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HARMONIC MAP HEAT FLOW FOR AXIALLY SYMMETRIC DATA

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We examine the harmonic map heat flow problem for maps between the three-dimensional ball and the two-sphere. We give blow-up results for certain initial data. We establish convergence results for suitable axially symmetric initial data, and discuss generalizations to higher dimensions.

1. Introduction

We consider a compact smooth m-dimensional Riemannian manifold (M, g), possibly with nonempty boundary ∂M, and a compact smooth n-dimensional Riemannian manifold (N, h). Given $u \in C^1(M, N)$, we define the *energy density* of u at x by

$$e(u(x)) = \tfrac{1}{2} g^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij} ,$$

and the *energy* of u by

$$E(u) = \int_M e(u) dvol . \tag{1.1}$$

It is natural to seek a suitable domain of functions for which the integral (1.1) makes sense, in order to be able to find critical points. Via Nash's embedding theorem, we can consider N to be isometrically embedded in $\mathbb{R}^k$ for some k, so N is defined by a system of constraint equations

$$f_i(u) = 0 \quad i = 1, \ldots, k-n .$$

We can then consider (1.1) for u belonging to $\mathcal{D}$, where

$$\mathcal{D} = L^\infty(M,\mathbb{R}^k) \cap H^1(M,\mathbb{R}^k) \cap \{u \mid f_i(u(x)) = 0 \text{ for a.a. } x, i = 1,\dots,k-n\} .$$

The energy integral (1.1) then reduces to the standard Dirichlet integral

$$E(u) = \tfrac{1}{2} \int_M |\nabla u|^2 dvol , \tag{1.2}$$

where $|\nabla u|$ denotes the Hilbert-Schmidt norm. We obtain the Euler-Lagrange equation

$$\tau(u^i) = \Delta_M u^i - g^{\alpha\beta} A^i_u(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}) = 0, \quad i = 1,\dots,k , \tag{1.3}$$

where Δ_M is the Laplace-Beltrami operator on N, and A is the second fundamental form of N (refer to [16] for details). The field $\tau(u)$ is referred to as the tension field of u.

We call a solution u to (1.3) a *harmonic map*. For (1.3) to make sense we must have $u \in C^2(M,N) \cap \mathcal{D}$, although we can define a *weak harmonic map* to be $u \in \mathcal{D}$ such that u satisfies (1.3) in the weak sense, i.e.

$$\sum_{i=1}^k \int_M [g^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \xi^i}{\partial x^\beta} + g^{\alpha\beta} \xi^i A^i_u(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta})] dvol = 0 ,$$

for any $\xi \in C_0^\infty(M,\mathbb{R}^k)$. We have the following result for higher regularity of weak harmonic maps:

Theorem 1.1. *If $u \in C^0(M,N) \cap \mathcal{D}$ is weakly harmonic, then u is smooth.*

Proof. See [15], Chapter 8, Theorem 2.1. See also [16], Lemma 2.1, and [2], section 3.10.

The relationships of the energy given in (1.1) to the bulk energy of a nematic liquid crystal, and hence of harmonic maps to equilibrium configurations of liquid crystals are well documented: see for example [9].

In section 2 we summarize the nomenclature associated with the harmonic map heat flow. In section 3 we show that blow up can occur for the solution to this system in the case where we map from B^3 to S^2. The main theorem, presented in section 4, is also concerned with mappings from B^3 to S^2. This theorem asserts that given sufficiently regular,axially symmetric nonsurjective initial and boundary data we will have a unique regular solution to the harmonic map heat flow equation, which subconverges as $t \to \infty$ to a smooth harmonic map. We also discuss extensions to higher dimensions in section 5.

2. Heat Flow for Harmonic Maps

One approach to determining the existence of harmonic maps from M to N homotopic to a given u_0 is the so called "heat flow" method, whereby one considers the system

$$\partial_t u(x,t) = \tau(u(x,t)) \quad \text{on } M \times \mathbb{R}_+ \tag{2.1}$$

with initial conditions

$$u(x,0) = u_0(x) \quad \text{on } M \ . \tag{2.2}$$

We refer to t as the time variable, and x as the space variable. If ∂M is nonempty, then given $\varphi : \partial M \to N$ it is natural to ask whether or not there exists a harmonic map $u : M \to N$ such that $u\Big|_{\partial M} = \varphi$. One way of studying this question is to consider the problem (2.1), (2.2) for u_0 a suitable extension of φ to M, together with the Dirichlet boundary condition

$$u(x,t)\Big|_{\partial M} = \varphi(x) \quad \text{on } M \times \mathbb{R}_+ \ . \tag{2.3}$$

The problem (2.1), (2.2) has two useful properties. The first is short-term existence for suitable u_0.

Theorem 2.1. *Given $u_0 \in C^{2,\alpha}(M,N)$, there exists $T > 0$ depending only on (M,g) and (N,h) and u_0 such that (2.1), (2.2) possesses a unique solution $u(x,t)$ for $0 \le t < T$: the solution is of class $C^{2,\alpha;1,\alpha}$.*

Proof. See [11], p. 72; see also [8], p. 105 for uniqueness.

This result can be extended to the Dirichlet problem (2.1), (2.2), (2.3) for ∂M nonempty: see [8] p. 122.

The second property is the fact that the energy is nondecreasing. This follows directly from integrating the expression $\frac{d}{dt}E(u(\cdot,t))$ by parts: we obtain

$$\frac{d}{dt}E(u(\cdot,t)) = -\int_M \left(\frac{\partial u}{\partial t}\right)^2 dvol \ . \tag{2.4}$$

There are three broad classes of behaviour for the system (2.1), (2.2), or (2.1), (2.2), (2.3). We say that the solution $u(x,t)$ *blows up* at $t = T$ if

$$\limsup_{t \to T^-} \|\nabla u(\cdot,t)\|_\infty = \infty \ .$$

Then the system (2.1), (2.2) or (2.1), (2.2), (2.3) will behave in one of the following three ways:

i) the solution blows up in finite time, i.e. there exists $T > 0$ such that $u(\cdot, t)$ is regular for $0 \leq t < T$, but blows up at $t = T$;

ii) the solution is regular for all finite time, but blows up for $T = \infty$;

iii) the solution is regular for all time, and subconverges to a smooth harmonic map u_∞ (i.e. there exists a sequence $\{t_k\} \to \infty$ such that $u(\cdot, t_k)$ converges to a smooth $u_\infty(\cdot)$). Indeed, the solution may possibly converge to such a map. The resultant u_∞ is homotopic to u_0 (relatively homotopic in the event that ∂M is nonempty).

Examples exist for each type of behaviour. Eells and Sampson [3] introduced the heat flow method in 1964: they showed convergence of the solution of (2.1), (2.2) to a smooth harmonic map homotopic to the original ($t = 0$) map in the case where N has everywhere nonpositive sectional curvature. Hamilton [8] extended this to the case where ∂M is nonempty. Similar results were obtained by Jost [12] and von Wahl [18] in the case that the image of $u_0(M)$ lies in a geodesically small ball in N, and again by Hamilton [8] in the case that the image of $u_0(M)$ lies in a geodesically convex set in N.

There can be topological restrictions which prevent the system from having any chance of converging or subconverging. For example it was found by Eells and Wood [4] that there exist no harmonic maps $T^2 \to S^2$ with degree ± 1, whatever metrics are put on T^2 and S^2. It follows that the solution to (2.1), (2.2) for any initial data of degree ± 1 must blow up, although it may do so in finite or infinite time.

Coron and Ghidaglia [1] gave symmetric initial data $\mathbb{R}^n \to S^n$ and $S^n \to S^n$, $n \geq 3$ for which the heat flow (2.1), (2.2) blows up in finite time. In contrast Grayson and Hamilton [7] construct symmetric initial data $B^2 \to S^2$ such that the system (2.1), (2.2), (2.3) has a solution which is regular for all time, and has image S^2 for all time: the solution is prevented from converging to a smooth harmonic map because the only such map with the appropriate boundary data is a constant. It follows that blow up occurs at $T = \infty$.

3. Heat Flow from B^3 to S^2

We consider B^3 and S^2 as the (closed) unit ball and unit sphere respectively in $\mathbb{R}^3$, with the induced (standard) metric. In this case (2.1)–(2.3) takes the form

$$\left.\begin{aligned} u_t - \Delta u &= u|\nabla u|^2 \ , \\ u(0,t) &= u_0(\cdot) \ , \\ u(\cdot,t)\big|_{\partial B^3} &= \varphi(\cdot) \ . \end{aligned}\right\} \tag{3.1}$$

Firstly we show that, for suitable initial data, blow-up occurs for the harmonic heat flow.

Define S^3_+ to be the upper half unit hemisphere in $\mathbb{R}^4$, i.e.

$$S^3_+ = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ \Big| \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1, \quad x_1 > 0\}.$$

Note that $\overline{S}^3_+$ is diffeomorphic to B^3, and ∂S^3_+, the "equator" on S^3, is diffeomorphic to S^2.

We have the following Lemma:

Lemma 3.1. *Let $\varphi : \overline{S}^3_+ \to S^2$ be a smooth harmonic map which is constant on ∂S^3_+. Then φ is itself constant.*

Proof. This is a special case of Theorem 1.4 of [13]. See also [17], Lemma 2.5.

In an analogous manner to S^3_+, we can define

$$S^3_- = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ \Big| \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1, \quad x_1 < 0\}.$$

Let $S^\sharp = \partial S^3_+ = \partial S^3_-$.

We now state

Theorem 3.2. *There exists smooth initial data u_0 on B^3 such that the solution to the problem (2.1), (2.2), (2.3) blows up.*

Proof. Note that $S^3 = \frac{\overline{S}^3_+ \cup \overline{S}^3_-}{S^\sharp}$, so by the above diffeomorphism relationships, a map $v : S^3 \to S^2$ induces maps $v_1 : B^3 \to S^2$ (consider $v\big|_{\overline{S}^3_+}$) and $v_2 : B^3 \to S^2$ (consider $v\big|_{\overline{S}^3_-}$). Conversely given $v_1, v_2 : B^3 \to S^2$, with $v_1\big|_{\partial B^3} = v_2\big|_{\partial B^3}$, we can form $v = v_1 * v_2$ mapping $S^3 \to S^2$ by considering v_1 to map $\overline{S}^3_+$ and v_2 to map $\overline{S}^3_-$.

Now choose a point $\{p\}$ on S^2. Let u be a smooth map from S^3 to S^2 with nonzero Hopf invariant, such that $u\Big|_{S^3_-}$ is the constant map to p. Let $u_0 = u\Big|_{\overline{S}^3_+}$: by the above argument u_0 induces a map from B^3 to S^2, which we shall also call u_0. Let u_2 be the constant map from S^3_- to p. Then u_0 and u_2 agree on $S^\sharp$, so $u = u_1 * u_2$ is well defined.

On B^3, we consider $u_1(\cdot,t)$ to be the solution to (2.1), (2.2), (2.3) with initial data $u_0(\cdot)$, i.e. u_1 solves

$$\left.\begin{aligned} \frac{\partial}{\partial t}u_1 - \Delta u_1 &= u_1|\nabla u_1|^2 \\ u_1(\cdot,0) &= u_0(\cdot) \\ u_1(\cdot,t)\Big|_{\partial B^3} &= u_0\Big|_{\partial B^3} = p \end{aligned}\right\} \tag{3.2}$$

This induces a smooth family of homotopic maps on S^3, viz

$$u(\cdot,t) = u_1(\cdot,t) * u_2(\cdot).$$

If the solution to (3.2) does not blow up, it must subconverge for suitable $\{t_k\} \to \infty$ to a smooth harmonic map $u_\infty(\cdot)$ homotopic to $u_1(\cdot,0) = u_0(\cdot)$. By Lemma 3.1 the only such map is the constant map to p, so $u(\cdot,t_k)$ converges to a constant map, which has Hopf invariant 0.

However $u(\cdot,0)$ has nonzero Hopf invariant. As our deformation respects homotopy type and the Hopf invariant is integral, $u(\cdot,t_k)$ has nonzero Hopf invariant for all t_k, so $u_\infty(\cdot)$ also has nonzero Hopf invariant.

This is a contradiction, so the solution to (3.2) must blow up: this completes the proof of Theorem 3.2 □

4. Axially Symmetric Heat Flow from B^3 to S^2

We wish to consider the harmonic map heat flow from B^3 to S^2, (3.1), with specific coordinates, viz cylindrical coordinates on B^3 and spherical coordinates on S^2.

In rectangular coordinates,

$$B^3 = \{(x,y,z) \in \mathbb{R}^3 \quad \Big| \quad x^2+y^2+z^2 \le 1\}.$$

Introducing cylindrical coordinates by the equations

$$\left.\begin{aligned} x &= \mathrm{r}\cos\theta \\ y &= \mathrm{r}\sin\theta \\ z &= z \end{aligned}\right\}$$

we have

$$B^3 = \{(r,\theta,z) \quad \Big| \quad r^2+z^2 \le 1, \quad r \ge 0\},$$

where we identify (r,θ,z) with $(r,\theta+2\pi,z)$. In rectangular coordinates,

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \quad \Big| \quad x^2+y^2+z^2 \le 1\}.$$

Introducing spherical coordinates by the equations

$$\left.\begin{aligned} x &= \sin\psi\cos\chi \\ y &= \sin\psi\sin\chi \\ z &= \cos\psi, \end{aligned}\right\}$$

we have

$$S^2 = \{(\chi,\psi)\}$$

where we identify (χ,ψ) with $(\chi+2\pi,\psi)$ and with $(\chi,\psi+2\pi)$.

We can thus calculate the Christoffel symbols, and use the expression for the tension field in local coordinates, (1.3)to express (3.1) in our chosen coordinates, obtaining the equations

$$\chi_t = \Delta\chi + 2\cot\psi\ \langle\nabla\chi,\nabla\psi\rangle \tag{4.1}$$

$$\text{and} \qquad \psi_t = \Delta\psi - \frac{\sin 2\psi}{2}|\nabla\chi|^2, \tag{4.2}$$

with initial conditions

$$\chi(\cdot,0) = \chi_0(\cdot), \quad \psi(\cdot,0) = \psi_0(\cdot), \tag{4.3}$$

and boundary conditions

$$\chi(\cdot,t)\Big|_{\partial B^3} = \chi_0(\cdot)\Big|_{\partial B^3}, \quad \psi(\cdot,t)\Big|_{\partial B^3} = \psi_0(\cdot)\Big|_{\partial B^3}. \tag{4.4}$$

Note that in deriving these equations we allowed each variable in S^2 to depend upon all of the coordinates in B^3, i.e. $\chi_0 = \chi_0(r,\theta,z)$, $\chi = \chi(r,\theta,z,t)$, $\psi_0 =$

$\psi_0(r,\theta,z)$ and $\psi = \psi(r,\theta,z,t)$. We wish to restrict this considerably by searching for an axially symmetric harmonic map. These maps are studied in, for example, [19] and [10].

A function $u : B^3 \to S^2$ given by

$$(r,\theta,z) \mapsto (\chi,\psi)$$

is *axially symmetric* if

$$\psi = \psi(r,z) \quad \text{and} \quad \chi \equiv \theta.$$

Here the boundary data can be expressed in a simple form, for at $(r,\theta,z) \in \partial B^3$ we have $r = \sqrt{1-z^2}$, so defining a function g by

$$g(z) = \psi(\sqrt{1-z^2}, z),$$

we have

$$u\Big|_{\partial B^3} = (\theta, g(z)).$$

Consider the region D in (r,z) space given by

$$D = \left\{(r,z) \;\Big|\; r > 0,\ r^2 + z^2 < 1\right\}.$$

Let

$$\Gamma_1 = \left\{(r,z) \;\Big|\; r \geq 0, r^2 + z^2 = 1\right\},$$
$$\Gamma_2 = \left\{(0,z) \;\Big|\; -1 < z < 1\right\}.$$

Note that $\partial D = \Gamma_1 \dot{\cup} \Gamma_2$.

For such axially symmetric u, the energy is given by

$$\begin{aligned} E(u) &\overset{\text{def}}{=} \tfrac{1}{2}\int_{B^3} |\nabla u|^2 dvol \\ &= \pi\mathcal{E}(\psi)\ . \end{aligned}$$

where

$$\mathcal{E}(\psi) = \int_D \left(\psi_r^2 + \psi_z^2 + \frac{\sin^2\psi}{r^2}\right) r dr dz\ , \tag{4.5}$$

The L^2-gradient flow associated with the functional $\mathcal{E}$ is

$$\psi_t = \psi_{rr} + \psi_{zz} + \frac{\psi_r}{r} - \frac{\sin 2\psi}{2r^2} \ . \tag{4.6}$$

Suppose that we can solve the evolution problem (4.6) with initial and boundary conditions

$$\left.\begin{aligned} \psi(r,z,0) &= \psi_0(r,z) \\ \psi(r,z,t)\Big|_{\Gamma} &= g(z). \end{aligned}\right\} \tag{4.7}$$

Then the function $u(r,\theta,z,t)$ given by

$$(r,\theta,z,t) \mapsto (\theta, \psi(r,z,t))$$

will satisfy (4.2) and (4.3) for $r \neq 0$. If we can show that this extends to a solution for $r = 0$ also, then we will have found an axially symmetric solution to (4.2) and (4.3). Also the initial and boundary conditions (4.4) will be satisfied by

$$\begin{aligned} u(r,\theta,z,0) &= (\theta, \psi_0(r,z)) \quad \text{and} \\ u(r,\theta,z,t)\Big|_{\partial B^3} &= (\theta, g(z)), \end{aligned}$$

by (4.7).

There are two potential sources of difficulty in dealing with the problem (4.6), (4.7). The first is the singularity of the coefficients of ψ_r and $\sin 2\psi$ along Γ_2. The second is that we do not have a well posed boundary problem, as the data is not explicitly specified along Γ_2. We show now that the second problem can be alleviated by imposing the additional boundary condition

$$\psi(\cdot,t)\Big|_{\partial B^3} = 0 \ . \tag{4.8}$$

We consider continuous boundary data $(\theta, g(z))$ on ∂B^3. We firstly note that the North and South Poles of $B^3 (N = (0,\theta,1),\ S = (0,\theta,-1))$, must each be mapped to either the North or South Pole of S^2, i.e.

$$g(\pm 1) = j\pi, \quad j \in \mathbb{Z}.$$

This follows from the fact that the choice of θ is arbitrary in assigning coordinates $(0,\theta,1)$ to N, so

$$u(N) = u(0,\theta,1) = (\theta, g(1))$$

is a circle, and hence u is not well defined, unless the coordinates of the image of N are similarly independent of the choice of θ: the only points on S^2 for which this occurs are the North and South Poles. The same argument holds for S.

Suppose that the initial map $(\theta, \psi_0(r,z))$ is continuous. The above argument shows that

$$\psi_0(0,z) = j\pi \quad j \in \mathbb{Z}, -1 \leq z \leq 1 \ .$$

In particular

$$\psi_0(0,1) = \psi_0(0,-1), \quad \text{i.e. } g(1) = g(-1) \ .$$

We suppose now that our initial boundary data are continuous and axially symmetric, and further that our boundary data is of degree zero. We also assume that the image of the boundary is not all of S^2, so that we may assume that there exists a $\delta > 0$, such that

$$0 \leq \psi_0(r,z) < \pi - \delta \ .$$

By the above argument ,

$$\psi_0(0,z) = 0 \quad -1 \leq z \leq 1 \ .$$

We then have

Theorem 4.1. *Given axially symmetric boundary data $\varphi : \partial B^3 \to S^2$ which does not cover S^2, and which is Lipschitz continuous and of degree zero, and an axially symmetric initial extension $u_0 \in C^{2,\alpha}(B^3, S^2)$, which does not cover S^2 either, (3.1) subconverges as $t \to \infty$ to an axially symmetric smooth harmonic map homotopic to u_0. This map is also nonsurjective.*

The result follows from intermediate steps showing:

i) short term existence;

ii) no blow-up on the z-axis;

iii) no blow-up for $r > 0$; and

iv) convergence considerations.

Step i).

Lemma 4.2. *The problem (4.1)–(4.4) for axially symmetric initial and boundary data*

$$\chi_0(\cdot,0)=\theta, \qquad \psi_0(\cdot,0)=\psi_0(r,z,0)\in C^{2,\alpha}(D)$$
$$\chi(\cdot,t)\Big|_{\partial B^3}=\theta, \qquad \psi(\cdot,t)\Big|_{\partial B^3}=\psi(\sqrt{1-z^2},z,0)=g(z)\in C^{0,1}(\Gamma_1)$$

satisfying $0\le\psi_0<\pi$ *has a unique regular (i.e.* $C^{2,\alpha;1,\alpha}\left(IntB^3\times[0,T)\right)\cap C^{0,1;1,\alpha}$ $(B^3\times[0,T)))$ *axially symmetric solution for* $0\le t<T$ *for some* $T>0$.

Proof. We know by Theorem 2.1 and the regularity of the initial and boundary data that we have a unique regular solution (χ,ψ) to the evolution problem of Lemma 4.2 for $0\le t<T$, for some $T>0$. A priori this solution need not be axially symmetric. Consider then $\widetilde{\chi}$ and $\widetilde{\psi}$ defined by

$$\widetilde{\chi}(r,\theta,z,t)=\chi(r,\theta+\alpha,z,t)-\alpha$$
$$\widetilde{\psi}(r,\theta,z,t)=\psi(r,\theta+\alpha,z,t)$$

for α a fixed (arbitrary) real number. We can check that $(\widetilde{\chi},\widetilde{\psi})$ also solves the evolution problem of Lemma 4.2 : by the uniqueness result in Theorem 2.1 we can thus conclude that $(\widetilde{\chi},\widetilde{\psi})=(\chi,\psi)$. Since α is arbitrary, this shows that ψ is independent of θ, i.e.

$$\psi=\psi(r,z,t)\ . \tag{4.8}$$

Further we see that χ can be written in the form

$$\chi(r,\theta,z,t)=\theta+\beta(r,z,t)\ . \tag{4.9}$$

Using (4.8) and (4.9), we can reduce (4.1) and (4.2) to

$$\beta_t=\Delta\beta+2\cot\psi\ (\beta_r\psi_r+\beta_z\psi_z) \tag{4.10}$$

and

$$\psi_t=\Delta\psi-\frac{\sin 2\psi}{2}(\beta_r^2+\frac{1}{r^2}+\beta_z^2), \tag{4.11}$$

where Δ is the Laplacian in $\mathbb{R}^3$ (i.e. $\Delta=\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}$ as both β and ψ are independent of θ).

We see from (4.8), (4.9) and the initial and boundary conditions in Lemma 4.2 that β satisfies

$$\left.\begin{aligned} \beta(\cdot,0) &= 0 \\ \beta(\cdot,t)\Big|_{\partial B^3} &= 0 \end{aligned}\right\} \tag{4.12}$$

We shall use a maximum principle argument to show that $\beta(\cdot,t)$ is identically zero on $B^3 \times [0,T)$. We define γ by

$$\gamma(\cdot,t) = e^{-t}\beta(\cdot,t) :$$

then by (4.10) γ satisfies

$$\gamma + \gamma_t = \Delta\gamma + 2\cot\psi\ (\gamma_r\psi_r + \gamma_z\psi_z). \tag{4.13}$$

Suppose $\beta(\cdot,t)$ takes on a positive maximum on $B^3 \times [0,T']$, for some $T' > 0$. Then $\gamma(\cdot,t)$ must also take on a positive maximum (although not necessarily at the same point). This maximum for γ cannot be achieved on $\partial B^3 \times [0,T']$ nor on $B^3 \times \{t^{-1}(0)\}$ by (2.31). By (4.11) we see that this maximum also cannot be achieved at (r',θ',z',t') contained in $IntB^3 \times (0,T']$, unless possibly $\psi(r',\theta',z',t') = k\pi$, and

$$\lim_{(r,\theta,z,t)\to(r',\theta',z',t')}[\cot\psi\ (\gamma_r\psi_r + \gamma_z\psi_z)] > 0. \tag{4.14}$$

We note from our initial conditions and the regularity of the initial data that

$$0 \leq \psi(\cdot,t) \leq \pi\ . \tag{4.15}$$

To see this, we suppose $\inf\limits_{B^3}\psi(\cdot,t) < 0$ for some $t > 0$. Since the initial data is non-negative and regular, for T_2 sufficiently small

$$0 > \inf_{IntB^3\times[0,T_2]}\psi(\cdot,t) > -\frac{\pi}{2}\ . \tag{4.16}$$

Then ψ must take on a negative minimum on $IntB^3 \times (0,T_2]$: we see from (4.11) that this is not possible, given that at this minimum, by (4.16), we have

$$\frac{\sin 2\psi}{2}(\beta_r^2 + \frac{1}{r^2} + \beta_z^2) < 0\ .$$

Similarly since the initial data is less than π, we see that for any point $(\cdot,t_1)$ at which $\psi(\cdot,t_1) = \pi$, $\psi_t(\cdot,t_1) \leq 0$. This establishes (4.15).

Thus if $\psi(r', \theta', z', t') = 0$, we can conclude that ψ achieves a local minimum at (r', θ', z', t'). Since then γ achieves a maximum at (r', θ', z', t') and ψ a minimum a first derivative sign analysis in a small neighbourhood of (r', θ', z', t') yields a contradiction with (4.14) If $\psi(r', \theta', z', t') = \pi$, ψ achieves a local maximum at (r', θ', z', t'), and a similar argument shows (4.14) cannot hold in this case.

Thus $\gamma(\cdot, t)$, and hence $\beta(\cdot, t)$ cannot achieve a positive maximum on $B^3 \times [0, T']$ for any $T' < T$.

In the same way we show that $\beta(\cdot, t)$ cannot achieve a negative minimum on $B^3 \times [0, T']$ for any $T' < T$. We have thus shown $\beta(\cdot, t)$ is identically zero on $B^3 \times [0, T)$. By (4.9) then we have

$$\chi(r, \theta, z, t) = \theta.$$

Combining this with (4.8), we have shown that the solution to the initial-boundary value problem is axially symmetric for $0 \leq t < T$. □

It follows that we can restrict our attention to the behaviour of the system (4.6), (4.7), a semilinear parabolic equation. Further from (4.15) above, we know that the solution satisfies

$$0 \leq \psi(r, z, t) \leq \pi \qquad \text{for } (r, z) \in \overline{D}, \quad t \in [0, T).$$

Step ii). We need to show that the evolution problem (4.6), (4.7) cannot develop a singularity on the z-axis until after it develops one on $\overline{D} \setminus \Gamma_2$. Denote by T the time of "first blow-up", i.e.

$$T = \inf\{\widetilde{T} \;\Big|\; \limsup_{t \to \widetilde{T}^-} ||\nabla\psi(\cdot, t)||_\infty = \infty\}.$$

We note that if blow-up first occurs on Γ_2 then we have a sequence $(r_i', z_i', t_i) \to (0, z^*, T)$ for which $|\psi_r| \to \infty$. For if not, then we must have such a sequence for which $|\psi_z| \to \infty$. Since our solution is $C^{2,\alpha}$ for $t < T$, we can choose $\{\tilde{r}_i\}$ for which

$$||\psi(\cdot, t_i)||_{2,\alpha} < \frac{1}{\tilde{r}_i}. \tag{4.17}$$

In addition, since we have blow-up at $(0,z^*,T)$, for all i sufficiently large there exists $P=(r_i,z_i)$, $r_i<\tilde{r}_i$ with $|\psi_z(P,t_i)|>2$. But then

$$\begin{aligned}||\psi(\cdot,t_i)||_{2,\alpha} &\geq \max|\psi_{zr}(\cdot,t_i)| \\ &\geq \frac{|\psi_z(P,t_i)-\psi_z(0,z_i,t_i)|}{r_i} \\ &> \frac{2}{r_i} \quad \geq \frac{2}{\tilde{r}_i}\ .\end{aligned}$$

This contradicts (4.17), so we must have blow-up in the radial direction.

In particular if we can show that ψ is Lipschitz continuous on Γ_2 (uniformly in time), then we will have shown that first blow-up cannot occur on the z-axis. To do this, it suffices to show that ψ is bounded above pointwise on $\overline{D}\times[0,T)$ by a function $\xi(r,z,t)$ which is uniformly Lipschitz continuous on $\overline{D}\times[0,t)$.

We next give, via a maximum principle argument, sufficient conditions for a function to be a suitable barrier.

Lemma 4.3. *Suppose $\psi(\cdot,t)$ is a regular solution to (4.6), (4.7) on $[0,T)$. Let $\xi(\cdot,t)$ be a regular solution to (4.6) on $[0,T)$, i.e.*

$$\xi_t-\xi_{rr}-\xi_{zz}-\frac{\xi r}{r}+\frac{\sin 2\xi}{2r^2}=0\ . \tag{4.18}$$

In addition let ξ be Lipschitz continuous on Γ_2 (uniformly in time), and let ξ and ψ satisfy the initial and boundary relations

$$\left.\begin{aligned} &\xi(r,z,0)\geq\psi_0(r,z) \quad \text{on } D, \\ &\xi\Big|_{\Gamma_1\times[0,T)}\geq\psi\Big|_{\Gamma_1\times[0,T)}=\psi_0\Big|_{\Gamma_1}, \\ \text{and}\quad &\xi\Big|_{\Gamma_2\times[0,T)}=\psi\Big|_{\Gamma_2\times[0,T)}=0\ . \end{aligned}\right\} \tag{4.19}$$

Then $\xi\geq\psi$ on $\overline{D}\times[0,T)$.

Proof. Let $\eta=\psi-\xi$. By (4.19) $\eta\leq 0$ on $\overline{D}\times\{0\}$ and on $\partial D\times[0,T)$. By (4.6) and (4.18), η satisfies

$$\eta_t-\Delta\eta-\frac{\eta_r}{r}+\frac{a}{r^2}\eta=0 \tag{4.20}$$

on $D\times[0,T)$, where $\Delta=\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}$ is the Laplacian in (r,z) coordinates, and

$$\begin{aligned}a(r,z,t)=&\ \frac{\sin 2\xi-\sin 2\psi}{2(\xi-\psi)} \\ =&\ \int_0^1\cos 2[\sigma\xi+(1-\sigma)\psi]d\sigma\end{aligned}$$

is bounded and regular on $D \times [0,T)$: choose M such that $|a| < M$.

Suppose that the conclusion of the Lemma is false. Then there exists some time t', $0 < t' < T$, such that $\max_{\overline{D}\times[0,t']} \eta > 0$: by regularity we can choose t' sufficiently small that $0 < \max_{\overline{D}\times\{t\}} \eta \le \frac{\pi}{8}$. Using the regularity of ξ, we can choose $\rho > 0$ such that $\xi(r,z,t) \le \frac{\pi}{8}$ for $0 \le r \le \rho$, $t \in [0,T)$. Thus for $r \le \rho$ and $0 \le t < t'$,

$$\psi(r,z,t) - \xi(r,z,t) \le \frac{\pi}{8} \quad \text{and} \quad \xi(r,z,t) \le \frac{\pi}{8},$$

which means $\psi(r,z,t) \le \frac{\pi}{4}$, and thus by the definition of $a(\cdot,t)$ we have

$$a(r,z,t) \ge 0 \quad \text{for } r \le \rho,\ 0 \le t < T\ . \tag{4.21}$$

Multiplying (4.20) by $e^{-\frac{Mt}{2\rho^2}}$ and introducing $h(r,z,t)$ defined by $h = e^{-\frac{Mt}{2\rho^2}}\eta$, we have

$$h_t - \Delta h - \frac{hr}{r} + h\left(\frac{a}{2r^2} + \frac{M}{2\rho^2}\right) = 0\ . \tag{4.22}$$

As η takes on a positive maximum on $\overline{D} \times [0,t']$, h must do so as well. The function h is, like η, nonpositive on $\partial D \times [0,t']$ and on $D \times \{t^{-1}(0)\}$, so the positive maximum is achieved on $D \times (0,t']$, say at (r^*,z^*,t^*). By the regularity of η, and hence of h, we can conclude that $h_t(r^*,z^*,t^*) \ge 0$, $\Delta h(r^*,z^*,t^*) \le 0$ and $h_r(r^*,z^*,t^*) = 0$. From (4.22) then, we must have

$$\frac{a(r^*,z^*,t^*)}{2(r^*)^2} + \frac{M}{2\rho^2} \le 0\ . \tag{4.23}$$

If $r^* \le \rho$, by (4.21) $a(r^*,z^*,t^*) \ge 0$, so (4.23) is false. However if $r^* > \rho$, we have $\frac{a}{2(r^*)^2} > -\frac{M}{2\rho^2}$, so (4.23) is again false.

Thus the conclusion of the Lemma is valid. □

The formulation of $a(\cdot,t)$ as an integral to show its boundedness and regularity is used by Coron and Ghidaglia [1]: they use it in a barrier-type argument involving a self-similar supersolution to an equation analogous to (4.6) for dimension $n \ge 3$.

We have now reduced the problem of z-axis regularity to that of finding a regular uniformly Lipschitz continuous solution to (4.6) on $[0,\infty)$ which lies above the given initial and boundary data. This is easiest if we seek a static (time independent) solution of (4.6) : in fact we have a one-parameter family of such functions (which are functions of r only), as shown in section 5. We also show there that we can dominate any initial and boundary data by a member of this family, provided our data is sufficiently regular and bounded below π. Defining

$$D_\epsilon = \{(r,z) \in D \,\Big|\, r < \epsilon\} ,$$

we have

Lemma 4.4. *The evolution problem (4.6), (4.7) for initial and boundary data*

$$\psi_0 \in C^{2,\alpha}(D), \quad g \in \ Lip\,(\Gamma_1), \quad 0 \le \psi_0, g < \pi$$

satisfies

$$\sup_{\Gamma_2 \times [0,T)} ||\nabla\psi||_\infty < \infty \quad \textit{if} \quad \sup_{\overline{D}\setminus D_\epsilon \times [0,T_1)} ||\nabla\psi||_\infty < \infty, \quad \textit{for all } \epsilon > 0 \ ,$$

for all T_1, $0 < T_1 < T \le \infty$.

Proof. By Corollary 5.2 we can find a regular ξ which satisfies $\xi \ge \psi$ on $\overline{D} \times \{0\}$. This ξ is time independent, and has a bounded Lipschitz norm on Γ_2. By Lemma 4.3, as long as the solution is regular away from Γ_2, it does not cross the barrier ξ: but then the bound on the Lipschitz norm of ξ on Γ_2 gives the desired conclusion. □

Note that this Lemma precludes first blow up occurring simultaneously off and on the z-axis: the only possibility we need now consider is that of first blow-up off the z-axis.

Step iii). We now wish to show that the solution to (4.6), (4.7) cannot blow up off the z-axis before blowing up thereon.

Lemma 4.5. *Under the same conditions as in Lemma 4.4 if there exists* $\epsilon > 0$ *such that* $\sup_{D_\epsilon \times [0,T)} ||\nabla\psi||_\infty < \infty$, *then* $\sup_{D \times [0,T)} ||\nabla\psi||_\infty < \infty$, *for all* T, $0 \le T \le \infty$.

Proof. We need to show that there exists a C_1 such that

$$\sup_{D\setminus D_\epsilon \times [0,T)} ||\nabla\psi||_\infty < C_1 \ .$$

On $D \setminus D_\epsilon \times [0,T)$ we have

$$\psi_t - L\psi = f(\psi, r)$$

where $L\psi = \psi_{rr} + \psi_{zz} + \frac{\psi_r}{r}$ is an elliptic operator with bounded coefficients, and $f(\psi, r) = \frac{\sin 2\psi}{2r^2}$ satisfies

$$|f| \leq \frac{1}{2\epsilon^2} \quad \text{on } D \setminus D_\epsilon \ .$$

Thus we can apply the estimates for linear parabolic equations ([15] Ch. IV, Theorem 10.1) to conclude that there exists C_2 such that

$$\sup_{D\setminus D_\epsilon \times [0,T)} ||\psi||_{C^{2,\alpha}} < C_2 \ .$$

This implies the desired result. □

Step iv). From Lemma 4.4 and Lemma 4.5 then, it follows that there exists C such that

$$\sup_{t\in[0,\infty)} ||\nabla\psi(\cdot,t)||_\infty < C,$$

and further for any $\epsilon > 0$, there exists $C_1 = C_1(\epsilon)$ such that

$$\sup_{t\in[0,\infty)} ||\psi(\cdot,t)||_{C^{2,\alpha}} < C_1(\epsilon) \quad \text{on } D \setminus D_\epsilon \ .$$

It follows that there exists $\psi_\infty(\cdot)$ such that for any $\beta < \alpha$, $\psi_\infty \in C^{2,\beta}(D \setminus D_\epsilon)$, and $\{t_k\} \nearrow \infty$ such that

$$\psi(\cdot, t_k) \xrightarrow{C^{2,\beta}} \psi_\infty(\cdot) \quad \text{as } k \to \infty, \text{ on } D \setminus D_\epsilon \ . \tag{4.24}$$

Since $\mathcal{E}(\psi(\cdot,t))$ is finite and nonincreasing with t, integrating (4.6) from 0 to $T > 0$ gives us

$$\mathcal{E}(0) - \mathcal{E}(T) = \int_0^T \int_D \psi_t^2(x,t)dxdt \ ,$$

so

$$\psi_t(\cdot,t) \xrightarrow{L^2} 0 \quad \text{as } t \to \infty \ .$$

By (4.6) and (4.24) this gives us that

$$\psi_{rr}(\cdot,t_k) + \psi_{zz}(\cdot,t_k) + \frac{\psi_r(\cdot,t_k)}{r} - \frac{\sin 2\psi(\cdot,t_k)}{2r^2} \xrightarrow{L^2} 0 \quad \text{as } k \to \infty \ ,$$

for $r > 0$.

Thus u as given by $u(r,\theta,z) = (\theta, \psi(r,z))$, i.e.

$$u(x,y,z) = (\sin\psi_\infty \cos\frac{x}{r}, \ \sin\psi_\infty \sin\frac{y}{r}, \ \cos\psi_\infty) \tag{4.25}$$

is weakly harmonic for $r > 0$: since u is $C^{2,\beta}$ for $r > 0$, it is harmonic for $r > 0$.

We know by Lemma 4.4 that as $r \to 0$, $\psi_\infty \to 0$ and $||\nabla \psi_\infty||_\infty$ is bounded. Hence u as defined in (4.25) is uniformly Lipschitz continuous, in particular for $r = 0$. By using exactly the same method as Zhang ([19], (section 4)), we can show that u is weakly harmonic on all of B^3, so by Theorem 1.1 u is harmonic on B^3.

Note also that since $\psi(\cdot, t) \in C^{2,\alpha}(D \setminus D_\epsilon)$ for any $\epsilon > 0$, by (4.6) $\psi_t(\cdot, t) \in C^\alpha(D \setminus D_\epsilon)$, i.e. the solution to (4.6), (4.7) belongs to $C^{2,\alpha;1,\alpha}(D \setminus D_\epsilon)$ for any $\epsilon > 0$. Since $\psi_t \equiv 0$ for $r = 0$, $\psi(\cdot, t_1)$ is homotopic to $\psi(\cdot, t_2)$ for any t_1, t_2 (i.e. homotopic relative to g), and so $\psi(\cdot)$ is homotopic to $\psi(\cdot, t)$ for any t, in particular to $\psi(\cdot, 0) = \psi_0(\cdot)$.

This completes the proof of Theorem 4.1. □

5. Barrier Functions

We look for specific solutions to the evolution problem (4.6) on $D \times [0, \infty)$ which have finite energy, i.e. from (4.5)

$$\int_D [(\psi_r^2 + \psi_z^2)r + \frac{\sin^2 \psi}{r}] dr dz < \infty \tag{5.1}$$

We will look for solutions that are independent of r and t.

If we can find φ satisfying

$$\varphi'' + \frac{\varphi'}{r} - \frac{\sin 2\varphi}{2r^2} = 0 , \qquad ' = \frac{d}{dr} \tag{5.2}$$

for $r \in (0, 1)$, with

$$\int_0^1 [(\varphi')^2 r + \frac{\sin^2 \varphi}{r}] dr < \infty , \tag{5.3}$$

then $\psi(r, z, t) = \varphi(r)$ will satisfy (4.6) and (5.1). As we are interested in these barriers as they pertain to axially symmetric maps of degree zero from B^3 to S^2, we impose the boundary condition

$$\varphi(0) = 0. \tag{5.4}$$

To solve (5.2), (5.4) we make the substitution $r = e^\tau, \quad -\infty < \tau \leq 0$. Then we wish to solve

$$\ddot{\varphi} - \frac{\sin 2\varphi}{2} = 0, \qquad \dot{} = \frac{d}{d\tau} \tag{5.5}$$

$$\text{with} \qquad \lim_{r \to -\infty} \varphi(r) = 0. \tag{5.6}$$

Multiplying (5.5) by $\dot{\varphi}$ and integrating, we obtain

$$\dot{\varphi}^2 = \sin^2 \varphi + c \ , \tag{5.7}$$

and we see $c = 0$ from the boundary condition (5.6). If then we have a solution to

$$\dot{\varphi} = \sin \varphi \ , \tag{5.8}$$

by (5.7) it will also solve (5.5).

Equation (5.8) can be solved directly by integrating: we obtain

$$\operatorname{cosec} \varphi + \cot \varphi = \lambda e^{-\tau}$$

where λ is a constant of integration yet to be determined. This yields

$$\varphi(r) = \arccos\left(\frac{\lambda^2 - r^2}{\lambda^2 + r^2}\right) = 2 \arctan \frac{r}{\lambda} \ , \tag{5.9}$$

if $0 \leq \varphi \leq \pi$, upon reverting to the original r variable.

We can check directly that this solution satisfies (5.3). If we set $\lambda = \operatorname{cosec} \varphi_1 + \cot \varphi_1$, where $\varphi_1 = \varphi(1)$ is additional given boundary data satisfying $0 < \varphi_1 < \pi$, then $0 < \lambda < \infty$, and by (5.8), (5.9) gives us a solution of (5.2) which is monotone increasing from 0 at $r = 0$ to φ_1 at $r = 1$. We observe that $\varphi'(0) = \frac{2}{\lambda}$: this says that we can find a solution of (5.2) which grows as rapidly as is desired at $r = 0$. Further we note that $\varphi(r)$ is strictly convex for $r > 0$. In terms of problem (4.6) this means we can find a regular solution of finite energy which lies above given, "nice", initial and boundary data. The following, which makes this notion precise, is immediate:

Lemma 5.1. *Suppose $\eta(r) \in Lip\ [0,1]$, $\eta(0) = 0$, $\max_{[0,1]} \eta = \pi - \delta$, $\delta > 0$. Then there exists $\lambda > 0$ such that $\varphi(r) = 2 \arctan \frac{r}{\lambda}$ satisfies $\varphi \geq \eta$ on $[0,1]$, with strict inequality holding for $r > 0$.*

Corollary 5.2. *Consider problem (4.6), with initial and boundary conditions*

$$\psi(r,z,0) = \psi_0(r,z), \qquad \psi\Big|_{\partial D} = \psi_0\Big|_{\partial D},$$

with $\psi_0 \in Lip\ (\overline{D})$, $\psi_0(0,z) = 0$, $\psi_0 < \pi$. *This problem has a regular solution* $\xi(r,z,t)$ *which exists for all time, and which satisfies*

$$\xi(r,z,0) \geq \psi_0(r,z), \quad \xi\Big|_{\partial D} \geq \psi\Big|_{\partial D}.$$

Proof. Take $\eta(r) = \max_r \psi_0(r,z)$ in Lemma 5.1: $\xi(r,z,t) = \varphi(r)$ suffices. □

We wish to comment briefly on the extension of these methods to higher dimensions. We consider cylindrical coordinates on B^n, where $(x_1,\ldots,x_n) \in B^n$ is written as (r,z,θ), for

$$r = (x_1^2 + \ldots + x_{n-1}^2)^{\frac{1}{2}} ,$$

$$z = x_n$$

$$\text{and} \quad \theta \in S^{n-2} .$$

Then we consider axially symmetric $u : B^n \to S^{n-1}$ given by

$$u : (r,\theta,z) \mapsto (\psi,\theta) ,$$

where

$$\psi = \psi(r,z) \in S^1 .$$

In this case the energy of the map u is given by

$$E(u) \stackrel{\text{def}}{=} \tfrac{1}{2}\int |\nabla u|^2 dvol \quad = \frac{n\omega_n}{2}\mathcal{E}(\psi)$$

where ω_n is the volume of B^n, and

$$\mathcal{E}(\psi) = \int_D [\psi_r^2 + \psi_z^2 + \frac{n-2}{r^2}\sin^2\psi] r^{n-2} drdz .$$

The Euler-Lagrange equation of $\mathcal{E}(\psi)$ is

$$T\psi = \psi_{rr} + \psi_{zz} + \frac{n-2}{r}\psi_r - \frac{n-2}{2r^2}\sin 2\psi = 0 . \tag{5.10}$$

In fact for our barrier arguments, we really only need that ψ is a supersolution of (5.10).We look for a time and z independent barrier, i.e. φ satisfying

$$\varphi'' + \frac{n-2}{r}\varphi' - \frac{n-2}{2r^2}\sin 2\varphi \leq 0 \ , \tag{5.11}$$

where $' = \frac{d}{dr}$. Making the change of variable $\tau = \frac{r^{3-n}}{n-3}$, (5.11) becomes

$$\ddot{\varphi}\tau^2 - \frac{j^2}{2}\sin 2\varphi \leq 0 \ , \tag{5.12}$$

where $\dot{} = \frac{d}{d\tau}$, and $j = \sqrt{n-2}\ (n-3)$.

Multiplying through by $2\dot{\varphi}$ we obtain

$$(\dot{\varphi}^2\tau^2)^{\cdot} - 2\tau\dot{\varphi}^2 - j^2 \sin 2\varphi\dot{\varphi} \leq 0 \ . \tag{5.13}$$

Since $2\tau\dot{\varphi}^2 \geq 0$, (5.13) will certainly hold if

$$(\dot{\varphi}^2 t^2)^{\cdot} - j^2 \sin 2\varphi\dot{\varphi} = 0 \ \ : \tag{5.14}$$

as (5.13) is equivalent to (5.11), it suffices to solve (5.14).

Working as in (5.5)–(5.9), we obtain

$$\varphi(r) = 2\arctan\frac{r^j}{\lambda} \ . \tag{5.15}$$

For any $\lambda > 0$, the function given in (5.15) satisfies $\varphi(0) = 0$, and φ is strictly increasing for $0 \leq r \leq 1$. As was the case in dimension n=3, given $\varphi_1 : 0 < \varphi_1 < \pi$, setting $\lambda = \operatorname{cosec}\varphi_1 + \cot\varphi_1$ in (5.15) will yield $\varphi(1) = \varphi_1$.

We do not, however, have a full analogue of Lemma 5.1, because

$$\varphi(r) = 0(r^j) \quad \text{as } r \downarrow 0 \ ,$$

and so we only guarantee to dominate initial data which is also of order r^j as $r \downarrow 0$ (and which in addition, as in Lemma 5.1, lies below π).

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