\mu, -i\mu) + \begin{pmatrix} 0 & \varepsilon q \\ \bar{\varepsilon} q & 0 \end{pmatrix} + B\right] z \qquad \varepsilon = 1, i \tag{17}$$

q differentiable and $q \in \mathcal{L}^p$ for some $p \geq 1$.

The eigenvalues of the leading matrix in (17) are $\pm i\nu$ where

$$\nu = (\mu^2 - q^2)^{\frac{1}{2}} \tag{18}$$

and we choose the sign of the square root such that sign $Im\nu$ = sign $Im\mu$. This can be achieved by using the power series expansion of the square root. In fact this expansion is even finite modulo $\mathcal{L}^1$, because $q \in \mathcal{L}^p$ and this applies to all such analytic functions. The eigenvectors corresponding to $\pm i\nu$ are $(1, \alpha_+)^t$ and $(\alpha_-, 1)^t$ with

$$\alpha_+ = \frac{i(\nu - \mu)}{\varepsilon q} \quad \text{and} \quad \alpha_- = -\frac{i(\nu - \mu)}{\bar{\varepsilon} q}$$

Thus the matrix

$$A_1 = \begin{pmatrix} \psi_1 & \psi_2\alpha_- \\ \psi_1\alpha_+ & \psi_2 \end{pmatrix}$$

will diagonalize the leading part of (17) and we shall choose ψ_1 and ψ_2 such that $A_1^{-1} \cdot A'_1$ is approximately zero diagonal. A computation as above shows that

$$\psi_1 = \psi_2 = (1 - \alpha^2)^{-\frac{1}{2}}, \qquad \alpha^2 = \alpha_+\alpha_-$$

has the desired properties. Then the transformed system, $w = A_1^{-1}z$ becomes

$$w' = \left[\operatorname{diag}(i\nu, -i\nu) + (1-\alpha^2)^{-1}\begin{pmatrix} 0 & \alpha'_- \\ \alpha'_+ & 0 \end{pmatrix} + R + A_1^{-1}BA_1\right] w$$

This is again of the form (17). Thus this transformation can be applied repeatedly. If the system, which is obtained by k diagonalizations, is written as

$$w' = \left[\operatorname{diag}(i\mu_k, -i\mu_k) + \begin{pmatrix} 0 & \varepsilon q_k \\ \bar{\varepsilon} q_k & 0 \end{pmatrix} + R_k + B_k^{-1}\mathcal{W}B_k\right] w \qquad \varepsilon = 1, i \tag{19}$$
with $B_k = AA_1 \ldots A_{k-1}$

we have

i) $\mu_k(t) - \mu(t) \to 0$ for $t \to \infty$ and $\mu_k(t)$ is real for real λ.

ii) $q_k \in \mathcal{L}^{\frac{n}{k}}$ near infinity.

iii) The k-th diagonalizing matrix A_k is 1 modulo $\mathcal{L}^{\frac{n}{k}}$.

iv) μ_k and A_k depend continuously on λ.

v) Each q_k is a rational function of $\mu, P; P_{1,l}^{[j]}$ and $P_{3,l}^{[j]^*}$ $j < l$, where each summand contains at least k derivatives.

vi) There is an $r \in \mathcal{L}^1$ such that $|R_{ki,j}(t)| \leq r(t)$ near infinity for all $\lambda \in K$. Thus near infinity R_k depends continuously on λ and can be estimated uniformly for $\lambda \in K$ with respect to $||\ \ ||_1$.

Because of (15) the diagonalization may be terminated after n steps.

Summing up we have shown:

Theorem 2: The system (1) with the potentials V_1, V_2 and V_3 satisfying (15) and $W_i = 0 \quad i = 1,2,3$ has a fundamental matrix U of the form

$$U(t) = A(t)(1 + B(t))(1 + o(1))\exp(\operatorname{diag}(i\sigma_n(t), -i\sigma_n(t)) \tag{20}$$

where $\sigma_n(t) = \int_c^t \mu_n(s)ds$. A and$B$ depend continuously on λ and $B(t) \to 0$ for $t \to \infty$ uniformly in K. The $1+o(1)$ term depends continuously on λ and can be estimated uniformly for $\lambda \in K$. For real λ in $[-m, m]^c$ (20) can be written as

$$U(t) = A(t)(1 + C(t))\exp(\operatorname{diag}(i\sigma_n(t), -i\sigma_n(t))$$
with $C(t) \to 0$ for $t \to \infty$ uniformly for $\lambda \in K \cap \mathbf{R}$ (20')

Proof: We have shown above that n diagonalizations lead to the system $w' = (\operatorname{diag}(i\mu_n, -i\mu_n) + R_n)w$, whose fundamental matrix is by Levinson's theorem $(1 + o(1))\exp(\operatorname{diag}(i\sigma_n, -i\sigma_n))$. Thus the result follows, because $A_1A_2 \ldots A_{n-1}$ can be written as $(1 + B_1(t))$ with $B_1(t) \to 0$ for $t \to \infty$ as above. It remains to check the dichotomy condition of Levinson. For $\lambda = \lambda_1 + i\lambda_2$ we see from (i) above

$$\mu_n(t) \to \mu(\infty) = ((-m-\lambda)(m-\lambda))^{\frac{1}{2}} \approx \pm\left[(\lambda_1^2 - m^2)^{\frac{1}{2}} + \frac{i\lambda_1\lambda_2}{(\lambda_1^2 - m^2)^{\frac{1}{2}}}\right]$$

and thus this condition holds for $\lambda_2 > 0$. For real λ however μ_n is real and Levinson's condition holds trivially.

This theorem has the following consequences.

Corollary 1: On $(0, \infty)$ consider the Dirac differential expression

$$\tau_D u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} u' + \begin{pmatrix} V_1 - m & V_2 \\ V_2 & V_3 + m \end{pmatrix} u \tag{21}$$

with V_1, V_2 and V_3 satisfying the conditions of Theorem 1. Then for $\lambda \in [-m, m]^c$ any λ-eigenfunction behaves asymptotically like a plane wave. The spectrum of any Dirac Hamiltonian $\mathcal{H}_D$ derived from τ_D is absolutely continuous in $[-m, m]^c$. If τ is regular at 0 any Titchmarsh Weyl m-function associated to τ_D is continuous in K and $\lim_{\lambda_2 \to 0+} m(\lambda_1 + i\lambda_2) = m_+(\lambda_1)$ exists boundedly with $0 < Im\ m_+(\lambda_1) < \infty$. If τ_D is regular at 0 $\mathcal{H}_D$ satisfies a limiting absorption principle i.e.

$$\sup_{\lambda \in \overset{\circ}{K}} \left(\| M_s (\lambda - \mathcal{H}_D)^{-1} M_s \| \right) < \infty \qquad \text{for } s > \frac{1}{2} \tag{22}$$

where M_s is the operator of multiplication by $(1 + t^2)^{-s}$. Moreover

$$\lim_{\lambda_2 \to 0+} M_s (\lambda_1 + i\lambda_2 - \mathcal{H}_D)^{-1} M_s = M_s R^+ M_s \tag{23}$$

exists in the norm topology.

Proof: The asymptotics of the eigenfunctions is a restatement of the result of Theorem 2. For real $\lambda \in [-m, m]^c$ all eigenfunctions are thus bounded. Thus the absolute continuity of the spectrum follows from Theorem 1.

For simplicity assume τ_D to be regular at 0 and let $u_1(t, \lambda), u_2(t; \lambda)$ be λ-eigenfuntions for τ_D with $u_1(0, \lambda) = (\cos\alpha, \sin\alpha)^t$ and $u_2(0, \lambda) = (-\sin\alpha, \cos\alpha)^t$. Then u_1 and u_2 are real for real λ and depend analytically on λ. Thus

$$u_1 = d(\lambda)(1 + o(1)) \begin{pmatrix} \cos(\sigma_n + \gamma) \\ \mp a \sin(\sigma_n + \gamma) \end{pmatrix} \text{ and } u_2 = e(\lambda)(1 + o(1)) \begin{pmatrix} \sin(\sigma_n + \gamma_1) \\ \pm a \cos(\sigma_n + \gamma_1) \end{pmatrix}$$

where $a = |a_+(\infty)|$ and where the $o(1)$ term depends continuously on λ and can be estimated uniformly for $\lambda \in K$. Since $1 = [u_1, u_2](x) = e(\lambda)d(\lambda)(1 + o(1))\cos(\gamma - \gamma_1)$ and

$$m(\lambda) = \lim_{t \to \infty} -\frac{u_2(t)}{u_1(t)}$$

m depends continuously on λ and $Im\ m_+(\lambda_1) \neq 0$ for $\lambda_1 \in [-m, m]^c$.

For $\lambda \in K$ and $Im\lambda > 0$ the resolvent $(\lambda - \mathcal{H}_D)^{-1}$ is an integral operator with a 2 by 2 matrix kerned [16, ch. 7]. Thus this is also true for $M_s(\lambda - \mathcal{H}_D)^{-1} M_s$ and it suffices to prove (22) and (23) for the matrix component operators only. Since (23) follows from (22) as in [5], it remains to show the uniform boundedness of the matrix component operators. For this we use the Schur test [14, Th 4.1.2] with $p = r = 2$ and test function $\phi_1 = \phi_2 = 1$. Because of Theorem 2 it suffices to apply the test only to the functions $\exp(\pm i\sigma_n)$ in the resolvent kernel of $(\lambda - \mathcal{H}_D)^{-1}$.

Remark 1: It should be noted that the absolute continuity of the spectrum is valid regardless of the behaviour of the potential near 0. The limiting absorption principle can also be shown for potentials which are singular near 0, e.g. the Coulomb potential. In many of these cases u_∞ and u_0 behave like $t^{-\rho}$ or t^{ρ} respectively near 0. Then the Schur test function should be chosen as t^{β} near 0. In addition it should be noted that the absolute continuity of the spectrum follows from the continuity of the m-function and from the limiting absorption principle as well.

Remark 2: Above we have shown the absolute continuity by using the subordinacy principle. Likewise we could have employed the technique of Weidmann [16] as in [2]. In this case the result in Lemma 15.4 respectively 16.8 in [16] follows from Theorem 2, while the generalized Prüfer angle θ is approximately σ_n.

Corollary 2: Corollary 1 is valid analogously for Schrödinger differential expressions on $(0,\infty)$

$$-y'' + Vy = \tau_s y \tag{24}$$

if $V \in \mathcal{D}_n(\mathbf{R})$.

Remark 1: The above remarks also apply to this situation. Corollary 1 and 2 extend the well known results of Heinz [11] and Weidmann [16]. Corollary 2 also extends the main theorem of Ben-Artzi [3] and gives another proof for Theorem 1 in [17] in the case $n = 2$.

Remark 2: In the proof of the theorem no essential use was ever made of $P_1(t), P_2(t), P_3(t) \to 0$ for $t \to \infty$. Thus Theorem 2 and the Corollaries remain valid for potentials in $\mathcal{E}_n(\mathbf{R})$ provided

$$(P_1(t) - m - \lambda)(P_3(t) + m - \lambda) - P_2^2(t) \geq \delta > 0 \quad \text{for all } t \geq c \text{ for some } c > 0. \tag{25}$$

For Schrödinger operators (25) becomes

$$\lambda - P(t) \geq \delta > 0 \qquad \text{for all } t \geq c > 0 \text{ for some } c \tag{25'}$$

Corollary 3: Assume $V_i = S_i + P_i \quad i = 1,3$ with $S_i \in \mathcal{L}^1$ and $P_i \in \mathcal{E}_n(\mathbf{R})$. Moreover let $V_2 = P_2 + S_2$ with P_2 differentiable satisfying $S_2, P_2', P_2 \cdot P_i' \in \mathcal{L}^1$ and assume that (25) holds for some c. Then the spectrum of the corresponding Dirac Hamiltonian $\mathcal{H}_D$ is absolutely continuous in a neighborhood of λ.

Proof: Again the absolute continuity follows from the boundedness of the solutions.

Remark: A corresponding result also holds for Schrödinger operators. It is also possible to show a continuity result for m and a limiting absorption principle in this case.

The previous corollary and the following result were prompted by the analysis of

$$-y'' + \sin t^{\alpha} y = \tau_s y \qquad \alpha < 1$$

Corollary 4: On $(0,\infty)$ consider the Schrödinger operators

$$\tau_s y = -y'' + (V + S)y \qquad \text{with } V \in \mathcal{E}_n$$

and $S \in \mathcal{L}^2 \cap \mathcal{L}^1$. Then any Hamiltonian $\mathcal{H}_s$ corresponding to τ_s satisfies

i) $\sigma_{ess}(\mathcal{H}_s) = [\alpha, \infty) \quad$ where $\alpha = \liminf V(t)$

ii) $(\beta, \infty) \subset \sigma_{ac}(\mathcal{H}_s)$ where $\beta = \limsup V(t)$

Proof: For $\lambda > \beta$ Corollary 3 shows that any λ eigenfunction behaves asymptotically like a plane wave. Thus Theorem 1 is applicable.

Lemma 2: Consider a Schrödinger differential expression as above and assume V satisfies for some $\lambda \in \mathbf{R}$ the following condition:.

For each $\varepsilon > 0$ there exist an interval I_ε with length $I_\varepsilon \to 0$ for $\varepsilon > 0$ and

$$|V(t) - \lambda| < \varepsilon \text{ for } t \in I_\varepsilon$$

Then $[\lambda, \infty) \subset \sigma_{ess}(\mathcal{H}_s)$ for any Hamiltonian $\mathcal{H}_s$ associated to τ.

Proof: Let φ_n be a smooth, almost constant function with support $I_{\frac{1}{n}}$. Then $e^{ikx} \cdot \varphi_n$ is a singular sequence for $\lambda + k^2$.

IV

We shall now extend Theorem 2 to include von Neumann Wigner potentials W. Such potentials are assumed to be decaying and oscillatory. Mathematically they are thus conditionally integrable. If the decay at infinity is improved by integration we call the corresponding terms rapidly oscillating. In the next theorem such potentials will be denoted by X. It is clear from the work of Harris Lutz [10] and [1] that the $1+Q$-transformation is particularly suited to handle Wigner von Neumann potentials, in particular the rapidly oscillating ones.

Since μ and σ depend on λ we shall also write $\mu(t,\lambda)$ and $\sigma(t,\lambda)$ in order to indicate this dependence. As in [1] we write $\int_\infty^t f(s)ds = \tilde{f}(t)$ for a conditionally integrable f.

Theorem 3: Consider the Dirac differential expression (1) such that near infinity

$$V_i = P_i + S_i + W_i + X_i \qquad i = 1,2,3 \tag{26.1}$$

$$P_1, P_3 \in \mathcal{D}_2(\mathbf{R}), P_2 \text{ differentiable}, \quad P_2(t) \to 0 \quad \text{for } t \to \infty \tag{26.2}$$

$$P_2', P_2P_1, P_2P_3, \quad S_i \in \mathcal{L}^1 \tag{26.3}$$

Moreover assume that $W_i, X_i, W_i \cos 2\sigma$ and $W_i \sin 2\sigma$ are conditionally integrable and that

$$h_i = (W_i \cos 2\sigma)^\sim \text{ and } h_{i+3} = (W_i \sin 2\sigma)^\sim$$

satisfy

$$h_iW_j, h_iP_j' \in \mathcal{L}^1 \qquad i = 1,2,...,6, j = 1,2,3 \tag{26.4}$$

and for $\tilde{X}_i$ we demand

$$\tilde{X}_i, \tilde{X}_iX_j, X_ih_k \in \mathcal{L}^1 \qquad i,j = 1,2,3, k = 1,2,...,6 \tag{26.5}$$

Then a fundamental matrix U of (1) is given by

$$U(t) = A(t)(1 + B(t))\Sigma(1 + C(t))$$

where $\Sigma = \exp\operatorname{diag}(i(\sigma+L), -i(\sigma+L)), iL(t) = \int_c^t (a_+ - a_-)^{-1}(W_1 + (a_+ + a_-)W_2 + a_-a_+W_3)$ and $B(t), C(t) \to 0$ for $t \to \infty$. If (26.4) and (26.5) hold for all $\lambda \in (\lambda_1, \lambda_2)$ this set belongs to the absolutely continuous spectrum of any Dirac Hamiltonian associated to (1).

Proof: It follows from the proof of Theorem 2 that (1) with $W_i, X_i = 0$ has a fundamental matrix U of the form $U = AA_1\Sigma_1(1 + C(t))$ with $C(t) \to 0$ for $t \to \infty$ and $\Sigma_1 = \exp\operatorname{diag}(i\sigma_1, -i\sigma_1)$. Thus the transformation

$$z = \Sigma_1^{-1}A_1^{-1}A^{-1}u$$

will lead to

$$z' = \left[\Sigma_1^{-1}A_1^{-1}A^{-1}WAA_1\Sigma_1 + R\right] z$$

Now one sees easily with $W_i + X_i \to W_i$

$$A^{-1}WA = (a_- - a_+)^{-1}\begin{pmatrix} -W_1 - (a_- + a_+)W_2 - a_-a_+W_3 & -W_1 - 2a_-W_2 - a_-^2W_3 \\ W_1 + 2a_+W_2 + a_+^2W_3 & W_1 + (a_- + a_+)W_2 + a_+a_-W_3 \end{pmatrix} \tag{27}$$

Moreover A_1 is of the form $A_1 = \begin{pmatrix} 1 & i\alpha \\ -i\alpha & 1 \end{pmatrix}$ with $\alpha = \frac{q}{2\mu}, q \in \mathcal{L}^2$ and $q' \in \mathcal{L}^1$. Thus we have $\mu_2 - \mu_1 = \mu_2 - \mu \in \mathcal{L}^1$ and this implies that $e^{2i\sigma_1} = e^{2i\sigma} \cdot k$ with $k' \in \mathcal{L}^1$.

Writing now

$$\Sigma_1^{-1}A_1^{-1}A^{-1}WAA_1\Sigma_1 = \begin{pmatrix} l & q_1e^{-2i\sigma} \\ q_2e^{2i\sigma} & -l \end{pmatrix}$$

we see that $q_1e^{-2i\sigma}$ and $q_2e^{-2i\sigma}$ are sums of terms of the form $hkW_ie^{\pm 2i\sigma}$ and $hkX_ie^{\pm 2i\sigma}$ where h is a rational expression in μ, P_1, P_2 and P_3 and where $k' \in \mathcal{L}^1$. If the corresponding expressions in the $1 + Q$-transformation are chosen as $hk(W_ie^{\pm 2i\sigma})^\sim$ respectively $hk\tilde{X}_ie^{\pm 2i\sigma}$ the transformation $w = \left(1 + \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix}\right)^{-1} z$ leads to

$$w' = \{\operatorname{diag}(-l, l) + R\}w$$

Here we may take $l = -(a_- - a_+)^{-1}[W_1 + (a_- + a_+)W_2 + a_-a_+W_3]$, because all other terms are integrable (27). Even if l is not integrable, the solutions of this system are still bounded, because $Re\ l = 0$. Writing $iL(t) = \int_c^t l(s)ds$, the fundamental matrix of this system is given by $\exp\operatorname{diag}(iL, -iL) \cdot (1 + C(t))$ where $C(t) \to 0$ for $t \to \infty$. The remaining claims follow directly from this and the subordinacy principle.

Remark: Theorem 3 remains valid even without the assumption $X_ih_k \in \mathcal{L}^1$. In this case a further $1 + Q$-transformation is needed. It is obvious that a corresponding nonresonance result is also valid for Schrödinger operators. The above theorem extends the main result of Behncke, Rejto [2] as well as the principal theorem of Hinton and Shaw in [12].

In order to extend the main theorem of [1] we have to specify the W_i further. Following [1] we require

$$W_i = \Sigma f_j^{(i)} \sin g_j \quad \text{with } f_j^{(i)} = f_{j,1}^{(i)} + f_{j,2}^{(i)} \tag{28.1}$$

$$f_{j,k}^{(i)} \text{ is } k\text{-times differentiable with} \tag{28.2}$$

$$f_{j,k}^{(i)} \in \mathcal{L}^3, f_{j,k}^{(i)\prime} \in \mathcal{L}^{\frac{3}{2}}, f_{j,2}^{(i)\prime\prime}, f_{j,1}^{(i)\prime} \in \mathcal{L}^1 \tag{28.3}$$

$$g_j'(t) \to \infty \text{ or } g_j'(t) \to g_j'(\infty) > 0 \text{ for } t \to \infty \tag{28.4}$$

$$\frac{g''}{g'^2} \in \mathcal{L}^1 \tag{28.5}$$

$$\text{For } j \neq k \quad \liminf |g_j'(t) - g_k'(t)| > 0 \tag{28.6}$$

As in [1] define the resonance sets $\mathcal{R}_1$ and $\mathcal{R}_2$ by

$$\begin{aligned} \mathcal{R}_1 &= \{\lambda \in [-m,m]^c \mid \lim_{t\to\infty} g_j'(t) - 2\mu(t,\lambda) = 0 \quad \text{for some } j\} \\ \mathcal{R}_2 &= \{\lambda \in [-m,m] \mid \lim(g_j' \pm g_k') - 2\mu(t,\lambda) = 0 \quad \text{for some } j,k\} \end{aligned} \tag{29}$$

Theorem 4: Let V_1, V_2, V_3 be given by (15) with $P_i \in \mathcal{D}_3$ $X_i = 0$ and W_i satisfying (28). Then any Dirac Hamiltonian corresponding to (1) has an absolutely continuous spectrum in $[-m,m]^c \backslash (\mathcal{R}_1 \cup \mathcal{R}_2)$. The same result holds for corresponding Schrödinger operators.

Proof: Let $\lambda \in [-m,m]^c \backslash (\mathcal{R}_1 \cup \mathcal{R}_2)$. Arguing as in [1] or as above it suffices to show that the corresponding λ-eigenfunctions behave like plane waves. As above we see that the first three diagonalizing transformations, with respect to the P_i lead to

$$w' = \{\Sigma^{-1}(1+C)^{-1}A_2^{-1}A_1^{-1}A^{-1}WAA_1A_2(1+C)\Sigma\}w = \begin{pmatrix} l & q_1e^{-2i\sigma_2} \\ q_2e^{2i\sigma_2} & -l \end{pmatrix} w \tag{30}$$

where $C(t) \to 0$ for $t \to \infty$ and $C' \in \mathcal{L}^1$. A typical summand of $q_1e^{-2i\sigma_2}$ is now $h \cdot fe^{-2i\sigma_2 \pm ig}$ where h is a differentiable function built from the $P_{i,k}^{[j]}, j < k$ and integrals of $\mathcal{L}^1$ functions, while f arises from the W_i. Similar representations can be found for $q_2e^{2i\sigma_2}$ and l. Since $\lambda \notin \mathcal{R}_1$, or $\underline{\lim}|2i\mu_2 \pm ig| > 0$ a $1 + Q$-transformation whose corresponding terms are $h \cdot f\dfrac{e^{-2i\sigma_2 \pm ig}}{(-2i\mu_2 \pm ig')}$ can be applied to transform the off diagonal terms in (30). The resulting system $z = \begin{pmatrix} 1 & Q_1 \\ Q_2 & 1 \end{pmatrix}^{-1} w$ is then of the form

$$z' = \left\{ \begin{pmatrix} l_1 & q_3 \\ q_4 & -l_1 \end{pmatrix} + R \right\} z \tag{31}$$

where q_3 is a sum of terms $hh_1ff_1e^{-2i\sigma_2 \pm ig_1 \pm ig_2}$ and $h^{[1]}fe^{-2i\sigma_2 \pm ig}$. For q_4 a similar representation is valid. Thus the proof can be completed as in [1] with a second $1+Q$-transformation. In fact in this case one obtains the same expression as in the Theorem of [1]. In particular $Re\ l_1$ is integrable.

Altogether this shows that a fundamental matrix for (1) is given by

$$A(t)(1+B(t))\Sigma(1+C(t)) \exp \operatorname{diag}(iL_1, -iL_1)$$

where $B(t) \to 0, C(t) \to 0$ for $t \to \infty$ and where $iL = i\int_c^t Iml\ dt$.

Remark: By adapting this approach to the case studied in [1], we can also show the continuity of the m-fuction off $\mathcal{R}_1 \cup \mathcal{R}_2$ and a limiting absorption principle with the aid of the complex $1 + Q$-transformation. However the proofs are considerably more involved in this case. Likewise the result can be extended to Wigner von Neumann potentials which are infinite sums provided $\mathcal{R}_1 \cup \mathcal{R}_2$ consists of isolated points and the terms converge sufficiently rapidly. In addition it should be noted that we only used that $\pm g' + 2\mu_2$ is bounded away from 0 near infinity.

In order to apply the $1+Q$-transformation we used the invertibility of $(1+Q)$. So far we have achieved this by demanding $Q(t) \to 0$ for $t \to \infty$. However if Q is nilpotent, this condition is also satisfied. The Schrödinger differential expression

$$\tau_s y = -y'' + (q + h \sin g) y$$

with g real valued twice differentiable and

$$g'(t) \to \infty, \quad q, \frac{h}{g'} \in \mathcal{E}_n(\mathrm{R}); \frac{g''}{g'^2}, g'^{-1} \in \mathcal{L}^1, \tag{32}$$

is an example, albeit a strange one, for this. We note that h may be unbounded. But this is compensated by an extreme rapid oscillation.

As above we write $(\tau_s - \lambda)y = 0$ in systems for m

$$u' = \begin{pmatrix} 0 & 1 \\ q_0 & 0 \end{pmatrix} u \quad \text{with} \quad q_0 = q + h \sin g - \lambda$$

and transform it by $v = \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} u, \quad Q = \frac{h}{g} \cos g$. Then

$$v' = \begin{pmatrix} -Q & 1 \\ q_1 + Q_1 & Q \end{pmatrix} v \quad \text{where} \quad q_1 = q - \tfrac{1}{2}\left(\tfrac{h}{g'}\right)^2 - \lambda \text{ and}$$
$$Q_1 = \left(\tfrac{h}{g'}\right)' \cos g - \tfrac{1}{2}\left(\tfrac{h}{g'}\right)^2 \cos 2g$$

and another $1+Q$-transformation, $w = \begin{pmatrix} 1+Q_2 & 0 \\ Q_3 & 1-Q_2 \end{pmatrix} v$ transforms this system into

$$w' = \left[\begin{pmatrix} 0 & 1 \\ q_1 & 0 \end{pmatrix} + R\right] w$$

and to this system Theorem 2 can be applied. Thus for $\lambda > \limsup_{t\to\infty} q_{(t)} - \frac{1}{2}\left(\frac{h}{g'}\right)^2(t) = s$ the solutions of this equation look like plane waves and the spectrum of corresponding Schrödinger Hamiltonian is absolutely continuous in (s, ∞). With more effort the result can be extended to $g'^{-1} \in \mathcal{L}^p$ provided $q'g'^{-4}, \left(\frac{h}{g'}\right)^{[1]} g'^{-4}, \left(\frac{h}{g'}\right)^{[2]} g'^{-3}, \left(\frac{h}{g'}\right)^{[1]2} g'^{-2} \in \mathcal{L}^1$. It also holds for $q = \Sigma h_i \sin g_i$ if $g_i^{-1}, (g_i \pm g_j)^{-2}, (g_i \pm g_j \pm g_k)'^{-2} \in \mathcal{L}^2$. Even the boundedness of $\frac{h}{g'}$ may be dropped in some cases.

Above we have used the asymptotics of the eigenfunctions to show the absence of singular continuous spectrum. In addition it is possible to derive the density of states from our asymptotic formulae. For this one can use that σ_n is approximately the generalized Prüfer angle for the corresponding differential expression. Thus $\frac{1}{\pi}\sigma_n(d,\lambda)$ is approximately the number of zeroes of a λ eigenfunction of τ - remember $\sigma_n(c) = 0$ and the integrated density of states in $[\lambda_1, \lambda_2]$ is given by

$$\lim_{d\to\infty} \frac{1}{\pi}\frac{1}{d}[\sigma_n(d,\lambda_2) - \sigma_n(d,\lambda_1)]$$

If $\lim_{t\to\infty} \mu(t,\lambda) = \mu(\infty,\lambda)$ exists, this latter expression is just $\frac{1}{\pi}[\mu(\infty,\lambda_2) - \mu(\infty,\lambda)]$.

With the aid of several complex $1 + Q$-transformations it is also possible to derive the singularity of m at a resonance. For the simple case $P_i = S_i = W_2 = 0 \quad i = 1,2,3$ and $W_i = O\left(\frac{1}{t}\right)$ one finds that $m(\lambda_1 + i\lambda_2) \propto \lambda_2^{-2\alpha}, 0 < \alpha < \frac{1}{2}$ if the corresponding subordinate solution behaves like $t^{-\alpha}$ near infinity. Thus there is a close connection between the singularity of the m-function and the decay of the corresponding subordinate solution, which certainly holds in a much wider context.

Remark: After this paper was completed the author learned that this result had also been obtained independently by different methods by Hinton, Klaus and Shaw [18] and Atkinson (private communication).

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