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LOCAL BOUNDEDNESS OF MINIMIZERS IN A LIMIT CASE (*)

Nicola Fusco-Carlo Sbordone

We prove the local boundedness of minimizers of a functional with anisotropic polynomial growth. The result here obtained is optimal if compared with previously know counterexamples.

1. INTRODUCTION.

Let us consider the following functional

$$I(v) = \int_\Omega f(x, v, Dv)dx \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and

$$\sum_{i=1}^{n} |z_i|^{q_i} \le f(x,y,z) \le c(1 + \sum_{i=1}^{n} |z_i|^{q_i}) \tag{1.2}$$

with $q_i \geq 1$ and $q = \max\{q_i\}$, $p = \min\{q_i\}$. Some recent counterexamples ([3], [5], [7], [8]) have shown that, if

$$q > \bar{q}^\star, \tag{1.3}$$

where

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$$\frac{1}{\bar{q}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}, \quad \bar{q}^\star = \frac{n\bar{q}}{n-\bar{q}}, \qquad (\bar{q} < n) \tag{1.4}$$

the local minimizers of $I(v)$ are not necessarily continuos or even bounded. So the regularity results obtained when $p = q$ (see [4] e.g.) cannot be always extended to this case. Of course the growth condition (1.2) is a particular case of the following one:

$$|z|^p \leq f(x,y,z) \leq c'(1+|z|^q). \tag{1.5}$$

However, under the assumption (1.5), in [9] it has been proved that, if $q < p^\star = np/(n-p)$, the local minima are locally bounded. A sharper result can be proved when f satisfies (1.2). Namely in [1] it is shown that, if $u \in W^{1,p}(\Omega)$ minimizes $I(v)$ with bounded boundary data and if $q < \bar{q}^\star$, then $u \in L^\infty(\Omega)$. This result has almost completely filled the gap between the counterexamples and the regularity results, leaving out the case $q = \bar{q}^\star$ alone. In this short paper we prove that the above results can be extended up to the limit case. More precisely, we show that if f verifies (1.2) and $q \leq \bar{q}^\star$, then the local minimizers of $I(v)$ are locally bounded in Ω (theorem 1). We also show that the local boundedness of minima still holds under more general growth conditions. In particular, we may allow f to satisfy (1.5) with $p \geq 1$ and $q = p^\star$.

2. THE LOCAL BOUNDEDNESS RESULT

In the following we shall denote by Ω a bounded open set in $\mathbb{R}^n$. If $q_1, q_2, \ldots, q_n \geq 1$, we set

$$\frac{1}{\bar{q}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}, \qquad \bar{q}^\star = \frac{n\bar{q}}{n-\bar{q}} \qquad (if \quad \bar{q} < n) \tag{2.1}$$

$$q = \max\{q_i\} \qquad p = \min\{q_i\}. \tag{2.2}$$

We recall the following Sobolev-type inequality.

LEMMA 1. *Let $p, q_i, q, \bar{q}^\star$ be as in (2.1),(2.2). If $u \in W_0^{1,p}(\Omega)$, then*

$$||u||_{L^{\bar{q}^\star}(\Omega)} \leq c(\Omega)\sum_{i=1}^{n}||D_i u||_{L^{q_i}(\Omega)}.$$

If u is in $W^{1,p}_{loc}(\Omega)$ and $B_R \subset \Omega$ is any ball, we put

$$A_{k,R} = \{x \in B_R : u > k\}.$$

Then one can prove the following extension of Lemma 5.4, Ch. *II* of [6].

LEMMA 2. *If $u \in W^{1,p}_{loc}(\Omega)$ and for any ball $B_R \subset\subset \Omega$, $0 < \sigma < 1$, $k > 0$*

$$\sum_{i=1}^{n} \int_{A_{k,\sigma R}} |D_i u|^{q_i} dx \le c \left[\int_{A_{k,R}} \left| \frac{u-k}{(1-\sigma)R} \right|^{\bar{q}^*} dx + |A_{k,R}| \right] \tag{2.3}$$

then $u(x)$ is locally bounded from above in Ω.

PROOF. Let us fix a ball $B_R \subset\subset \Omega$. We may always suppose that B_R is centered at the origin. Let us define the following sequences:

$$\rho_h = \frac{R}{2} + \frac{R}{2^{h+1}}, \qquad \bar{\rho}_h = \frac{\rho_h + \rho_{h+1}}{2}$$

$$k_h = k\left(1 - \frac{1}{2^{h+1}} \right), \qquad h = 0, 1, 2, \ldots$$

where k is a positive number to be chosen later.

Moreover, we set

$$J_h = \int_{A_{k_h, \rho_h}} |u(x) - k_h|^{\bar{q}^*} dx.$$

We may suppose that

$$2^h J_h \ge 1 \qquad \text{for any} \quad h, \tag{2.4}$$

otherwise $J_{h_m} \to 0$ for a suitable subsequence, hence

$$\int_{A_{k,R/2}} |u-k|^{\bar{q}^*} dx \le J_{h_m} \to 0$$

and so $u \le k$ in $B_{R/2}$.

Let us fix a $C^1([0,\infty[)$ function $\xi(t)$ such that $0 \le \xi(t) \le 1$ for every $t \ge 0$ $, \xi(t) = 1$ for $t \le 1/2$, $\xi(t) = 0$ for $t \ge 3/4$ and $|D\xi| \le c$.

Set

$$\xi_h(x) = \xi\left(\frac{2^{h+1}}{R} \quad (|x| - R/2) \right),$$

so that $\xi_h \equiv 1$ in $B_{\rho_{h+1}}$ and $\xi_h \equiv 0$ outside $B_{\bar\rho_h}$.

Now, from Lemma 1 we have

$$J_{h+1} \le \int_{A_{k_{h+1},\bar\rho_h}} |u(x) - k_{h+1}|^{\bar q^*} \xi_h^{\bar q^*}(x)dx = \tag{2.5}$$

$$= \int_{B_R} |(u - k_{h+1})^+ \xi_h|^{\bar q^*} dx \le$$

$$\le c(R) \sum_{i=1}^{n} [\int_{B_R} |D_i((u - k_{h+1})^+ \xi_h)|^{q_i} dx]^{\bar q^*/q_i} \le$$

$$\le c(R) \sum_{i=1}^{n} [\int_{A_{k_{h+1},\bar\rho_h}} |D_i u|^{q_i} dx + \frac{2^{hq_i}}{R^{q_i}} \int_{A_{k_{h+1},\bar\rho_h}} |u - k_{h+1}|^{q_i} dx]^{\bar q^*/q_i}.$$

Now we may use assumption (2.3) and Young′ s inequality to get

$$J_{h+1} \le c \sum_{i=1}^{n} \left[\frac{2^{h\bar q^*}}{R^{\bar q^*}} \int_{A_{k_{h+1},\rho_h}} |u - k_{h+1}|^{\bar q^*} dx + |A_{k_{h+1},\rho_h}| \right]^{\bar q^*/q_i} \tag{2.6}$$

Now we observe that

$$\int_{A_{k_{h+1},\rho_h}} |u - k_{h+1}|^{\bar q^*} dx \le J_h$$

and

$$|A_{k_{h+1},\rho_h}| (k_{h+1} - k_h)^{\bar q^*} \le$$

$$\le \int_{A_{k_{h+1},\rho_h}} (u - k_h)^{\bar q^*} dx \le J_h$$

and so, if we choose $k \ge 1$,

$$|A_{k_{h+1},\rho_h}| \le \frac{2^{(h+2)\bar q^*}}{k^{\bar q^*}} J_h \le 2^{(h+2)\bar q^*} J_h$$

Then, by these estimates and (2.6) one has

$$J_{h+1} \le c(R) \sum_{i=1}^{n} [2^{h\bar q^*} J_h]^{\bar q^*/q_i} \le c 2^{h\bar q^{*2}/p} \quad J_h^{\bar q^*/p}, \tag{2.7}$$

since, by (2.4), $2^{h\bar{q}^\star} J_h \geq 1$.

Then, from Lemma 4.7, Ch.2, [6],setting

$$\epsilon = \frac{\bar{q}^\star}{p} - 1, \qquad b = 2^{q^{\star 2}/p},$$

from (2.7) we deduce that, if $k \geq 1$ is chosen such that

$$J_0 \leq c^{\frac{-1}{\epsilon}} b^{\frac{-1}{\epsilon^2}}$$

Then $\lim_n J_n = 0$ and thus

$$\sup_{B_{R/2}} u \leq k.$$

As a consequence of the previous lemma, we get the following regularity results.

First of all we suppose that $u \in W^{1,p}_{loc}(\Omega)$ is a minimizer of

$$I(v) = \int_\Omega f(x, v(x), Dv(x))dx \tag{2.8}$$

where $f = f(x,y,z) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,+\infty)$ is a Caratheodory function satisfying

$$\sum_{i=1}^n |z_i|^{q_i} \leq f(x,y,z) \leq c(1 + \sum_{i=1}^n |z_i|^{q_i}) \tag{2.9}$$

then we have

THEOREM 1. *If $u \in W^{1,p}_{loc}(\Omega)$ is a local minimizer of (2.8) and f satisfies (2.9) with $q \leq \bar{q}^\star$ then $|u|$ is locally bounded in Ω.*

PROOF. By a standard choice of test functions and the "hole filling" technique, one can prove that for any ball $B_R \subset\subset \Omega$ and $\sigma \in (0,1)$ if $\sigma R < s < t < R$,

$$\sum_{i=1}^n \int_{A_{k,s}} |D_i u|^{q_i} dx \leq$$

$$\leq \theta \sum_{i=1}^n \int_{A_{k,t}} |D_i u|^{q_i} dx + c \sum_{i=1}^n \int_{A_{k,R}} \left| \frac{u-k}{t-s} \right|^{q_i} dx + c|A_{k,R}|,$$

with $0 < \theta < 1, \quad k > 0$.

By using a straightforward extension of Lemma 3.1, Ch. V in [2], one gets

$$\sum_{i=1}^{n}\int_{A_{k,\sigma}}|D_iu|^{q_i}dx\le c\sum_{i=1}^{n}\int_{A_{k,R}}\left|\frac{u-k}{(1-\sigma)R}\right|^{q_i}dx+c|A_{k,R}|$$

$$\le c\int_{A_{k,R}}\left|\frac{u-k}{(1-\sigma)R}\right|^{\bar q^*}dx+c|A_{k,R}|.$$

Then, from Lemma 2 one has that u is locally bounded from above in Ω.

Similarly, $-u$ is a local minimizers of an integral satisfying the same assumptions, hence also $-u$ is bounded from above.

Moreover we may suppose that f verifies the following conditions:

$$\text{(2.10)}\qquad \begin{cases} f=f(x,z) \text{ is convex in } z \text{ and measurable in } x\\ f(x,z_1+z_2)\le c[f(x,z_1)+f(x,z_2)]\\ \sum_{i=1}^{n}|z_i|^{q_i}\le f(x,z)\le c'(1+|z|^{\bar q^*}) \end{cases}$$

Then we have

THEOREM 2. *If f verifies (2.10) and $q\le \bar q^*$, then any local minimizer $u\in W^{1,p}_{loc}(\Omega)$ of the functional*

$$\int_\Omega f(x,Dv)dx$$

is locally bounded in Ω.

PROOF. It is enough to observe that if we have (2.10_1) and (2.10_2) instead of (2.9), then we can always deduce (see [10])

$$\sum_{i=1}^{n}\int_{A_{k,s}}|D_iu|^{q_i}dx\le$$

$$\le\theta\sum_{i=1}^{n}\int_{A_{k,t}}|D_iu|^{q_i}dx+c\int_{A_{k,R}}\left|\frac{u-k}{t-s}\right|^{\bar q^*}dx+c|A_{k,R}|$$

with notations similar to those introduced in the proof of theorem 1. From this inequality one gets again (2.3) and therefore, by lemma 2, the result.

REMARK. We remark that in the special case $q_i=p$, for $i=1,\dots,n,(2.10_3)$ becomes

$$c|z|^p\le f(x,z)\le c'(1+|z|^{p^*})$$

and so theorem 2 extends the analogous result (theorem 4.2) proved in [9].

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N.Fusco: Ist. di Matematica, Fac. Scienze, Univ. Salerno, 84100 Baronissi (Salerno). **C.Sbordone:** Dip. di Matematica e Applicazioni "R. Caccioppoli", Via Mezzocannone 8, 80134 Napoli

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