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A stability estimate for the Aleksandrov-Fenchel inequality, with an application to mean curvature

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Abstract

In the quadratic Aleksandrov-Fenchel inequality for mixed volumes, stated as inequality (1) below, where $C_1, \dots, C_{n-2}$ are smooth convex bodies, equality holds only if the convex bodies K and L are homothetic. Under stronger regularity assumptions on $C_1, \dots, C_{n-2}$, a stability estimate is proved, expressing that K and L are close to homothetic if equality is satisfied approximately. This is applied to estimate explicitly the deviation of a closed convex hypersurface with mean curvature close to one from a unit sphere.

1 Introduction and results

In the theory of mixed volumes of convex bodies, the Aleksandrov-Fenchel inequality

$$V(K, L, C_1, \dots, C_{n-2})^2 \geq V(K, K, C_1, \dots, C_{n-2}) V(L, L, C_1, \dots, C_{n-2}) \quad (1)$$

is a central result (see, e.g., Busemann [3], Leichtweiß [9], Burago-Zalgaller [2]). Here $K, L, C_1, \dots, C_{n-2}$ are convex bodies in n-dimensional euclidean space E^n, and V denotes the mixed volume. We assume that $n \geq 3$. Equality in (1) holds if K and L are homothetic, but in general, depending on the properties of $C_1, \dots, C_{n-2}$, not only in this case (see [11] for a discussion of the equality problem). If $C_1, \dots, C_{n-2}$ are smooth, that is, if they have a unique supporting hyperplane at each boundary point, then equality in (1) holds only if K and L are homothetic. This was recently proved in [13]. In the special case where $C_1, \dots, C_{n-2}$ are balls, this uniqueness assertion admits a stronger version in the form of a stability result: if equality is almost satisfied, then K and L are almost homothetic, in the following precise sense.

We introduce the nonnegative deficit Δ by

$$\Delta(K, L, C_1, \ldots, C_{n-2}) :=$$

$$V(K, L, C_1, \ldots, C_{n-2})^2 - V(K, K, C_1, \ldots, C_{n-2})V(L, L, C_1, \ldots, C_{n-2})$$

and the L_2 metric δ_2 for convex bodies by

$$\delta_2(K, L) := \left(\int_{S^{n-1}} |h(K, u) - h(L, u)|^2 d\omega(u) \right)^{1/2},$$

where $h(K, \cdot)$ is the support function of K, S^{n-1} is the unit sphere of E^n and the integration is with respect to spherical Lebesgue measure ω. By B we denote the unit ball of E^n.

After applying a suitable homothety we may assume that the convex bodies K and L have coinciding Steiner points and equal mean widths. Under these assumptions, it was shown independently by Goodey and Groemer [5] and by Schneider [12] that

$$\Delta(K, L, B, \ldots, B) \geq \frac{n+1}{n(n-1)} a(K, L) \delta_2(K, L)^2 \tag{2}$$

with

$$a(K, L) := \max \{V(K, K, B, \ldots, B), V(L, L, B, \ldots, B)\}.$$

(This follows, e.g., from the first inequality for $V_{11}^2 - V_{02}V_{20}$ in [12], p. 55, where the roles of K and L can be interchanged.)

The L_2 metric δ_2 can be further estimated in terms of the Hausdorff metric δ, namely by

$$\delta_2(K, L)^2 \geq c\delta(K, L)^{n+1} \tag{3}$$

with a constant c depending only on n and the diameter of $K \cup L$ (see Vitale [15] and, for a better estimate if one of the bodies is a ball, Groemer and Schneider [6]).

For the general inequality (1), no strengthening of type (2) can exist, since equality in (1) is possible for non-homothetic convex bodies K and L. For the same reason, if a stability version of inequality (1) for $C_1, \ldots, C_{n-2}$ taken from a restricted class of convex bodies is to be proved, then this class cannot be dense in the space of convex bodies. In particular, the uniqueness result of [13] for smooth $C_1, \ldots, C_{n-2}$ cannot be improved in this way.

In the following, we show stability for the Aleksandrov-Fenchel inequality (1) under the assumption that the convex bodies $C_1, \ldots, C_{n-2}$ are η-smooth, for some fixed $\eta > 0$. A convex body C in E^n is called *η-smooth* if a ball of radius η can roll freely in C, that is, if to each point x in the boundary of C

there is a vector t such that $x \in \eta B + t \subset C$. Sufficient (but not necessary) for this is that the support function of C is twice continuously differentiable and all principal radii of curvature of C are $\geq \eta$. (This fact, known as "Blaschke's rolling theorem", follows from a standard convexity criterion, applied to the difference $h(C,\cdot) - h(\eta B,\cdot)$ of support functions. See also Koutroufiotis [8].)

By $\mathcal{K}^n(r,R)$ we denote the set of convex bodies in E^n that contain some ball of radius r and are contained in some ball of radius R.

Theorem 1. *Let positive numbers η, r, R and an integer $p \in \{1,\ldots,n-2\}$ be given. If $K, L, C_1,\ldots,C_p \in \mathcal{K}^n(r,R)$ and if $C_1,\ldots,C_p$ are η-smooth, then*

$$\Delta(K,L,C_1,\ldots,C_p,B,\ldots,B) \geq c_1\eta^{4(2^p-1)}\Delta(K,L,B,\ldots,B)^{2^p},$$

where the constant c_1 depends only on n,p,r,R.

Together with inequality (2) (and (3), if one prefers the Hausdorff metric), this theorem provides a stability estimate for the equality case in the Aleksandrov-Fenchel inequality (1) in the case of η-smooth bodies $C_1,\ldots,C_{n-2}$. An explicit value for the constant c_1 could be read off from the proof below, but this seems of minor interest since the order of the estimate, expressed by the exponents, is probably not optimal.

As an application of Theorem 1 we can prove a stability theorem for closed convex hypersurfaces with almost constant mean curvature.

Theorem 2. *Let $K \subset E^n$ be a convex body contained in some ball of radius R, let $0 < \epsilon_0 < 1$. Suppose that the boundary of K is twice continuously differentiable and that its mean curvature H satisfies*

$$1 - \epsilon \leq H \leq 1 + \epsilon$$

for some positive $\epsilon \leq \epsilon_0$. Then there is a ball B_1 of radius 1 such that

$$\delta(K,B_1) \leq c_2\epsilon^q \qquad \text{with } q = \frac{1}{(n+3)2^{n-3}},$$

where c_2 is a constant depending only on n, R, ϵ_0.

Stability of the sphere in the class of closed convex surfaces with almost constant mean curvature was treated by Diskant [4], Koutroufiotis [7], Moore [10], Treibergs [14]. These authors use completely different methods; their results are either restricted to threedimensional space, or need stronger assumptions, or are less explicit. We point out that the constant c_2 in our stability estimate involves an a-priori bound for the circumradius of K. Without further restriction of ϵ_0, this is inevitable, as shown by examples.

One should expect that there is a number $\epsilon_0 > 0$ such that the assumption $1 - \epsilon \leq H \leq 1 + \epsilon$ with $\epsilon \leq \epsilon_0$ implies an absolute bound for the diameter of K (if so, the convexity would be essential). For $n = 3$, this was proved by Diskant [4], but his method does not seem to extend to higher dimensions.

2 Proof of Theorem 1

By $\mathcal{K}^n$ we denote the space of convex bodies (nonempty, compact, convex subsets) of n-dimensional euclidean vector space E^n. We assume that $n \geq 3$. Let $K_0, K_1, \dots, K_n \in \mathcal{K}^n$. First we assume that all these bodies have interior points. The bodies $K_4, \dots, K_n$ will be kept fixed until the formulation of the Lemma below. By

$$V(K_1, K_2, K_3) := V(K_1, K_2, K_3, K_4, \dots, K_n)$$

we abbreviate the mixed volume of $K_1, \dots, K_n$. As long as also K_0, K_1, K_2, K_3 are fixed, we further use the abbreviations

$$W_{ij} := V(K_i, K_j, K_3), \quad i, j \in \{0, 1, 2\},$$

$$\Delta = \Delta(K_1, K_2, K_3) := W_{12}^2 - W_{11}W_{22},$$

$$\Lambda = \Lambda(K_1, K_2, K_3; K_0) := -\frac{W_{11}}{W_{01}^2} + \frac{2W_{12}}{W_{01}W_{02}} - \frac{W_{22}}{W_{02}^2}.$$

Then $\Delta \geq 0$ by the Aleksandrov-Fenchel inequality (1). Further known inequalities are

$$\Lambda \geq 0$$

and

$$(W_{00}W_{12} - W_{01}W_{02})^2 \leq (W_{01}^2 - W_{00}W_{11})(W_{02}^2 - W_{00}W_{22})$$

(compare (2.7) in [11], respectively (2.5) in [12], where references can be found). We derive some consequences. Rearranging the terms of the latter inequality, we get

$$\Delta \leq \frac{W_{01}^2 W_{02}^2}{W_{00}} \Lambda. \tag{4}$$

Using the identity $b^2 - ac = -c(a - 2b + c) + (b - c)^2$ with

$$a = \frac{W_{11}}{W_{01}^2}, \quad b = \frac{W_{12}}{W_{01}W_{02}}, \quad c = \frac{W_{22}}{W_{02}^2},$$

we see that

$$\frac{\Delta}{W_{01}^2 W_{02}^2} = \frac{W_{22}}{W_{02}^2}\Lambda + \left(\frac{W_{12}}{W_{01}W_{02}} - \frac{W_{22}}{W_{02}^2}\right)^2, \tag{5}$$

in particular

$$\Delta \geq W_{01}^2 W_{22} \Lambda. \tag{6}$$

Inequality (5) together with $\Lambda \geq 0$ yields

$$\left| \frac{W_{12}}{W_{01} W_{02}} - \frac{W_{22}}{W_{02}^2} \right| \leq \frac{\sqrt{\Delta}}{W_{01} W_{02}}.$$

Multiplication with W_{02}^2 / W_{12} gives

$$\left| \frac{W_{22}}{W_{12}} - \frac{W_{02}}{W_{01}} \right| \leq \frac{W_{02}}{W_{01} W_{12}} \sqrt{\Delta}, \tag{7}$$

while multiplication with $W_{01} W_{02} / W_{22}$ leads to

$$\left| \frac{W_{12}}{W_{22}} - \frac{W_{01}}{W_{02}} \right| \leq \frac{\sqrt{\Delta}}{W_{22}}. \tag{8}$$

Now let n-dimensional convex bodies $K, L, M, M', \bar{M} \in \mathcal{K}^n$ be given. For n-dimensional convex bodies $P, Q \in \mathcal{K}^n$, put

$$\alpha(Q) := \frac{V(L, M, Q)}{V(K, M, Q)}$$

and

$$\beta(P, Q) := V(K, K, P)\alpha(Q) - 2V(K, L, P) + V(L, L, P)\alpha(Q)^{-1}.$$

With the choice $K_1 = K$, $K_2 = L$, $K_3 = M$, inequality (7) gives

$$|\alpha(L) - \alpha(K_0)| \leq \frac{V(K_0, L, M)}{V(K_0, K, M)} \frac{\sqrt{\Delta(K, L, M)}}{V(K, L, M)}, \tag{9}$$

and inequality (8) yields

$$\left| \alpha(L)^{-1} - \alpha(K_0)^{-1} \right| \leq \frac{\sqrt{\Delta(K, L, M)}}{V(L, L, M)}. \tag{10}$$

Applying (6) to $K_0 = L$, $K_1 = K$, $K_2 = L$, $K_3 = M$, we get

$$\beta(M, L) \geq -\frac{\Delta(K, L, M)}{V(K, L, M)}. \tag{11}$$

For arbitrary convex bodies P we have

$$|\beta(P, P) - \beta(P, L)| \leq$$

$$V(K,K,P)|\alpha(P)-\alpha(L)|+V(L,L,P)\left|\alpha(P)^{-1}-\alpha(L)^{-1}\right|,$$

hence (9) and (10) yield

$$|\beta(P,P)-\beta(P,L)|\leq A(P)\sqrt{\Delta(K,L,M)} \tag{12}$$

with

$$A(P):=\frac{V(K,K,P)V(L,M,P)}{V(K,L,M)V(K,M,P)}+\frac{V(L,L,P)}{V(L,L,M)}. \tag{13}$$

Applying $\Lambda\geq 0$ to $K_0=M$, $K_1=K$, $K_2=L$, $K_3=M'$, we get $\beta(M',M')\leq 0$ and hence $\beta(M',L)\leq\beta(M',L)-\beta(M',M')$, thus by (12)

$$\beta(M',L)\leq A(M')\sqrt{\Delta(K,L,M)}. \tag{14}$$

From now on we assume that

$$M=\bar{M}+M'.$$

Then $\beta(\bar{M},L)=\beta(M,L)-\beta(M',L)$ by the linearity of the mixed volume in each argument, hence (11) and (14) yield

$$\beta(\bar{M},L)\geq-\frac{\Delta(K,L,M)}{V(K,L,M)}-A(M')\sqrt{\Delta(K,L,M)}. \tag{15}$$

Further, $\beta(\bar{M},\bar{M})=\beta(\bar{M},L)+[\beta(\bar{M},\bar{M})-\beta(\bar{M},L)]$, hence (15) and (12) give

$$\beta(\bar{M},\bar{M})\geq -B \tag{16}$$

with

$$B:=\frac{\Delta(K,L,M)}{V(K,L,M)}+[A(M')+A(\bar{M})]\sqrt{\Delta(K,L,M)}.$$

Dividing (16) by $V(K,M,\bar{M})V(L,M,\bar{M})$, we get

$$\Lambda(K,L,\bar{M};M)\leq\frac{B}{V(K,M,\bar{M})V(L,M,\bar{M})}.$$

Inequality (4) with $K_1=K$, $K_2=L$, $K_3=\bar{M}$, $K_0=M$ yields

$$\Delta(K,L,\bar{M})\leq\frac{V(K,M,\bar{M})^2V(L,M,\bar{M})^2}{V(M,M,\bar{M})}\Lambda(K,L,\bar{M};M).$$

Both inequalities together show that

$$\Delta(K,L,\bar{M})\leq \tag{17}$$

$$\frac{V(K,M,\bar{M})V(L,M,\bar{M})}{V(M,M,\bar{M})}\left[\frac{\Delta(K,L,M)}{V(K,L,M)}+[A(M')+A(\bar{M})]\sqrt{\Delta(K,L,M)}\right].$$

To simplify this estimate, we now assume that $K, L, M \in \mathcal{K}^n(r,R)$. Then K is contained in some translate of $(R/r)M$, hence the monotoneity of the mixed volume in each argument gives

$$\frac{V(K,M,\bar{M})}{V(M,M,\bar{M})} \leq \frac{R}{r}.$$

Since $M = \bar{M} + M'$, we may assume that $\bar{M} \subset M$ and $M' \subset M$; then

$$V(L,M,\bar{M}) \leq R^3 V(B,B,B).$$

From $\Delta(K,L,M) \leq V(K,L,M)^2$ we get

$$\frac{\Delta(K,L,M)}{V(K,L,M)} \leq \sqrt{\Delta(K,L,M)}.$$

Since $M' \subset M$ and M is contained in a translate of $(R/r)K$, we infer that

$$A(M') \leq \left(\frac{R}{r}\right)^2 + 1,$$

and the same inequality holds for $A(\bar{M})$. Now (17) yields the following result.

Lemma. *Let $K, L, M \in \mathcal{K}^n(r,R)$, let $K_4, \ldots, K_n \in \mathcal{K}^n$ be arbitrary convex bodies. If the convex body $\bar{M}$ is a summand of M, then*

$$\Delta(K,L,\bar{M}) \leq \left[2\left(\frac{R}{r}\right)^3 + 3\left(\frac{R}{r}\right)\right] R^3 V(B,B,B)\sqrt{\Delta(K,L,M)}.$$

Here the initial assumption that $K_4, \ldots, K_n, \bar{M}$ be n-dimensional is no longer necessary, since by an obvious approximation argument the inequality can be extended to the general case.

The lemma shows, in particular, that $\Delta(K,L,M) = 0$ implies $\Delta(K,L,\bar{M}) = 0$. This was proved in [11], Theorem 4.1, and the present proof can be considered as a quantitative elaboration of the argument given there.

If we now assume that also $K_4, \ldots, K_n \in \mathcal{K}^n(r,R)$ and that $\bar{M} = \eta B$ with $\eta > 0$, then the lemma gives

$$\Delta(K,L,\eta B) \leq \left[2\left(\frac{R}{r}\right)^3 + 3\left(\frac{R}{r}\right)\right] R^n \sqrt{\Delta(K,L,M)},$$

hence

$$\Delta(K,L,M) \geq c_2\eta^4\Delta(K,L,B)^2$$

with a constant c_2 depending only on n, r, R. Repeated application of this inequality, with $C_1, \ldots, C_p$ in turn playing the role of M, now yields Theorem 1, if we observe the fact that ηB is a summand of C_i if C_i is η-smooth.

3 Proof of Theorem 2

Let $K \in \mathcal{K}^n$. Writing, as usual,

$$W_i := V(\underbrace{K,\ldots,K}_{n-i},\underbrace{B,\ldots,B}_{i}),$$

we have

$$\begin{aligned} W_0 &= \frac{1}{n}\int p dS, \\ W_1 &= \frac{1}{n}\int pH dS = \frac{1}{n}\int dS \\ W_2 &= \frac{1}{n}\int H dS, \end{aligned} \tag{18}$$

where all integrations are over $\operatorname{bd} K$, p denotes the distance of the tangent plane from 0, and dS is the surface area element (see Bonnesen-Fenchel [1], p. 63). We may assume that $p > 0$. Under the assumptions of Theorem 2 we deduce

$$W_1 \leq (1+\epsilon)W_0, \quad W_2 \geq (1-\epsilon)W_1,$$

hence

$$W_1^2 - W_0W_2 \leq \frac{2\epsilon}{1-\epsilon}W_0W_2.$$

Denoting the (nonnegative) principal radii of curvature of $\operatorname{bd} K$ by $r_1, \ldots, r_{n-1}$ (where ∞ is allowed), we have

$$\frac{1}{r_i} \leq \frac{1}{r_1} + \ldots + \frac{1}{r_{n-1}} = (n-1)H \leq (n-1)(1+\epsilon),$$

hence

$$r_i \geq \frac{1}{(n-1)(1+\epsilon)} =: \eta.$$

As remarked before Theorem 1, this implies that K is η-smooth. Now Theorem 1 yields

$$\frac{2\epsilon}{1-\epsilon}W_0W_2 \geq W_1^2 - W_0W_2 = \Delta(K,B,K,\ldots,K) \geq$$

$$c_3 \left[\frac{1}{(n-1)(1+\epsilon)}\right]^{4(2^{n-2}-1)} \Delta(K,B,B,\dots,B)^{2^{n-2}}. \tag{19}$$

Here the constant c_3, like the constants $c_4,\dots,c_7$ below, depends only on n,ϵ_0,R (since K is η-smooth, its inradius is not less than $[(n-1)(1+\epsilon_0)]^{-1}$).

Let B_K denote the Steiner ball of K (the ball which has the same Steiner point and mean width $w(K)$ as K), then

$$\Delta(K,B_K,B,\dots,B) = \left(\frac{w(K)}{2}\right)^2 \Delta(K,B,B,\dots,B) \le c_4\Delta(K,B,\dots,B). \tag{20}$$

Now inequalities (19), (20), (2) together with the estimate

$$\delta_2(K,B_K)^2 \ge c_5\delta(K,B_K)^{\frac{n+3}{2}}$$

established in [6] yield

$$\delta(K,B_K) \le c_6\epsilon^{\frac{1}{(n+3)2^{n-3}}}. \tag{21}$$

We may assume that K has its Steiner point at the origin. Denoting the radius of B_K by ρ and the right-hand side of (21) by α, we then have

$$(\rho-\alpha)B \subset K \subset (\rho+\alpha)B.$$

From (18) we get

$$\int HdS \le (1+\epsilon)\int dS = (1+\epsilon)\int pHdS \le (1+\epsilon)(\rho+\alpha)\int HdS,$$

thus $(1+\epsilon)(\rho+\alpha) \ge 1$, and similarly $(1-\epsilon)(\rho-\alpha) \le 1$. We conclude that $|\rho-1| \le c_7\alpha$ and hence that

$$\delta(K,B) \le \delta(K,B_K) + \delta(B_K,B) \le \alpha + c_7\alpha.$$

This completes the proof of Theorem 2.

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