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# Systoles on Riemann surfaces

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## 1. Introduction

*1.1.*

By *surface* I denote an (oriented) connected Riemann surface of constant curvature $-1$. Its *signature* $(g, n)$ indicates the genus $g$ and the number of boundary components $n$ which here are simple closed geodesics or cusps. The ***systole*** of a surface is the shortest closed geodesic which is not a boundary component. $T(M)$ denotes the Teichmüller space of a surface with the additional condition that the lengths of the boundary components (if any) are always fixed for the surfaces of $T(M)$. A geodesic $u$ of a surface $M$ is considered as being marked. This means that the geodesic in the same marked homotopy class in another surface $M' \in T(M)$ is also denoted by $u$. Instead of simple closed geodesic, I will only write geodesic.

This paper treats the following two main problems, cited as (I) and (II) in the sequel:

(I) For a fixed signature $(g, n)$ search the surfaces with a systole of maximal length, they are called *global maximal surfaces*. Surfaces for which the length of the systole is a *local* maximum in the corresponding Teichmüller space are called *maximal surfaces*.

(II) For a fixed signature $(g, n)$ search the surfaces with the maximal number of systoles, these surfaces are called ***best kissing number surfaces***.

Here, a fixed signature means that $(g, n)$ and the length of each boundary geodesic are fixed. For every fixed signature $(g, n)$ with $2g+n \geq 4$ both problems have at least one solution. The proof of this basic result is in [13] for (I) and for (II) in Section 2 below.

The two problems (I) and (II) were introduced by the author, see [13],[14], [15],[16]. A theory is developed in [13] how maximal surfaces can be found and many examples are presented. The most important and interesting examples appear in [14], they are the surfaces corresponding to the principal congruence subgroups of PSL(2,$\mathbb{Z}$). [15] and [16] treat (II), but in a less general context. Surfaces with a very big number of systoles are constructed, see [15] for surfaces with cusps and [16] for closed surfaces. Section 2 below will introduce a general theory of (II) and examples of best kissing number surfaces are presented. In particular, problem (II) will be completely solved for genus 2 and an upper bound for the number of systoles of closed surfaces is proved.

*1.2.*

The two problems (I) and (II) have well known analoga in Euclidean spaces, namely the problem of finding the best lattice sphere packing and the problem of finding the lattice sphere packing with the bigest kissing number, compare [6],[10], see also [2],[12]. In fact, if $g = 1$, the correspondences are the following for flat tori of dimension 2. Let $Q$ be a Euclidean parallelogram which is the fundamental domain of a torus $T$ of a normalized area. The systole of $T$ then corresponds to a side or to a diagonal of $Q$. So, placing circles with centers in the corresponding vertices and with a radius being half of the length of the systole, we have a lattice circle packing in the Euclidean plane. The solution for problem (I), namely the torus with the longest systole, induces the best lattice circle packing. The solution of problem (II), namely the torus with the maximal number of homotopy classes of systoles (in the hyperbolic case we do not need to speak of homotopy classes since there is a unique geodesic in a homotopy class), induces the circle packing with the bigest kissing number. Of course, these Euclidean problems are trivial in dimension 2 and their solutions were already known by greek mathematicians, but for dimensions bigger than 2, they are non-trivial. Their analysis has led to important theories with many interesting applications. For the problems (I) and (II) (in the hyperbolic case) the genus serves in some sense as an analogon to the dimension in the Euclidean case. One can see that the theory developed in [13] has some rather near correspondences with Voronoï's theory of the bigest minimum of positive definite quadratic forms with integer coefficients, see [18], which is another formulation for the problem of the best lattice sphere packing.

The theory in the Euclidean case gives some ideas what can happen in the hyperbolic case.

(i) Only for small dimensions the same lattice sphere packing solves both problems, so one expects the same thing in the hyperbolic case. I conjecture that already for closed surfaces of genus 5, the best kissing number surface and the global maximal surface are different, see below Section 3c.

(ii) There exist dimensions such that the solution of one (or both) of the Euclidean problems is not unique. We have seen in [14], Section 5, that the same can happen for (I). Further, we will see in Section 3c that there exist two different closed surfaces of genus 4 with 36 systoles, and I conjecture that both are best kissing number surfaces.

(iii) The first idea that the solutions of the two problems should have many symmetries, is wrong in the Euclidean case. We will see a plausible argument that the same is true in (I).

(iv) In the Euclidean case, the kissing number and the bigest minimum of positive quadratic forms increase monotonically with the dimension. One is led to the idea that the same happens in the hyperbolic case, more precisely, if the number of boundary components and their lengths are fixed, then the maximal length of the systole and the maximal number of systoles should increase monotonically with the genus. The assertion is however wrong if we only compare with the area of the surfaces: There are global maximal surfaces $M_1$ and $M_2$ such that, in $M_2$, the area is bigger, but the length of the systole is smaller than in $M_1$, see [14], Section 5.

*1.3.*

Problem (I) has another closely related problem, namely the so called isostolic problem. Here, one is looking for bounds (depending on the topology, for example) for the systole of Riemannian manifolds with an arbitrare continuous metric. This problem of course is much more general, and important progresses have been made in the last years, in particular due to some papers of M. Gromov, see [8],[9], see also the papers of M. Berger [3] and of E. Calabi [4].

*1.4.*

Problem (I) already appears in a rudimental form in Fricke/Klein [7]. In Zweiter Abschnitt, Sechstes Kapitel of [7], Fricke/Klein treat the (2,3,7) triangle surface of genus 3 with an automorphism group of order 168 which they call $F_{168}$; this was certainly Klein's prefered surface. §11 on page 379 is intitled "Die 21 kürzesten Linien der $F_{168}$...", and in this paragraph they write "... und um für diese einen zweckmässigen Namen zu besitzen, benennen wir sie (auf Grund einer nahe liegenden Ueberlegung) als eine *kürzeste Linie auf der Fläche* $F_{168}$." So, Fricke/Klein identify the 21 systoles of $F_{168}$, but did not prove this fact. It seems that the first proof has appeared in [13] where it is also shown that $F_{168}$ is a maximal surface, but not the global maximal surface for genus 3.

*1.5.*

I already have described the content of Section 2. Section 3 treats some questions concerning problem (I). New maximal surfaces of small genus are presented and I show why a maximal surface does not need to have many symmetries. I also show that there exist non-congruence subgroups of PSL(2,$\mathbb{Z}$) which correspond to maximal surfaces.

## 2. Best kissing number surfaces

**Theorem 1.** *For every fixed signature* $(g,n)$ *there exists a best kissing number surface.*

*Proof.* By [13], Theorem 2.6 and Theorem 2.8, for every signature a maximal surface $M'$ exists and has at least $dim(T(M')) + 1$ systoles. Therefore, the best kissing number surface $M$ has at least $dim(T(M)) + 1$ systoles. It follows that $M$ has systoles which intersect and they thus must be longer than $2\sinh^{-1} 1$. Hence we can restrict the search for the best kissing number surface to a compact part of the moduli space which proves its existence. □

### *2.1. The notion of a local maximum for problem (II)*

While for problem (I) it also makes sense to look for local maxima, this is not the case for problem (II), at least not in a naïve way, since for problem (II) each surface is a local maximum (the number of systoles does not increase locally).

To avoid this problem we could look for *isolated* local maxima for (II) and we have the following result.

**Theorem 2.** *For every fixed signature $(g, n)$ there exists only a finite number of isolated local maxima for (II).*

*Proof.* Let $M$ be an isolated local maximum of (II), let $F$ be its set of systoles. Then $F$ fills up, this means that every inner closed geodesic of $M$ is intersected by a systole. Since if a simple closed geodesic $a$ is not intersected by a systole, a small twist deformation along $a$ cannot change the number of systoles contradicting thus the hypothesis that $M$ is an isolated local maximum. Therefore, all isolated local maxima of (II) lie in a compact subset of the moduli space and since among them, there is no cluster point, their number is finite. □

But also the notion of an isolated local maximum of (II) is not completely satisfying. The problem is that there exist non-trivial examples of local maxima which are not isolated. See for example the following closed surface $M$ of genus 2. The set of systoles $F$ of $M$ contains exactly seven systoles, one of them is dividing, call it $a$. It divides $M$ into two subsurfaces $S_1$ and $S_2$ of signature (1,1). Let $S_1$ and $S_2$ be isometric such that each of them contains three systoles. It follows that $M$ is well determinded up to a twist deformation along $a$, compare [13] Corollary 4.1. Let $\mathcal{M}$ be the set of surfaces in the moduli space of genus 2 which are constructed, from $M$, by a twist deformation along $a$. Then all surfaces of $\mathcal{M}$ have the same seven systoles and all surfaces in a small neighborhood of $\mathcal{M}$ have less than seven systoles.

Such examples exist for most signatures and we can even not exclude a priori that the global maximum of (II) is of this nature (I conjecture however that this cannot happen, see below). So, it is not easy to find a reasonable notion for a local maximum of (II).

**Theorem 3.** *A maximal surface $M$ is an isolated local maximum of (II).*

*Proof.* Let $F$ be the set of systoles of $M$. In a neighborhood of $M$ the set of systoles is a subset of $F$. If $M$ is not an isolated local maximum for (II), then there exist surfaces $M' \neq M$ in each neighborhood of $M$ such that all geodesics of $F$ are systoles. On the other hand, since $M$ is a maximal surface, there exists a neighborhood $U$ of $M$ such that for $M' \in U$, the geodesics of $F$ cannot all be longer than in $M$. It follows that there exists a surface $M' \in U$ such that $F$ is the set of systoles which, in $M'$, are shorter or of equal length than in $M$. But this contradicts the fact that $M$ is $F$–minimal (see [13]) which means that for each surface $M' \neq M$ in the moduli space of $M$, there is at least one geodesic of $F$ which is longer in $M'$ than in $M$. We therefore have proved that $M$ is an isolated local maximum for (II). □

The preceeding theorem leads to the interesting question if there exist isolated local maxima of (II) which are not local maxima of (I). Moreover, I conjecture that the global maximum of (II) is at least a local maximum of (I):

**Conjecture.** The best kissing number surface is a maximal surface.

### *2.2. Examples of best kissing number surfaces*

I now give examples of best kissing number surfaces.

**Theorem 4.** *The best kissing number surfaces for the signatures (1,1), (0,4) and (1,2) are identical to the global maximal surfaces of these signatures. Their respective numbers of systoles is three in the cases (1,1), (0,4) and five in the case (1,2).*

*Proof.* The proof of Theorem 4.2 in [13] shows that a surface of signature (1,1) has at most three systoles and it has three systoles if and only if it is the unique global maximal surface. The proof of Theorem 4.3 in [13] shows the analogon in the case (1,2), namely that a surface of signature (1,2) has five systoles if and only if it is the unique global maximal surface. Finally, Theorem 5.6 of [13] and its proof shows the assertion in the case (0,4). □

**Theorem 5.** *The best kissing number surface for genus 2 is the global maximal surface of genus 2, namely the (2,3,8) triangle surface with an automorphism group of order 48.*

*Proof.* Call M* the global maximal surface of genus 2. M* has 12 systoles (see [13]). It follows that the best kissing number surface $M$ has at least 12 systoles. Systoles of closed surfaces which intersect mutually, are non–dividing. If $M$ has a dividing systole, then it is intersected by no other systole and $M$ can have at most seven different systoles, compare the above described example. Surfaces of genus 2 are hyperelliptic and the simple closed non–dividing geodesics of these surfaces pass through exactly two Weierstrass points. There are six Weierstrass points, call them 1,2,3,4,5,6. Since systoles of closed surfaces intersect at most once, we can identify each systole with the two Weierstrass points where it passes through. In the following we speak of systoles 1–2, 4–6 and so on.

Assume now that through the Weierstrass point 1 four different systoles pass. Then there are two of them which intersect in 1 by an angle $\alpha \leq \pi/4$. It follows as in the proof of Lemma 5.1 of [13] that $\alpha = \pi/4$ and that $M =$ M*. I however note that in the proof of this Lemma 5.1 instead of $\pi/4$ it is written $\pi/8$ which is of course a fault. It now follows that either $M =$ M* or at most three different systoles pass through every Weierstrass point. In the latter case, $M$ has at most nine systoles. □

The following theorem is still more interesting and, surprisingly, it can be proved in more or less the same manner as Theorem 5.3 of [13] which assures that M* is the unique maximal surface of genus 2.

**Theorem 6.** *$M^*$ is the unique isolated maximum of (II) of genus 2.*

*Proof.* Assume that $M \neq M^*$ is an isolated maximum of (II). Then every closed geodesic of $M$ is intersected by a systole hence $M$ has no dividing systole. In the proof of the preceeding Theorem 5 we have seen that through each Weierstrass point of $M$ at most three systoles pass.

(i) **Claim 1:** Let 1–2 and 1–3 be systoles of $M$. Then $M$ has no systole 2–3. *Proof.* Assume firstly that $M$ has a systole 2–3 such that there exists a subsurface $S$ of signature (1,1) which contains the three systoles 1–2, 1–3, 2–3. Let $z$ be the boundary geodesic of $S$. Then $z$ is the boundary geodesic of another subsurface $S'$ with signature (1,1). $S'$ must be isometric to $S$, compare Lemma 5.2 of [13]. The automorphism group of $S$ contains a subgroup of order three. Since $S'$ is isometric, this subgroup also acts on $M$ and it follows that systoles which intersect $z$ appear as triples. The crucial point is that one such triple is not sufficient. We can increase the length of the three systoles of $S$ and of $S'$ be the same amount $\epsilon$. The result is a new surface $M'$. If the twist along $z$ has not been changed, then the triple of systoles intersecting $z$ will be shorter in $M'$, by calculation. Executing a twist deformation along $z$ we can increase the length of this triple by the same amount $\epsilon$ and $M'$ has again the same nine systoles as $M$. Since $M$ is an isolated maximum of (II), a second triple of systoles much intersect $z$, but then $M$ has 12 systoles which is impossible since $M^*$ is the unique surface with 12 systoles.

Assume now that $M$ has a systole 2–3, but there does not exist a subsurface of signature (1,1) which contains 1–2,1–3 and 2–3. This is impossible by the calculation made in the proof of Lemma 5.3 of [13] which thus finishes the proof of Claim 1.

(ii) **Claim 2:** Let $a$ be a simple closed non-dividing geodesic of $M$. Then there are at most three systoles of $M$ which do not intersect $a$.

*Proof.* Cut $M$ along $a$, the result is a hyperelliptic surface $N$ of signature (1,2). Assume that $N$ contains four systoles of $M$. Then $N$ has an involution which also acts on $M$ and it follows that systoles of $M$ which intersect $a$, appear as pairs, compare the proof of Lemma 5.4 in [13]. Again, the crucial point is that one pair is not sufficient by the same argument as above. It follows that there are two pairs of systoles which intersect $a$. It then follows as in the proof of Lemma 5.4 of [13] that $M$ must have three systoles of the type 1–2,1–3,2–3 which contradicts Claim 1 and proves therefore Claim 2.

(iii) By the proof of Theorem 5, $M$ has at most nine systoles. If $M$ has seven, eight or nine systoles then a contradiction follows by Claim 1 or Claim 2, compare the proof of Theorem 5.3 in [13].

(iv) By (i),(ii), and (iii) it follows that $M$ has at most six systoles and that every simple closed geodesic of $M$ is intersected by at least two systoles. Assume that through every Weierstrass point at most two systoles pass. Then, up to permutations of the Weierstrass points, there are two possibilities. Either $M = M_1$ has six systoles 1–2,2–3,3–4,4–5,5–6,6–1 or $M = M_2$ has five systoles 1–2,2–3,3–4,4–5,5–1.

Assume that $M$ has a Weierstrass point where three systoles pass through, for example 1–2,1–3,1–4. Then, by Claim 1, all other systoles pass through 5 or 6. By (ii), only one systole of the three possible 2–5,3–5,4–5 can exist, and by the same reason, only one systole of the three possible 2–6,3–6,4–6 can exist. But since each geodesic 5–6 must be intersected by at least two systoles, there exists a systole in both triples. We can therefore assume that 2–5 is a systole. If 2–6 is also a systole, then 5–6 cannot be a systole by (i). Hence a third possibility for $M$ is $M_3$ with the five systoles 1–2,1–3,1–4,2–5,2–6. If 2–6 is not a systole, we can assume that 3–6 is a systole. Then 2–5 is only intersected by 1–2, therefore 5–6 must also be a systole. The fourth possibility for $M$ is thus a surface $M_4$ with the six systoles 1–2,1–3,1–4,2–5,3–6,5–6. Up to permutations of the Weierstrass points the four mentioned surfaces $M_i$ cover all possible cases.

(v) The cases $M_1, M_3, M_4$ each contain two pairs of systoles which do not intersect, namely 1–2,2–3 and 4–5,5–6 in $M_1$, 1–3,1–4 and 2–5,2–6 in $M_3$, 1–3,1–4 and 2–5,5–6 in $M_4$. Hence there exists a dividing geodesic $z$ which separates $M$ into two isometric subsurfaces of signature (1,1). Therefore, a symmetry on $M$ is induced which implies that systoles intersecting $z$ appear as pairs. It follows by the argument of (i) that two pairs of systoles must intersect $z$ which shows that the three cases $M_1, M_3, M_4$ are impossible.

(vi) Case $M_2$ remains. The systoles of $M_2$ induce a hyperbolic pentagon $P$ with sides of equal length. Then there exists a pentagon $P'$ with sides of equal length which are a little bit longer than the sides of $P$ and such that the angles of both pentagons are almost the same. Moreover, we can construct by $P'$ a surface $M'$ of genus 2 which has the same five systoles as $M_2$ proving thus that $M_2$ is not an isolated maximum of (II). □

The topology induces an upper bound of the maximal number of systoles. For closed surfaces for example, there is a bounded number of non–dividing simple closed geodesics which mutually intersect at most once.

**Theorem 7.** *Let $M$ be a closed surface of genus $g$ and $a$ a systole of $M$. Then $a$ is intersected by at most $8(g-1)$ other systoles.*

*Proof.* Since in a closed surface systoles mutually intersect at most once, we assume that $a$ is non–dividing. In the sequel, a geodesic is a simple closed non–dividing geodesic which intersects $a$ once and all geodesics mutually intersect at most once. We will calculate the maximal number of such geodesics. This is a topological question, so we are free to choose the metric of $M$. Let $b$ a geodesic and cut $M$ along $a$ and along $b$. We can assume that the result is a Euclidean rectangle in which the opposite sides are identified and from which $2(g-1)$ circles are removed.

We firstly count the geodesics which do not intersect one of the circles and do not intersect $b$. Their maximal number is $2(g-1)$. There is also a maximal number of $2(g-1)$ geodesics which do not intersect a circle, but intersect $b$. So we have at most $4(g-1)$ geodesics which do not intersect a circle, compare Fig. 1.

Assume now that the geodesics intersect two circles. Then again, we have at most $2(g-1)$ geodesics which do not intersect $b$ and at most $2(g-1)$ which intersect $b$, compare Fig. 2.

Every geodesic which intersects more than four circles would intersect more than once one of the $8(g-1)$ geodesics mentioned. So, this set of $8(g-1)$ geodesics is a maximal set. Other maximal sets of geodesics are induced by other choices of $b$ and of the circles. □

**Corollary 1.** *A surface $M$ of genus $g$ has at most $12g^2 - 21g + 9$ systoles.*

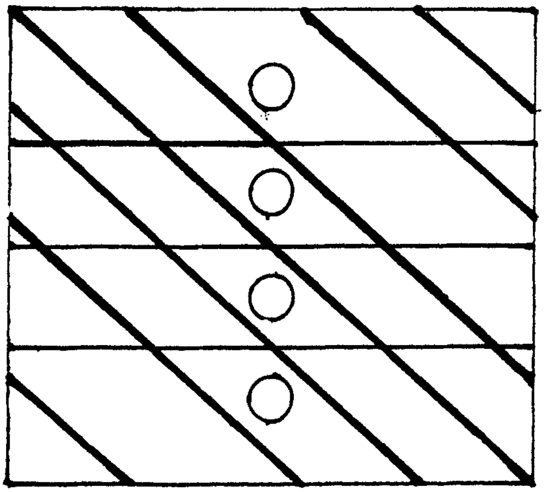

**Fig. 1.** The rectangle for $g = 3$ with the geodesics not intersecting a circle. Four geodesics do not intersect $b$ (thin lines), four geodesics intersect $b$ (thick lines). In the figure, the circle on the top must be identified with the circle on the bottom, and so on

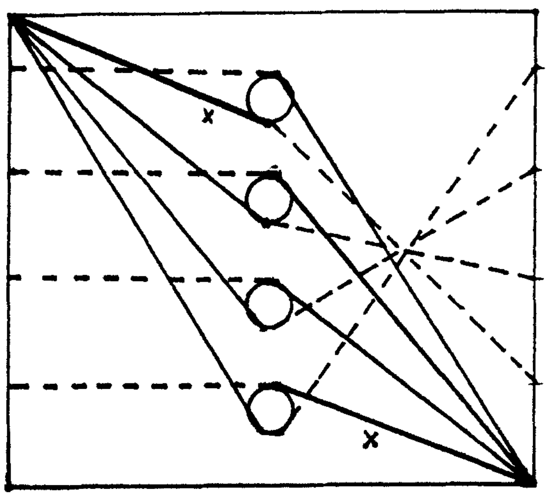


**Fig. 2.** The geodesics intersecting two circles (recall that the circle on the top is identified with the circle on the bottom, and so on). Four geodesics do not intersect $b$ (dotted lines), four geodesics intersect $b$ (plane lines, one is denoted by $x$)

*Proof.* We calculate the maximal number of simple closed geodesics of $M$ which mutually intersect at most once. Let $S$ be a set of mutually non intersecting simple closed geodesics of $M$. Let $a \in S$. Then $a$ is intersected by at most $8(g-1)$ systoles by Theorem 7. Each systole which intersects $a$, intersects at least a second element of $S$. It follows that $M$ has at most

$$3g-3+(3g-3)(4g-4)=12g^2-21g+9$$

systoles. □

**Remark.** Assume that for $g \to \infty$ the maximal number of systoles of a closed surface of genus $g$ increases with $g^k$ for a real number $k$. Corollary 1 shows that $k \leq 2$ and [16] proves that $k > 1$.

## 3. Some remarks on maximal surfaces

### *3.1. Maximal surfaces without symmetries*

In most cases, we can construct a maximal surface from a closed surface $M$ which admits a tesselation of triangles of equal sides such that around each triangle vertex the same number $N$ of triangles lies. The construction goes as follows. Replace each triangle by a right angled hexagon which has an isometry group of order three. This means that the hexagon has three sides $a, b, c$ of equal length $\epsilon$ such that these three sides are not neighbors. Construct a surface $M(N\epsilon/2)$ with the same tesselation structure as $M$ in which the triangles are replaced by hexagons and the triangle vertices by boundary geodesics of equal length $N\epsilon$. Let $M(0)$ be the surface with $\epsilon = 0$, this means that the boundary geodesics become cusps.

Let $a$ be a hexagon (triangle) side in $M(0)$ between two cusps $v$ and $w$. Then there exists a unique subsurface $S(a)$ of $M(0)$ of signature $(0,3)$ which contains $a$, $v$ and $w$. Denote by $z$ the third boundary geodesic of $S(a)$. Then the length of $z$ depends only on $N$, but not on the specific tesselation and we have (compare [14], Proposition 2)

$$\cosh z/4 = \frac{N}{2}.$$

The geodesics of type $z$ are the shortest dividing geodesics of $M(0)$ and they are also systoles in most cases. This depends on the following argument.

There exists an infinite number of (2,3,N) triangle surfaces, see [11]. Accept that, for a fixed $N$, the length of the systoles of the (2,3,N) triangle surfaces tends to infinity if the genus tends to infinity. Accept further that the length of each simple closed geodesic increases if we pass from $M$ to $M(0)$. It follows that the length of the shortest non–dividing closed geodesic in $M(0)$ tends to infinity and therefore, in most cases, the systole must be a dividing geodesic and its length depends only on $N$. It then follows by [14], Theorem 13, that $M(0)$ is a global maximal surface.

Now, there are many surfaces $M$ which admit a triangle tesselation in the above defined sense, which are however not (2,3,N) triangle surfaces. This means that the automorphism group of $M$ does not act on the triangles. M. Conder [5] has even shown that such surfaces can have trivial automorphism group. Therefore, by the above argument, it is very plausible that, in this way, one can construct global maximal surfaces with trivial automorphism group.

*3.2. Maximal surfaces and non-congruent subgroups of PSL(2,ℤ)*

The principal conguent subgroups of PSL(2,ℤ) are the most important maximal surfaces. This has led to the question [1] if the property of being a maximal surface has something to do with the distinction of congruent and non-congruent subgroups of PSL(2,ℤ). It however seems that this is not the case, at least not in a simple manner.

Let $P(n)$ be an Euclidean regular pyramid with the basis being a regular n-gon and in which the triangles all have three sides of equal length. To $P(n)$ corresponds a surface, also denoted by $P(n)$, where the vertices are replaced by cusps. Let $D(n)$ be the corresponding double pyramid which is the double of $P(n)$ with respect to the basis. Then, at least if $n$ is odd, $D(n)$ corresponds to a non-congruent subgroup of PSL(2,ℤ). This is a classical result, compare [17], pg 162.

**Theorem 8.** *$D(3)$ is a maximal surface.*

*Proof.* Denote by $T_1$ and $T_2$ the two vertices of $D(3)$ which correspond to the tops of the pyramids and by $S_i, i = 1, 2, 3$, the three vertices of the basis of the pyramid. To an edge $e$ of $D(3)$ there corresponds a unique closed geodesic $z(e)$ as was explained above in Subsection 3.1. $D(3)$ has three edges $s_i, i = 1, 2, 3$, between the vertices $S_i$ and six edges $t_i, i = 1, \ldots, 6$, such that one of the corresponding vertices is $T_1$ or $T_2$. The systoles of $D(3)$ are either the $z(s_i)$ or the $z(t_i)$. A simple calculation implies that the $z(t_i)$ are shorter and hence the systoles of $D(3)$. Denote this set of systoles by $F$.

It is easy to see that $D(3)$ is the unique surface of the fixed signature (0,5) which has the enlarged isometry group of the double pyramid (the enlarged isometry group also contains the orientation inversing isometries). It thus follows by [13], Corollary 3.3, that $D(3)$ is strongly $F$-minimal.

Let us make the notation of the systoles more precise. Let $t_i, i = 1, 2, 3$, be the systoles which are induced by the edge connecting $T_1$ and $S_i, i = 1, 2, 3$. Let $t_{i+3}$ be the systole which intersects $t_i, i = 1, 2, 3$. Then, for example, $t_1$ intersects $t_2, t_3$ and $t_4$.

Let $\xi_i$ be the vector corresponding to a twist deformation along $t_i$ (for the definition of vectors and the following notation compare Section 2 of [13]). Let $\xi_i(t_j)$ be the coefficient of $\xi_i$ corresponding to $t_j$. We have

$$\xi_i(t_j) = \sum_k \cos \alpha_k$$

where the angles $\alpha_k$ are the directed angles from $t_i$ to $t_j$ in the intersection points. By the symmetry of $D(3)$ we have $\xi_i(t_{i+3}) = 0, i = 1, 2, 3$. Moreover, $\xi_1(t_2) = -\xi_1(t_3) \neq 0$. It follows that the four vectors $\xi_i, i = 1, 2, 4, 5$, are independent in the vector space $TR_F(D(3))$. This implies that $D(3)$ is $F$-regular.

The theorem now follows by Theorem 2.7 of [13]. □

**Conjecture.** $D(3)$ is the global maximal surface of signature (0,5) with cusps.

*3.3. Maximal surfaces constructed as twisted doubles*

Many maximal surfaces can be constructed as twisted doubles, see [13], Section 7, or Section 5 of [14]. Here some new maximal surfaces of small genus are presented.

In [13] it was shown that among the twisted doubles of the principal congruence subgroups $\Gamma(N), N = 2,3,4,5$, there exists a maximal surface. For $N = 2,3,4,5$, the surface corresponding to $\Gamma(N)$ is the unique $(2,3,\infty(N))$ triangle surface. But for $N \geq 6$ there exist infinitely many different $(2,3,\infty(N))$ triangle surfaces. For $N = 6$ they have been classified by Coxeter/Moser [6]. Here we construct twisted doubles of $(2,3,\infty(6))$ triangle surfaces. I repeat the construction. Let $M$ be a (2,3,6) Euclidean torus. Replace the vertices by cusps, the result is a $(2,3,\infty(6))$ triangle surface $M(0)$. Let $M(y)$ be defined as above in Subsection 3.1, namely the cusps are replaced by boundary geodesics of length $2y$. Let $N(y,0)$ be the double of $M(y)$. Execute a twist deformation of the same amount $\theta$ along all boundary geodesics of $M(y)$. Denote the result by $N(y,\theta)$. This is the twisted double of $M$ or of $M(0)$. If $M$ is a $(2,3,N)$ triangle surface, then $N(y,\theta)$ is a $(2,2,2,3)$ quadrilateral surface.

**Definition.** In the sequel the notation of [13], Section 7, will be used, in particular:
(i) The set of the boundary geodesics of $M$ is denoted by $Y$. There length is $2y$.
(ii) To a triangle side in $M$ corresponds a closed geodesic $a$ in $N(y,\theta)$ such that the order of the isometry group which leaves $a$ invariant, is four. The set of these geodesics is denoted by $X$ and its length by $2x$. Let $b \in Y$ be intersected by $a$. Then there exists a unique shortest simple closed geodesic $c$ such that $a$,$b$ and $c$ are contained in a subsurface of signature (1,1) and such that $c$ intersects once $a$ and once $b$. The set of the geodesics of type $c$ is denoted by $X'$ and their length by $2x'$.
(iii) The length of a hexagon side in $N(y,\theta)$ which corresponds to a triangle side in $M$ is denoted by $t$. The length of the second shortest common orthogonal between two geodesics of $Y$ is denoted by $s$.
(iv) Let $a$ be a simple closed geodesic in a twisted double. Then $N(a)$ is the number of intersection points of $a$ with the geodesics of $Y$. If $a \in Y$ then $N(a) = 0$ by convention.

**Definition.** $M_k$ denotes the $(2,3,\infty(6))$ triangle surface with $6k$ $(2,3,\infty(6))$ triangles.

**Lemma 1.** *For $k = 6$, $k = 8$, $k = 24$ there exists a surface $M_k$. Its twisted double has genus 4,5 and 13, respectively.*

*Proof.* The existence follows by the classification of Coxeter/Moser [6]. In their notation these surfaces correspond to $(b,c) = (1,1),(0,2),(2,2)$, respectively. Since the area of one triangle (hexagon) is $\pi$, the genus of the twisted double is evident. □

**Lemma 2.** *$M_6$ is a maximal surface with 9 systoles (compare Fig. 3).*

*Proof.* $M_6$ has signature (1,3). It corresponds to the surface denoted by $M_2$ in Section 4 of [13]. Theorem 4.5 of [13] then says that $M_6$ is a maximal surface if the systoles are non–dividing (which is the case as can easily been verified). Moreover, in this case $M_6$ has 9 systoles. □

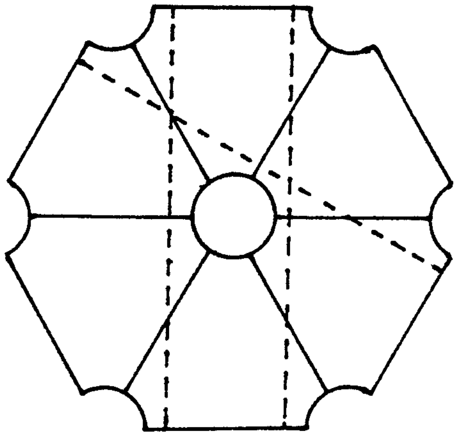

**Fig. 3.** $M_6$ with three of its nine systoles (dotted lines). Opposite sides of the figure must be identified. The geodesics of the set $Y$ are drawn as circles. The common orthogonals between two circles have length $t$

**Definition.** The set of the systoles of $M_6$ is denoted by $J$. In a twisted double of $M_6$ the set of the 18 geodesics corresponding to the geodesics of $J$ in the two copies of $M_6$ is also denoted by $J$. The length of these geodesics is denoted by $2j$.

**Lemma 3.** *Let $T$ be the Teichmüller space of $M_6$ which contains all surfaces of signature (1,3), also those which have boundary geodesics of different length as $M_6$. Then $M_6$ is $J$-regular in $T$.*

*Proof.* In the notation of Section 4 of [13] $M_6$ contains a geodesic $e$. By Lemma 4.6 of [13], the set $J \cup \{e\}$ is a parametrizing set of $T$. So, if $M_6$ is $J$-singular then there exists a, up to a scaling factor, unique vector $\xi$ such that the $J$-coefficients of $\xi$ are 0 and the coefficient corresponding to $e$ is not zero. Therefore, $\xi$ must respect the automorphism group of $M_6$. But there is only one non-zero vector which respects this group, namely the vector which corresponds to the family of maximal surfaces $M_2$ (in the notation of Section 4 of [13]) in $T$ and this vector is non-zero in its $J$-coefficients. □

**Definition.** The set of geodesics of type $b$ of a twisted double of $M_6$ (see Fig. 4) is denoted by $U$, their length is denoted by $2u$.

**Theorem 9.** *There exists a, up to isometry, unique twisted double $S_4$ of $M_6$ with $x = u = i$. $S_4$ is a maximal surface. Its set of systoles $F$ contains exactly the 36 geodesics of $X \cup U \cup J$.*

*Proof.* (i) Let the twist in the twisted doubles of $M_6$ be $y/3$. It then follows that $x = u$. The twisted doubles with this twist are contained in a one parameter family which is parametrized by $y$. Let $a \in X$ and $b \in U$ mutually intersect. Then there exists a $c \in J$ such that $a, b, c$ are contained in a subsurface of signature (1,1). The three geodesics induce a hyperbolic triangle, denote by $\gamma$

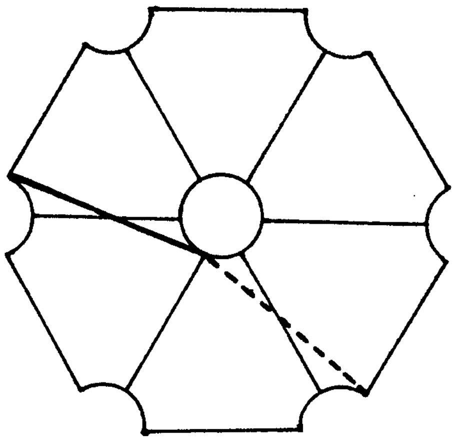

**Fig. 4.** $M_6$ with one geodesic of type $b$ (thick line). One half of $b$ lies in one copy of $M_6$ (plane thick line), the other half lies in the second copy of $M_6$ (dotted thick line)

the angle between $a$ and $b$ in this triangle. Now if $y$ increases, then $t$ decreases and $\gamma$ increases. If $y$ — 0 then $x > j$. If $y$ — $\infty$ then $\gamma$ — $\pi$ and therefore $x < j$. It follows that there exists a surface $S_4$. It is unique since if $x = u = j$ increase then $y$ and $t$ increase which is impossible.

(ii) We now prove that $F = X \cup U \cup J$ is the set of systoles of $S_4$. By calculation we have
$y = 2.373, \quad x = 2.264, \quad x' = 3.37, \quad t = 2.072.$
Let $a$ be a systole of $S_4$. Then $N(a) \leq 2$ by the preceeding list. Moreover, if $N(a) = 2$ then the two segments of $a$ are homotopic to common orthognals of length $t$ since $s > 3$. Therefore, if $N(a) = 2$, then $a \in X \cup U$. If $N(a) = 0$ then $a \in J$ by Lemma 2 and the preceeding list.

(iii) By Corollary 3.3 of [13] we have only to prove that $S_4$ is strongly $F$-minimal with respect to the set of twisted doubles of $M_6$. This can be done by calculation in an analogous manner as we have done it in Section 7 of [13] for several examples.

(iv) We finally show that $S_4$ is $F$-regular. By Lemma 3, a vector $\xi$ which is 0 in its $F$-coefficients is induced by twist deformations along the geodesics of $Y$. Let $a \in X$ intersect the geodesics $b$ and $c$ of $Y$. It follows that $\xi(b) = -\xi(c)$. Let $d$ be the third element of $Y$. The same argument then shows that $\xi(d) = -\xi(b)$ and $\xi(d) = -\xi(c)$, hence $\xi(d) = 0$ and $\xi$ is the zero vector. Therefore, $S_4$ is $F$-regular.

The theorem now follows by Theorem 2.7 of [13]. □

**Remark.** In [13], Section 8, another maximal surface of genus 4 with 36 systoles has been described, it was denoted by M(4). The systoles of M(4) are longer than the systoles of $S_4$. I conjecture that, for genus 4, M(4) is the global maximal surface and M(4) and $S_4$ are best kissing number surfaces.

**Lemma 4.** *$M_8$ is a surface of signature (1,4) with six systoles.*

*Proof.* The systoles of $M_8$ are the geodesics of type $j$ of Fig. 5. □

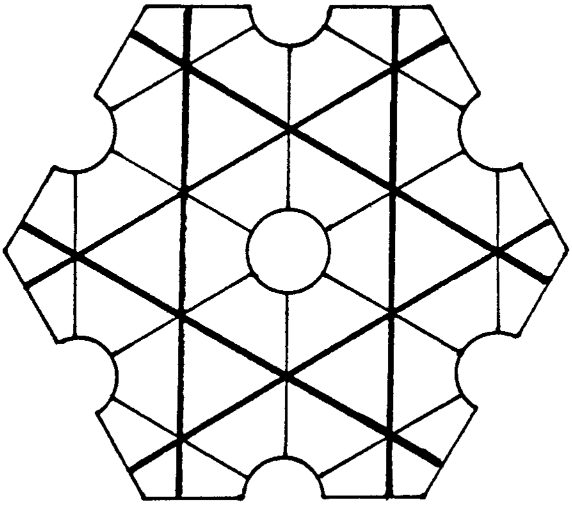

**Fig. 5.** $M_8$ with its six systoles (thick lines). Opposite sides of the figure must be identified

**Definition.** The set of the systoles of $M_8$ is denoted by $J$. In a twisted double of $M_8$ the set of the 12 geodesics corresponding to the geodesics of $J$ in the two copies of $M_8$ is also denoted by $J$. The length of these geodesics is denoted by $2j$.

**Definition.** The set of geodesics of type $b$ of a twisted double of $M_8$ (see Fig. 6) is denoted by $U$, its length is denoted by $u$.

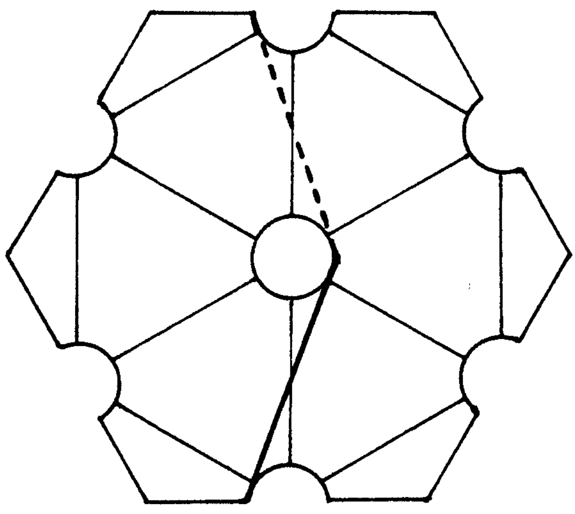

**Fig. 6.** $M_8$ with one geodesic of type $b$ (thick line). One half of $b$ lies in one copy of $M_8$ (plane thick line), the other half lies in the second copy of $M_8$ (dotted thick line)

**Theorem 10.** *There exists a, up to isometry, unique twisted double $S_5$ of $M_8$ with $x = u = i = y$. $S_5$ is a maximal surface. Its set of systoles $F$ contains exactly the 40 geodesics of $X \cup U \cup J \cup Y$.*

*Proof.* (i) In the twisted doubles of $M_8$ we let the twist be $y/2$. We then have a one parameter family of twisted doubles which is parametrized by $y$ and in this family we have $x = u$. As in the proof of Theorem 9, $x > y$ if $y \to 0$ and $x < y$ if $y \to \infty$. Therefore, there exists a surface $S_5$ with $x = u = y$. It is unique by the same argument as in the proof of Theorem 9.

(ii) We have to show that $x = j$ in $F_5$. Let $j_1$ and $j_2$ be two systoles of $M_8$ which do not intersect. Let $j_3$ and $j_4$ be the same systoles in the second copy of $M_8$. Then the four geodesics $j_i, i = 1, \ldots, 4$, separate $F_5$ into two isometric subsurfaces of signature (1,4). One of them, call it $M'$, contains two geodesics of $Y$, two geodesics of $X$ and two geodesics of $U$ and these six geodesics are the systoles of $M'$. In fact $M' = M_8(j)$. In the family $M_8(v)$, $2v$ being the length of the boundary geodesics, $v$ increases faster than the length of the six systoles. It follows that $M'$ is isometric to $M_8$ which implies $j = x$.

(iii) We now show that $F = X \cup U \cup Y \cup J$ is the set of systoles of $S_5$. By calculation, we have the following values

$x = 2.457, \quad t = 2.016, \quad s = 3.77, \quad x' = 3.238.$

For a systole $a$ in $F_5$ we therefore have $N(a) \leq 2$ and it follows by this list and Lemma 4 that $F$ is the set of systoles of $F_6$.

(iv) I again remark that it can be shown be calculation that $F_5$ is strongly $F$-minimal.

(v) We have to show that $S_5$ is $F$-regular. Assume that there exists a non-zero vector $\xi$ such that all $F$-components are zero. $\xi$ is induced by a combination of twist deformations along the geodesics of $X$. For each $a \in X$ there exists a geodesic $a'$ which intersects $a$ twice, but intersects no other element of $X$; moreover a twist deformation along $a$ in the good direction decreases the length of $a'$. Therefore, $\xi$ is determined by the coefficients corresponding to the geodesics of type $a'$. Since $\xi \neq 0$ we can assume that $\xi(a') := \alpha \neq 0$ (I recall that $\xi(a')$ is the coefficient of $\xi$ with respect to $a'$). $F_5$ has two automorphisms which leave $a \cup a'$ invariant. So we can assume that $\xi$ is invariant with respect to these automorphisms. It follows that the coefficients of $\xi$ can be denoted as in Fig. 7. Now, there is a systole $h$ of $J$ as in Fig. 7 and since $\xi(h) = 0$ it follows that $\beta = -\gamma$. Since $\xi(b) = 0$ for $b \in Y$, it follows that $\alpha = -\delta$ and $\eta = 0$. Let $c \in U$ be the geodesic of Fig. 7. Since $\xi(c) = 0$ we must have $\delta = \alpha = 0$, a contradiction. Hence $S_5$ is $F$-regular.

The theorem now follows by Theorem 2.7 of [13]. □

**Theorem 11.** *The (2,3,8) triangle surface $O(x|z)$ of genus 5 is a twisted double of $M_8$. $O(x|z)$ is a maximal surface.*

*Proof.* We have seen in [13], Section 7, that $O(x|z)$ is a maximal surface. We have thus to show that $O(x|z)$ is a twisted double of $M_8$. We continue with the one parameter family of the proof of Theorem 10, with $x = u$. Since the twist is $y/2$, all these surfaces have geodesics of the length $t + s$. Their number is 24. It is easy to see that there exists a unique twisted double in our family with $2x = t + s$. By calculation we can see that this surface is $O(x|z)$. □

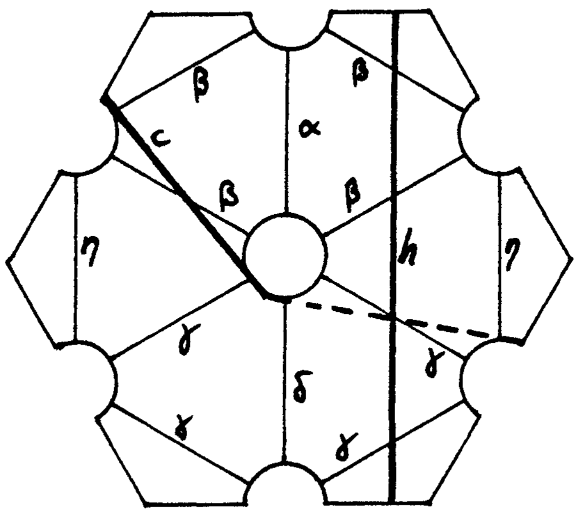


**Fig. 7.** $M_8$ with the coefficients of $\xi$. The geodesics $h$ of $J$ and $c \in U$ are drawn by the thick lines

**Remark.** For genus 5, there are thus at least the three maximal surfaces $S_5$, $O(x|z)$ and $O(x|y)$ (for the third see [13]). $S_5$ has the longest systole of these three surfaces. On the other hand, $O(x|z)$ has the bigest number of systoles. So, genus 5 may be an example that the global maximal surface and the best kissing number surface do not need to be identical.

**Remark.** For a fixed signature, one can construct a graph called *maximal graph* such that the vertices are the maximal surfaces. The edges correspond to one parameter families in the sense of that one considered in the proofs of Theorem 10 and 11. Therefore, the maximal surfaces $S_5$ and $O(x|z)$ are connected by an edge. The third maximal surface $O(x,y)$ is connected to $O(x|z)$ by an edge, compare Section 7 of [13]. I conjecture that the maximal graphs are connected.

**Lemma 5.** *Let $a$ be a systole of a surface $M_{24}(y)$. Then $a$ is longer than $2y$.*

*Proof.* By the symmetry of the surface, $a$ must pass through at least six different right angled hexagons (corresponding to triangles in $M_{24}(0)$) and is therefore longer then $6y/3$. □

**Definition.** The set of geodesics of type $b$ of a twisted double of $M_{24}$ (see Fig. 8) is denoted by $U$. Their length is denoted by $u$.

**Theorem 12.** *There exists a, up to isometry, unique twisted double $S_{13}$ of $M_{24}$ with $x = x' = u$. $S_{13}$ is a maximal surface. $X \cup X' \cup U$ is its set of 126 systoles.*

*Proof.* (i) The condition $x = x'$ defines a one parameter family of twisted doubles of $M_{24}$ which is parametrized by $y$, compare Lemma 7.2 in [13]. In this family, if $y \to 0$ then $u > x$ and if $y \to \infty$ then the twist tends to $y/3$, hence $u - 2t \to 0$. This proves the existence of $F_{13}$.

(ii) By calculation, we have the following values

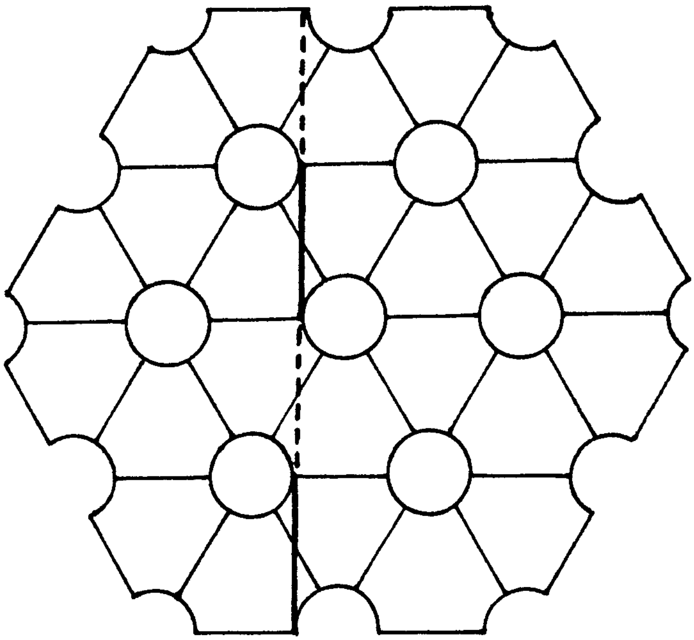

**Fig. 8.** $M_{24}$ with one geodesic of type $b$ (thick line). Two quarts of $b$ lie in one copy of $M_{24}$ (plane thick lines), the other two quarts lie in the second copy of $M_{24}$ (dotted thick lines). Opposite sides of the figure must be identified

$y = 3.329, \quad x = 3.2145, \quad t = 1.5526, \quad s = 3.560.$

It follows by this list that $N(a) \leq 4$ for a systole $a$ of $S_{13}$. $N(a) = 0$ is excluded by the list and by Lemma 5. If $N(a) = 2$ and both segments of $a$ are homotopic to a common orthogonal of length $t$ then $a \in X \cup X'$. It is easy to see that $a$ is not a systole if one segment of $a$ is homotopic to a common orthogonal longer than $t$. Since by the list $N(a) \leq 4$ the case $N(a) = 4$ remains. By the list, the four segments of $a$ must be homotopic to a common orthogonal of length $t$. If $a$ does not intersect the boundary of Fig. 8 then we have seen in [13], Lemmata 7.8–7.12, that there are two possibilities for $a$. It is easy to see by calculation that in these two cases $a$ is longer than $2x$. Hence $a$ must intersect the border of Fig. 8 and it follows that $a \in U$. We thus have shown that $F = X \cup X' \cup U$ is the set of systoles of $S_{13}$.

(iii) By calculation one can see that $S_{13}$ is strongly $F$–minimal.

(iv) The argument of the proof of Proposition 7.2 in [13] implies that $S_{13}$ is already $X \cup X'$–regular hence also $F$–regular.

The theorem now follows by Theorem 2.7 of [13]. □

**Remark.** (i) I conjecture that for all $(2, 3, \infty(6))$ triangle surfaces there exists at least one twisted double which is a maximal surface. The length of the systoles however becomes stable. To see this consider a "fundamental domain" which is symmetric in the sense of Fig. 8. If this "fundamental domain" of a $(2, 3, \infty(6))$ triangle surfaces becomes big enough, then the systoles which are nearest to the center of the domain cannot intersect the boundary of the domain. So, the search for the systoles becomes in some sense a local question, this means that all different isometry classes of systoles have a representative in a small part of the "fundamental domain" and all "fundamental domains" which are big enough contain this small part. It follows that the systoles are determined by the small part and its length becomes constant. By calculation, I conjecture that this length is 7.75 for the $(2, 3, \infty(6))$ triangle surfaces.

Interesting maximal surfaces are such with long systoles (long with respect to the area of the surface). Therefore, only $(2, 3, \infty(6))$ triangle surfaces of small genus can give interesting maximal surfaces.

(ii) I conjecture that the same phenomena appear for $(2, 3, \infty(N))$ triangle surfaces, $N > 6$. This means that they have a twisted double which is a maximal surface and that the length of the systoles of these maximal surfaces becomes constant if the genus is big enough (with respect to $N$). So, also in this general case, only $(2, 3, \infty(N))$ triangle surfaces of small (with respect to $N$) genus can give interesting maximal surfaces.

## References

1. M. Baker. Private communication
2. K. Ball. *A lower bound for optimal density of lattice packings*. Duke Math. J. 68 (1992), Research Notices 217–221
3. M. Berger. *Systoles et applications selon Gromov*. Séminaire Bourbaki 45 1992–93, no 771
4. E. Calabi. *Extremal isosystolic metrics for compact surfaces*. Preprint 1993
5. M. Conder. Asymmetric combinatorially–regular maps. Preprint 1994
6. J. Conway; N. Sloane. *Sphere packings, lattices and groups*. Springer Berlin Heidelberg New York Tokyo (1988)

7. R. Fricke; F. Klein. *Vorlesungen über die Theorie der elliptischen Modulfunktionen.* Band 1, Teubner Leipzig (1890)
8. M. Gromov. *Filling Riemannian manifolds.* J. of Diff. Geometry **18** (1983), 1–147
9. M. Gromov. *Systoles and intersystolic inequalities.* Preprint 1992
10. P.M. Gruber; C.G. Lekkerkerker. *Geometry of numbers.* North–Holland Amsterdam New York Oxford Tokyo (1987)
11. R.S. Kulkarni. *Infinite families of surface symmetries.* Israel J. of Math. **76** (1991), 337–343
12. J. Oesterlé. *Empilement de sphères.* Astérisque **189–190** (1990), 375–398
13. P. Schmutz. *Riemann surfaces with shortest geodesic of maximal length.* Geometric and Functional Analysis GAFA **3** (1993), 564–631
14. P. Schmutz. *Congruence subgroups and maximal Riemann surfaces.* The Journal for Geometric Analysis **4** (1994), 207–218
15. P. Schmutz. *Arithmetic Fuchsian groups and the number of systoles.* Preprint (1993)
16. P. Schmutz. *Compact Riemann surfaces with many systoles.* Preprint (1993)
17. A. Venkov. *Spectral theory of automorphic forms and its applications.* Kluwer Academic Publishers Dordrecht Boston London (1990)
18. A. Voronoï. *Sur quelques propriétés des formes quadratiques positives parfaites.* J. reine angew. Math. **133** (1908), 97–178

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