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Eigenfunctions and Nodal Sets

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§0. Introduction

The purpose of this paper is to study the nodal sets, i.e. zero sets of the eigenfunctions of the Laplacian operator on a Riemannian manifold. We first study it locally. The result of Lipman Bers [2] concerning the local behaviour of solutions of elliptic equations is our main tool. It tells us that the nodal set locally looks like the nodal set of a spherical harmonic. Hence, we can prove in §2 that, except on a closed set of lower dimension, the nodal set is a C^∞ submanifold. This regularity result enables us to prove in §1 the well-known Courant's nodal domain theorem for high dimensions.

Courant's nodal domain theorem is the only known global theorem about nodal sets. We use it in §3 to prove that there is a global restriction to multiplicities of eigenvalues. Specifically, we prove the following theorem: Suppose that M is a Riemann surface of genus g, the multiplicity of the i-th eigenvalue is less than or equal to $(2g+i+1)(2g+i+2)/2$.

The results in §3 show that when M is homeomorphic to S^2 the multiplicity of the 1-st eigenvalue is at most 3. This phenomenon of relatively low multiplicity makes it feasible to study the geometry of the nodal lines of some special surfaces. We show that: If M is homeomorphic to S^2 and is isometric to a surface of revolution then we can find a basis for the space of 1-st eigenfunctions such that the nodal lines of each eigenfunction in the basis is a line of constant geodesic curvature.

Part of the results in this paper has been announced in [4].

§1. Courant's Nodal Domain Theorem

Suppose that (M, g) is an n-dimensional C^∞ Riemannian manifold. The Laplacian operator, denoted by Δ, acting on functions is locally given by

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij}\frac{\partial}{\partial x_j}\right),$$

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where as usual g_{ij} is the fundamental tensor, g^{ij} is its inverse, and $g = \det(g_{ij})$. We shall consider two kinds of eigenvalue problems.

FIXED MEMBRANE PROBLEM. Suppose that D is a compact domain of M. We shall study the following:

$$\Delta\phi + \lambda\phi = 0, \qquad \phi = 0 \text{ on } \partial D.$$

It is well-known that when ∂D is reasonably regular, e.g. piecewise C^1, the fixed membrane problem has discrete eigenvalues and we list them as $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$. Therefore, $\lambda_i(D)$ shall mean the i-th eigenvalue of the domain D w.r.t. the fixed membrane problem. We shall also show the well-known fact that $\lambda_1 < \lambda_2$, i.e. λ_1 has simple multiplicity. Also the term i-th eigenfunction is a function satisfying the fixed membrane problem with $\lambda = \lambda_i(D)$.

FREE MEMBRANE PROBLEM. Suppose that M is a compact Riemannian manifold without boundary. We shall study the following eigenvalue problem on M: $\Delta\psi + \mu\psi = 0$.

This problem also has discrete eigenvalues and clearly constant functions are eigenfunctions with $\mu = 0$. We list the eigenfunctions of the free membrane problem as $0 = \mu_0 < \mu_1 \leqslant \mu_2 \leqslant \mu_3 \cdots$. Therefore, $\mu_i(M)$ shall mean the i-th eigenvalue of the compact manifold M w.r.t. the free membrane problem. The term i-th eigenfunction will be used to mean a function on M satisfying the above differential equation with $\mu = \mu_i(M)$.

DEFINITION. Suppose that f is a solution of an elliptic equation on a manifold M. $f^{-1}(0)$ is called the nodal set of f, when $\dim M = 2$ it is also called the nodal lines. Every connected component of $M \setminus f^{-1}(0)$ is called a nodal domain of f.

One should notice immediately that if f is an eigenfunction of the Laplacian operator then f is the 1-st eigenfunction of each of its nodal domains. This observation suggests that we can reduce the problems about the i-th eigenvalues to problems about the 1-st eigenvalue of the fixed membrane problem.

Courant's nodal domain theorem. For the fixed membrane problem:

# of nodal domains of the i-th eigenvalue $\leqslant i$.

For the free membrane problem:

# of nodal domains of the i-th *eigenvalue* $\leqslant i + 1$.

In [3], this theorem is stated and proved in the two dimensional case. Using results in §2 about the regularity of nodal sets, we can follow the same method to prove this theorem.

The proof goes as follows: Suppose that ϕ_i is the i-th eigenfunction of the domain D, and $D_1, \ldots, D_{i+1}, \ldots$ are all the nodal domains of ϕ_i. Define functions ϕ_i^j, $1 \leqslant j \leqslant i$, on D as

$$\phi_i^j = \phi_i \text{ on } D_j \text{ and } \phi_i^j = 0 \text{ outside } D_j.$$

We can find real numbers $a_1, \ldots, a_i$ not all zero such that $\phi = \sum_{j=1}^{i} a_j \phi_i^j \perp$ the space generated by $\phi_1, \ldots, \phi_{i-1}$. Then, we have

$$\lambda_i(D) \leqslant \frac{\int_D (d\phi, d\phi)}{\int_D \phi^2} = \frac{\sum_{j=1}^{i} a_j^2 \int_{D_j} (d\phi_i, d\phi_i)}{\sum_{j=1}^{i} a_j^2 \int_{D_j} \phi_i^2}.$$

However, ϕ_i is the i-th eigenfunction and it satisfies $\Delta\phi_i + \lambda_i(D)\phi_i = 0$. The results in §2 shows that except on a closed set of lower dimension the nodal sets of ϕ_i form a C^∞ manifold. Thus, we have

$$\int_{D_j} (d\phi_i, d\phi_i) = \int_{D_j} -\Delta\phi_i \phi_i = \lambda_i(D) \int_{D_j} \phi_i^2 .$$

Consequently,

$$\lambda_i(D) = \frac{\int_D (d\phi, d\phi)}{\int_D \phi^2} .$$

Then ϕ is C^∞ and satisfies $\Delta\phi + \lambda_i(D)\phi = 0$. However, the fact that $\phi \equiv 0$ on an open set of D implies $\phi \equiv 0$, a contradiction. This completes the proof of the theorem for fixed membrane problems. The proof for free membrane problem is the same.

Notice that we have $\phi_2 \perp \phi_1$ and $\psi_1 \perp \psi_0$, where ψ_0 is a constant function. Hence ϕ_2 and ψ_1 must change sign. This proves the following well-known proposition:

PROPOSITION 1.1. *For the case of fixed membrane problems:*
# of nodal domains of $\phi_2 = 2$.
For the case of free membrane problems:
# of nodal domains of $\psi_1 = 2$.

§2. Local Behaviour of Nodal Sets

The nodal sets are a very "unstable" object. Slight changes of the metric or the domain would result in a violent change of the nodal sets, (see [6]). Therefore, the global behavior of the nodal sets is a quite difficult subject. We shall use a theorem of Lipman Bers [2] concerning the local behaviour of solutions of elliptic equations to study the nodal sets.

THEOREM 2.1 (Lipman Bers [2]). *Suppose that*

$$L\phi(x)=\sum_{v=0}^{m}\sum_{i+\cdots+i_n=v} a_{i_1\ldots i_n}(x)\frac{\partial^v\phi(x)}{\partial x_1^{i_1}\ldots\partial x_n^{i_n}}=0$$

is an elliptic equation with C^∞ coefficient defined in a neighborhood of the origin.

If a solution $\phi(x)$, $L\phi\equiv 0$, vanishes at the origin, but not of infinite order, then there exists a homogeneous polynomial of degree N, $p_N(x)\not\equiv 0$ such that

$$\frac{\partial^l\phi(x)}{\partial x_1^{i_1}\ldots\partial x_n^{i_n}}=\frac{\partial^l p_N(x)}{\partial x_1^{i_1}\ldots\partial x_n^{i_n}}+O(|x|^{N-l+\varepsilon})$$

for $l=0,\ldots,m$, $l=i_1+\cdots+i_n$, where ε is any number in the open interval (0, 1). *Also, $p_N(x)$ satisfies the "osculating equation" with constant coefficients*

$$L_0 p_N(x)=\sum_{i_1+\cdots+i_n=m} a_{i\ldots i_n}(0)\frac{\partial^m p_N(x)}{\partial x_1^{i_1}\ldots\partial x_n^{i_n}}=0.$$

When we are dealing with Laplacian operator on a manifold, we shall pull back the equation to the tangent space and apply Theorem 2.1.

THEOREM 2.2. *Suppose that M is an n-dim C^∞ Riemannian manifold without boundary (not necessarily compact). If $f\in C^\infty(M)$ satisfies $(\Delta+h(x))f=0$, $h\in C^\infty(M)$, then except on a closed set of lower dimension (i.e.* dim $<n-1$*) the nodal set of f forms an $(n-1)$-dim C^∞ manifold.*

Proof. Let $x_0\in M$, and $f(x_0)=0$. It is clear that we can assume M is within a very small neighborhood of x_0. We use normal coordinates around x_0 and hence we can assume we are working in a small open set of the origin in $\mathbf{R}^n$. The equation $(\Delta+h(x))f=0$ pulls back to a second order elliptic equation in a small neighborhood of $0\in\mathbf{R}^n$. By the results of N. Aronsajn [1], f can vanish only up to finite order around the origin. Hence we can apply Theorem 2.1. It tells us that

$$f(x)=p_N(x)+0(|x|^{N+\varepsilon})$$

where p_N is a homogeneous polynomial of degree N and $\varepsilon\in(0, 1)$.

Also, p_N satisfies the osculating equation at the origin. Since we are using normal coordinates, the osculating equation is the usual Laplace equation in Euclidean space, i.e.,

$$\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right) p_N=0 .$$

Thus, p_N is a spherical harmonic of degree N.

If $N=1$, $p_N(x)$ is a linear polynomial and this shows that $df(0)\neq 0$, then the nodal set around 0 is a very nice piece of C^∞ manifold.

When $N>1$, the situation is more complicated. We shall extend the method of T. C. Kuo [5] in Lemma 2.4 to prove that $f(x)=p_N(\Phi(x))$ where Φ is a C^1 diffeomorphism between two small neighborhoods of $0\in\mathbf{R}^n$ and $\Phi(0)=0$. Thus, the nodal set of f around the origin is C^1 diffeomorphic to the nodal set of a spherical harmonic around the origin. However, there is not much information about nodal sets of sphereical harmonics. The following simple observation will be useful.

LEMMA 2.3. *Suppose that p_N is a spherical harmonic of degree N, $N>1$. Then, the nodal set of p_N around the origin has a singularity at* 0.

Proof. Notice that if S^{n-1} is the sphere of radius 1 in $\mathbf{R}^n$ then $p_N|_{S^{n-1}}$ is an eigenfunction of S^{n-1}. Since $N>1$, $p_N|_{S^{n-1}}$ is not the 1-set eigenfunction. $p_N|_{S^{n-1}}$ must have zeros on S^{n-1} and the homogeneity of p_N shows that if $x\in S^{n-1}$ with $p_N(x)=0$ then $p_N(tx)=0$ for all $t>0$. The only case where the nodal sets of p_N around the origin is a smooth manifold is when the nodal set of $p_N|_{S^{n-1}}$ lies on a great circle of S^{n-1}. Since great circles are nodal sets of 1-st eigenfunctions on S^{n-1} and $N>1$, the assertion of the lemma is immediately seen to be true.

We now prove the theorem by induction on the dimension n.

If $n=1$, it is trivial.

Suppose that it is true for $n-1$.

We now prove it for n:

We shall show that the nodal set of f around the origin is C^1 diffeomorphic to the nodal set of a spherical harmonic p_N of degree N around the origin in $\mathbf{R}^n$. However, the nodal set of p_N around the origin is equal to $\{tx: t>0,\ p_N|_{S^{n-1}}(x)=0\}$. Remember that $p_N|_{S^{n-1}}$ is an eigenfunction on the $(n-1)$-dim sphere S^{n-1}. Our inductive assumption then applies and shows that Theorem 2.2 is true for the nodal set of p_N. Now recall that we have the relation $f(x)=p_N(\Phi(x))$, where Φ is a C^1 diffeomorphism keeping the origin fixed. Suppose that $p_N^{-1}(0)\setminus\pi=M_0$ around the origin, where π is a closed set of lower dimension and M_0 is an $(n-1)$-dim C^∞ manifold. Then $f^{-1}(0)\setminus\Phi^{-1}(\pi)=\Phi^{-1}(M_0)$. Thus $\Phi^{-1}(M_0)$ is a C^1 manifold. We now want to show further that $\Phi^{-1}(M_0)$ is C^∞. Indeed, let $y\in\Phi^{-1}(M_0)$. Then $f(y)=0$, and $\Phi(y)\in M_0$. Apply our previous argument to a small neighborhood of y, we have

$f(x) \sim p_{N'}(x)$, near y, where $p_{N'}$ is a spherical harmonic of degree N' in $\mathbf{R}^n$. We claim that $N'=1$. If this is true, then an open set of $\Phi^{-1}(M_0)$ around y is a piece of smooth manifold. $N'>1$ would lead to a contradiction. Note that a small neighborhood of $\Phi^{-1}(M_0)$ around y is C^1 diffeomorphic to the nodal set of $p_{N'}$ around the origin. Lemma 2.3 shows that if $N'>1$, $p_{N'}^{-1}(0)$ has a singularity at 0. The C^1 diffeomorphism transfers this singularity to $\Phi^{-1}(M_0)$ and hence results in a contradiction. This completes the proof of Theorem 2.2.

We now follow the method of T. C. Kuo [5] to prove the following lemma.

LEMMA 2.4. *Suppose that f, p are smooth functions in* $\mathbf{R}^n$

$$f(x)=p(x)+O(|x|^{N+\varepsilon}),$$

$$\frac{\partial f(x)}{\partial x_i}=\frac{\partial p(x)}{\partial x_i}+O(|x|^{N-1+\varepsilon}), \quad N\geqslant 1, \quad \varepsilon\in(0,1)$$

$$\frac{\partial^v}{\partial x_1^{i_1}\ldots\partial x_n^{i_n}}\,p(0)=0 \qquad 0\leqslant v\leqslant N-1$$

and

$$|\operatorname{grad} p|\geqslant \text{const}\,|x|^{N-1}.$$

Then, there exists a local C^1 diffeomorphism Φ fixing the origin such that

$$f(x)=p(\Phi(x)).$$

Proof. We may suppose $N>1$.

Set $F(x,a)=(1-a)f(x)+ap(x)$, $a\in\mathbf{R}$. Notice that

$$\operatorname{grad} F(0,a)=\left(\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n},\frac{\partial F}{\partial a}\right)=0 \quad \text{for all } a.$$

Define

$$X(x,a)=\begin{cases} |\operatorname{grad} F|^{-2}(p(x)-f(x))(\operatorname{grad} F) & \text{when } x\neq p \\ 0 & \text{when } x=0. \end{cases}$$

Outside $(0,a)$, $a\in\mathbf{R}$, $X(x,a)$ is a C^∞ vector field. $X(x,a)$ is C^1 at $(0,a)$. Indeed,

$$|(p(x)-f(x))\operatorname{grad} F|=0(|x|^{N+\varepsilon})\,|\operatorname{grad} F|$$

Note that

$$\begin{aligned}|\operatorname{grad} F|&\geqslant|(1-a)\operatorname{grad}(f)+a\operatorname{grad}(p)|-|(p-f)|\\&\geqslant|(1-a)\operatorname{grad}(f-p)+\operatorname{grad}(p)|-|(p-f)|\\&\geqslant\text{const.}\,|x|^{N-1}.\end{aligned}$$

So $|\text{grad} F|^2 \geqslant |\text{grad} F|$ const. $|x|^{N-1}$. Thus, $X(x, a) = (O|x|^{1+\varepsilon})$. This shows that $X(x, a)$ is a C^1 vector field.

Define $v(x, a) = (0, \dots, 0, 1) - X(x, a)$. $v(x, a)$ is also C^1 and we can assert that local solutions of $v(x, a)$ exist and are unique and depend in a C^1 way on the initial value and time.

Let $\phi(t; x_0, a_0)$ denote the solution with initial condition $\phi(0; x_0, a_0) = (x_0, a_0)$. Observe that, the dot product

$$\begin{aligned}(v(x, a), (0, \dots, 0, 1)) &= 1 - |\text{grad} F|^{-2}(p(x) - f(x))^2 \\ &\geqslant 1 - O(|x|^{2+2\varepsilon}) \\ &> 0 \text{ when } x \text{ is small.}\end{aligned}$$

So the a-component of any solution $\phi(t; x, 0)$ increases monotonically with t.

Hence $\phi(t; x, 0)$ meets the hyperplane $a = 1$, when x is small, at a unique point $\Phi(x)$. The mapping $x \to \Phi(x)$ is a C^1 local diffeomorphism. Moreover, as $\Phi(t; 0, 0) = (0, t)$ we have $\Phi(0) = 0$.

Now the dot product

$$\begin{aligned}(v(x, a), \text{grad} F) &= ((0, 0, \dots, 0, 1) - X(x, a), \text{grad} F) \\ &= (p(x) - f(x)) - (X(x, a), \text{grad} F) \\ &= 0.\end{aligned}$$

This shows that F is constant along the trajectories of $v(x, a)$. Hence

$$f(x) = F(x, 0) = F(\phi(t; x, 0)) \quad \text{for all } t.$$

As $(\Phi(x), 1) = \phi(t'; x, 0)$ for certain t' we have

$$f(x) = F((\Phi(x), 1)) = p(\Phi(x)).$$

This completes the proof of Lemma 2.4.

One readily sees that the topology of the nodal sets will be very complicated. In order to say more, we assume the dimension of M is two. The nodal set is then a set of lines. This becomes more manageable.

THEOREM 2.5. *Suppose that M is a* 2-dim *manifold. Then, for any solution of the equation $(\Delta + h(x))f = 0$, $h \in C^\infty(M)$, the following are true:*

i) *The critical points on the nodal lines are isolated.*

ii) *When the nodal lines meet, they form an equiangular system.*

iii) *The nodal lines consist of a number of C^2-immersed one dimensional closed submanifolds. Therefore, when M is compact, they are a number of C^2-immersed circles. (A C^2-immersed circle means $\Phi(S^1)$, where $\Phi: S^1 \to M$ is a C^2 immersion).*

iv) *When the nodal lines meet, the geodesic curvatures are zero there.*

Proof. i) is obvious from Theorem 2.2.

ii) is clearly true when the nodal lines are free of critical points. Around a critical point on the nodal lines, the nodal lines are C^1 diffeomorphic to the nodal lines of a spherical harmonic in $\mathbf{R}^2$. The nodal lines of a spherical harmonic in R^2 are quite simple. If p_N is a spherical harmonic in R^2, then $p_N|_{S^1}$ is an eigenfunction. The zeroes of $p_N|_{S^1}$ on S^1 are isolated and they divide S^1 into $2N$ arcs with equal length. Remember that if $p_N|_{S^1}(x)=0$ then $p_N(tx)=0$ for all $t>0$. Thus, the nodal lines of p_N consist of $2N$ straight lines passing through the origin. Moreover, the straight lines from an equiangular system at the origin. Observe that straight lines passing through the origin of the tangent plane map to geodesic lines under the exponential map. The deiivative of the exponential map at the origin is the identity map. These observations show that ii) is valid.

To prove iii) and iv), we first recall that at a critical point x_0 on the nodal lines, the spherical harmonic describing the local behaviour of the eigenfunction around x_0 has degree greater than or equal to 2. The error term is $O(|x|^{N+\varepsilon})$, $\varepsilon\in(0, 1)$, $N\geqslant 2$. So the order of contact of the nodal lines around x_0 and an equiangular system of geodesics is equal to 2. This observation proves iv) immediately. iii) also follows immediately because the nodal lines of a spherical harmonic in $\mathbf{R}^2$ are a set of straight lines through the origin.

§3. Global Restrictions to Eigenfunctions

We have studied the local behaviour of the nodal lines in §2. The nodal lines are also subject to a global restriction, namely, Courant's nodal domain theorem. If we have many closed curves on a surface, we can disconnect the surface into many components by deleting these curves.

We need the following topological lemma:

LEMMA 3.1. *Suppose that M is a compact Riemann surface with genus g, and $\phi_j: S^1\to M$ $1\leqslant j\leqslant 2g+k$, $k\geqslant 1$, is an injective piece-wise C^1 map such that $\phi_i(S^1)\cap\cap\phi_j(S^1)$, $i\neq j$, consists of a finite number of points. Then, $M\setminus\phi_1(S^1)\cup\ldots\cup\phi_{2g+k}(S^1)$ has at least $k+1$ connected components.*

Proof. It suffices to prove that when $k=1$, $M\setminus\phi_1(S^1)\cup\ldots\cup\phi_{2g+1}(S^1)$ is not connected. Note that

$$H_1(M;\mathbf{Z})=\underbrace{\mathbf{Z}\oplus\cdots\oplus\mathbf{Z}}_{2g\text{ times}},$$

where $\mathbf{Z}$ is the ring of integers. Each $\phi_j: S^1\to M$ defines a cycle in M. Therefore, there exists $n_1,\ldots,n_{2g+k}\,\mathbf{Z}$ not all zero such that the homology class represented by $\sum_{j=1}^{2g+1} n_j\phi_j$

is zero. Observe that $n_j\phi_j$ can be represented by $\phi_j \circ \beta_{n_j}$ where $\beta_{n_j}: S^1 \to S^1$ is defined by $\beta_{n_j}(e^{i\theta}) = e^{in_j\theta}$. Thus, $\phi_j \circ \beta_{n_j}(S^1) = \phi(S^1)$. Since n_j, $1 \leqslant j \leqslant 2g+k$, are not all zero, we may assume that $n_1 \neq 0$. We can assume there exists $x_0 \in \phi_1(S^1)$ such that ϕ_1 is a C^1 diffeomorphism in a neighborhood of $\phi_1^{-1}(x_0)$ and $x_0 \notin \phi_2(S^1) \cup \ldots \cup \phi_{2g+1}(S^1)$. Let $\alpha: (-1, 1) \to M$ be an injective C^1 map such that

$$\alpha((-1, 1)) \cap (\phi_1(S^1) \cup \ldots \cup \phi_{2g+1}(S^1)) = \{\alpha(0)\} = \{x_0\}$$

and the tangent vector of α at x_0 is perpendicular to the tangent vector of ϕ_1 at x_0. Suppose $M \setminus \phi_1(S^1) \cup \ldots \cup \phi_{2g+1}(S^1)$ is not disconnected. Since it is an open set, we can find a C^1 curve $\beta: [-1, 1] \to M \setminus \phi_1(S^1) \cup \ldots \cup \phi_{2g+1}(S^1)$ with $\beta(-1) = \alpha(-\frac{1}{2})$ and $\beta(1) = \alpha(\frac{1}{2})$. This implies that there exists an injective C^1 map $\Phi: (-1, 1) \times S^1 \to$ $\to M \setminus \phi_2(S^1) \cup \ldots \cup \phi_{2g+1}(S^1)$ and that $\Phi((-1, 1) \times S^1) \cap \phi_1(S^1)$ is a small neighborhood of $\phi_1(S^1)$ around x_0.

Let f be a non-trivial non-negative function belonging to $C_0^\infty((-1, 1))$ such that $\int_{-\infty}^{\infty} f(t)\, dt = 1$. Then $f(t)\, dt$ is a closed form in $(-1, 1) \times S^1$. Therefore, $(\Phi^{-1})^*(f(t)\, dt)$ is a C^1 closed form of M. Now since the homology class represented by $\sum_{j=1}^{2g+1} n_j\phi_j$ is zero, we have

$$((\Phi^{-1})^* f(t)\, dt)\left(\sum_{j=1}^{2g+1} n_j\phi_j\right) = 0.$$

However,

$$((\Phi^{-1})^* f(t)\, dt)\left(\sum_{j=1}^{2g+1} n_j\phi_j\right) = n_j \int_{-1}^{1} f(t)\, dt \neq 0,$$

a contradiction.

Thus, $M \setminus \phi_1(S^1) \cup \ldots \cup \phi_{2g+1}(S^1)$ has more than one component.

Remark. We can relax the condition: $\phi_i(S^1) \cap \phi_j(S^1)$, $i \neq j$, has only a finite number of points. The condition can be replaced by

$$\phi_i(S^1) \neq \phi_1(S^1) \cup \ldots \cup \phi_{i-1}(S^1) \cup \phi_{i+1}(S^1) \cup \ldots \cup \phi_{2g+k}(S^1).$$

DEFINITION. Suppose that ψ satisfies $(\Delta + h(x))\psi = 0$, $h \in C^\infty(M)$. We say that the order of vanishing of ψ at $x_0 \in M$ is equal to N iff when we pull back ψ to the tangent space at x_0 via the exponential map there is a homogeneous polynomial p_N of degree N such that $\psi \sim p_N$ near the origin.

THEOREM 3.2. *Suppose that M is a compact Riemann surface of genus g, and ψ is the i-th eigenfunction. Let $x_0 \in M$ and $\psi(x_0) = 0$. Then, the order of vanishing of ψ at x_0 is less than or equal to $2g+i$.*

Proof. The proof is an immediate consequence of Lemma 3.1, the following lemma, and the observation that an eigenfunction changes sign around any of its zeroes.

LEMMA 3.3. *Suppose that M is a compact Riemann surface, and ψ is an eigenfunction. Let $x_0 \in M$ and the order of vanishing of ψ at x_0 is k. Then, we can find $\phi_i : S^1 \to \to M$, $1 \leqslant i \leqslant k$, satisfying the assumption of Lemma* 3.1 *and $\phi_1(S^1) \subseteq \psi^{-1}(0)$.*

Proof. This follows from the observation that the set of nodal lines of a spherical harmonic of order k in $\mathbf{R}^2$ consist of k straight lines passing through the origin.

Theorem 3.2 shows that there is a topological restriction to the order of vanishing of an eigenfunction. We then derive in the following theorem that there is a topological restriction to multiplicities of eigenfunction.

THEOREM 3.4. *Suppose that M is a compact Riemann surface of genus g, and $\mu_i(M)$ is the i-th eigenvalue. Then, the multiplicity of $\mu_i(M)$ is less than or equal to* $(2g+i+1)(2g+i+2)/2$.

Proof. We first indicate the proof when $g=0$ and $i=1$. Then the order of vanishing on the nodal lines is less than or equal to 1. If the multiplicity of $\mu_1(M)=4$, then we have $\phi_1, \ldots, \phi_4$ linearly independent and $\Delta\phi_i + \mu_1(M)\,\phi_i = 0$, $1 \leqslant i \leqslant 4$. We can find $a_i, b_i \in \mathbf{R}$, $1 \leqslant i \leqslant 3$, such that $a_i^2 + b_i^2 \neq 0$ and $(a_i\phi_{i+1} - b_i\phi_1)(x_0) = 0$, $i = 1, 2, 3$. See that $a_i\phi_{i+1} - b_i\phi_1$ are again linearly independent. Then consider $d(a_i\phi_{i+1} - b_i\phi_1)(x_0)$. The dimension of the tangent space is equal to 2. Hence, we have $C_1, \ldots, C_3$ not all zero such that

$$\sum_{i=1}^{3} C_i d(a_i\phi_{i+1} - b_i\phi_1)(x_0) = 0.$$

Since we also have

$$\sum_{i=1}^{3} C_i (a_i\phi_{i+1} - b_i\phi_1)(x_0) = 0,$$

the order of vanishing of this non-trivial 1-st eigenfunction at x_0 is greater than or equal to 2. This contradicts the result of Theorem 3.2.

The general case goes the same by noting that on $\mathbf{R}^2$ the dimension of the space of constant coefficient partial differential operator of order less than or equal to k is equal to $\sum_{i=1}^{k+1} i$.

COROLLARY 3.5. *Suppose that M is homeomorphic to S^2, i.e., $g=0$. Then, the nodal line of a* 1*-st eigenfunction is a C^∞ simple closed curve and the multiplicity of $\mu_1(M)$ is less than or equal to* 3.

Remarks. (i) The bound of the multiplicity $\mu_1(M)$ in Corollary 3.5 is sharp because the coordinate function of spheres in $\mathbf{R}^3$ with center at the origin are 1-st eigenfunctions. However, when $g>0$ we don't know whether $(2g+2)(2g+3)/2$ is a sharp bound for the multiplicity $\mu_1(M)$ or not.

(ii) The Almgren-Calabi theorem states that every minimal immersion of S^2 into S^3 must lie on a great circle. Therefore, if we know that every minimal immersion of S^2 into S^3 is by the 1-st eigenfunctions then we can obtain the Almgren-Calabi theorem by Corollary 3.5.

§4. Geometry of Nodal Lines

One of the difficulties in studying the nodal sets is the presence of multiple eigenvalues. This is in some sense a singular case and non-generic. The results of K. Uhlenbeck [6] show that generically all eigenvalues have simple multiplicity. We gave an upper bound of the multiplicity of $\mu_i(M)$, when M is a compact Riemann surface of genus g. In general the multiplicities can be pretty big. We shall study the case when $g=0$ and $i=1$.

THEOREM 4.1. *Suppose that M is homeomorphic to S^2, and is isometric to a surface of revolution in $\mathbf{R}^3$. Then, we can find a basis $\{\psi_i\}$ of the space of 1-st eigenfunctions such that the nodal line of each ψ_i is a curve with constant geodesic curvature.*

Proof. Corollary 3.5 shows that the multiplicity of $\mu_1(M)\leqslant 3$ and that the nodal line of a 1-st eigenfunction is a C^∞ simple closed curve. Let E_1 denote the linear space of 1-st eigenfunction endowed with the usual L^2 inner product.

Note that S^1 acts on E_1 as a group of isometry and preserves the orientation.

When $\dim E_1=1$, we have a non-trivial $\psi_1\in E_1$ such that it is invariant under S^1. The famous theorem of H. Hopf on vector fields shows that there are only two fixed points under the action of S^1. Then we can find a point x_0 which is not a fixed point and that $\psi_1(x_0)=0$. Therefore, ψ_1 also vanishes on the orbit of x_0 under S^1. The orbit of x_0 is a C^∞ simple closed curve. Thus, we must have $\psi_1^{-1}(0)$ is equal to the orbit of x_0. Moreover, S^1 acts as isometry implies the orbit of x_0 has a constant geodesic curvature.

Suppose that $\dim E_1=3$. Results from linear algebra supply us with an orthonormal basis $\{\psi_1, \psi_2, \psi_3\}$ of E_1 such that ψ_1 is invariant under S^1 and S^1 rotates on the space spanned by $\{\psi_2, \psi_3\}$. Consequently $\psi_1^{-1}(0)$ is a simple closed curve of constant geodesic curvature. Notice that once we prove the theorem for $\psi_2^{-1}(0)$ and $\psi_3^{-1}(0)$, we also settle that case when $\dim E_1=2$. This is seen from results in linear algebra that we can find an orthonormal basis of E_1 such that S^1 acts as the usual rotation.

Now let us study $\psi_2^{-1}(0)$ and $\psi_3^{-1}(0)$.

We claim that $\psi_2^{-1}(0)\cap\psi_3^{-1}(0)\neq\emptyset$. This is a special case of the following lemma.

LEMMA 4.2. *Suppose that M is a compact Riemannian manifold, f and h are two linearly independent eigenfunctions of the same eigenvalue μ. If either $f^{-1}(0)$ or $h^{-1}(0)$ is connected then $f^{-1}(0)\cap h^{-1}(0)\neq\emptyset$.*

Proof. Suppose that $f^{-1}(0)\cap h^{-1}(0)=\emptyset$. Assume $h^{-1}(0)$ is connected.

Note that $\{x: f(x)>0\}\cap\{x: f(x)<0\}=\emptyset$. We can assume $h^{-1}(0)\subseteq\{x: f(x)>0\}$. One immediately sees that one of the nodal domains of h is contained in $\{x: f(x)>0\}$.

Courant's minimum principle immediately shows that $\mu=\lambda_1$ of a nodal domain of $h>\lambda_1(\{x: f(x)>0\})=\mu$.

This is a contradiction and the proof of the lemma is completed.

Now $\psi_2^{-1}(0)\cap\psi_3^{-1}(0)\neq\emptyset$. Let $x_0\in\psi_2^{-1}(0)\cap\psi_3^{-1}(0)$. Then, $\psi_2(x_0)=\psi_3(x_0)=0$. Note that if $\alpha\in S^1$, then there exists real numbers a, b such that

$$\psi_2(\alpha(x))=a\psi_2(x)+b\psi_3(x) \quad \text{for all } x\in M.$$

This shows that $\psi_2(\alpha(x_0))=a\psi_2(x_0)+b\psi_3(x_0)=0$. Since α is arbitrary, ψ_2 vanishes on the orbit of x_0 and so does ψ_3. This forces x_0 to be a fixed point of S^1. We claim that $\psi_2^{-1}(0)\cap\psi_3^{-1}(0)$ has more than two points. This is proved in the following lemma.

LEMMA 4.3. *Suppose that M is the same as Theorem 4.1, f and g are two linearly independent 1-st eigenfunctions. Then $f^{-1}(0)\cap h^{-1}(0)$ has more than two points.*

Proof. Lemma 4.2 shows that $f^{-1}(0)\cap h^{-1}(0)\neq\emptyset$. We first observe that when $f^{-1}(0)$ and $h^{-1}(0)$ meet at x_0 they must be transversal to each other at x_0. Suppose the contrary. If $f^{-1}(0)$ and $h^{-1}(0)$ are tangent to each other at x_0, then there exist a, b not all zero such that $d(af+bh)(x_0)=0$. Recall that $(af+bh)(x_0)=0$, x_0 is then a critical point along the nodal line of the non-trivial eigenfunction $af+bh$, a contradiction. Now the lemma is a consequence of the Jordan curve theorem in $\mathbf{R}^2$.

Actually, $\psi_2^{-1}(0)\cap\psi_3^{-1}(0)$ has exactly two points. This follows from the observation that $\psi_2^{-1}(0)\cap\psi_3^{-1}(0)$ is a fixed point set of S^1 acting on M.

Let $\{p, q\}=\psi_2^{-1}(0)\cap\psi_3^{-1}(0)$. The nodal lines of ψ_2 and ψ_3 are simple closed curves passing through p and q. Note that any two points on an orbit of S^1 have the same distance to p and q. Gauss's lemma implies immediately that the orthogonal trajectories of orbits of S^1 are closed geodesic loops passing through p and q. $f^{-1}(0)$ and $h^{-1}(0)$ are also orthogonal to the orbits of S^1 because the existence of involutive isometries fixing p and q and the result of Lemma 4.3. This shows that $\psi_2^{-1}(0)$ and $\psi_3^{-1}(0)$ are closed geodesic loops. Thus the proof of Theorem 4.1 is complete.

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