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Uniqueness of least energy solutions to a semilinear elliptic equation in $\mathbb{R}^2$

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Summary. In this paper, we prove that solutions minimizing the nonlinear functional

$$\frac{\int |\nabla \varphi|^2}{(\int \varphi^{p+1})^{2/p+1}}$$

among the Sobolev space $H_0^1(\Omega)$ are unique when Ω is bounded convex domain in $\mathbb{R}^2$. This uniqueness's result is equivalent to saying that solutions obtained from the Mountain Pass Lemma for the equation $\Delta u + u^p = 0$ are unique. We also prove that the level set of the unique solution is strictly convex.

Key words: Semilinear elliptic equation, solutions at the least energy, uniqueness, Pohozaev's identity

1. Introduction

In this paper, we want to study the question of uniqueness of solutions of

$$(1)\quad \begin{cases} \Delta u + u^p = 0, & \text{and } u > 0 \text{ in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian, and $p > 1$. It is well-known that (1) has a unique solution when Ω is a ball. Although a domain Ω could easily be constructed so that (1) has more than one solution (See [D1]), the question whether solutions of (1) are unique or not for convex domains is still open. In this paper, we will give a partial answer to this problem.

Consider the following minimizing problem,

$$(2) \qquad m_p \equiv \inf\{\int |\nabla \phi|^2 \mid \phi \in H_0^1(\Omega), \quad \text{and} \quad \int \phi^{p+1} = 1\}$$

By Kondrachov's compactness theorem, the infinimum of (2) can always be achieved by some positive function. Our main result is concerned with the uniqueness of functions which achieve the infinimum.

Theorem 1. *Suppose that Ω is bounded and convex in $\mathbb{R}^2$. Then solutions which achieve the infinimum of (2) are unique.*

Suppose v is one solution minimizing the L^2 norm of the gradient of functions among the Sobolev space $H_0^1(\Omega)$. Set $u = (m_p)^{-1/p-1}v$. Then u is a solution of (1). In fact, this solution u can be also obtained by the Mountain Pass Lemma if u is considered as a critical point of the functional F which is defined by

$$F(\phi) = \frac{1}{2}\int |\nabla \phi|^2 - \frac{1}{p+1}\int (\phi^+)^{p+1}$$

where $\phi^+ = \max(\phi, 0)$. Hence Theorem 1 implies that solutions of (1) obtained by the Mountain Pass Lemma are unique provided that Ω is convex.

The proof of Theorem 1 will be given in the next section. After establishing the uniqueness's result, and using a result due to Korevaar and Lewis [KL], we can prove that the level set of the unique solution is always strictly convex.

2. Proof of Theorem 1

Let v be a solution which achieves the infinimum (2). Then v satisfies

$$(3) \qquad \begin{cases} \Delta v + m_p v^p = 0, v > 0 & \text{in } \Omega, \\ \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

Lemma 1. *The linearized equation of (3) at v has a nonnegative second eigenvalues.*

Proof. For any $\phi \in C_0^\infty(\Omega)$, define

$$f(t) = \int |\nabla (v + t\phi)|^2 / [\int_\Omega (v + t\phi)^{p+1}]^{2/p+1}.$$

Since f has its minimum at $t = 0$, we have $f'(0) = 0$, and $f''(0) \geq 0$. A straightforward computation shows that

$$f''(0) = 2\{\int |\nabla (\phi)|^2 - pm_p \int v^{p-1}\phi^2 + \frac{p+1}{2}(\int v^p\phi)^2\}.$$

Applying the minimax principle for the second eigenvalue λ_2 for the linearized operator $\Delta + pm_p v^{p-1}$, we have

$$\lambda_2 \geq \inf_{\phi \perp v^p} \{ \int |\nabla \phi|^2 - pm_p \int v^{p-1}\phi^2 \} \geq 0.$$

Q.E.D.

Set $u = (m_p)^{-\frac{1}{p-1}} v$, then u is a solution of (1). Lemma 1 is equivalent to saying that the second eigenvalue λ_2 of $\Delta + pu^{p-1}$ is nonnegative. Next, we will show that $\lambda_2 \neq 0$.

Lemma 2. *Suppose that u is a solution of (1), and the linearized equation at u has a nonnegative second eigenvalue λ_2, then $\lambda_2 > 0$*

Proof. Assume that $\lambda_2 = 0$, and ϕ is a second eigenfunction; namely,

$$(4) \qquad \begin{cases} \Delta\phi + pu^{p-1}\phi = 0, & \text{in } \Omega \\ \phi = 0, & \text{on } \partial\Omega. \end{cases}$$

Fix a point $Q = (x_0, y_0) \in \mathbb{R}^2$ which will be chosen later. Let T be a differential operator of the first order defined by

$$T = (x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}.$$

Let $w = Tu$. And, by $\Delta T = T\Delta + 2\Delta$, we have

$$(5) \qquad \Delta w + pu^{p-1}w = -2u^p$$

Applying Green's Theorem and (3) & (4), we have

$$(p-1)\int u^p\phi = \int_\Omega (\phi\Delta u - u\Delta\phi) = 0.$$

Also, by (4) and (5), we have

$$(6) \qquad -\int_{\partial\Omega} w\frac{\partial\phi}{\partial\nu}ds = \int_\Omega (\phi\Delta w - w\Delta\phi) = -2\int_\Omega u^p\phi = 0.$$

By Courant's nodal line theorem, the nodal line $\{x \in \Omega \mid \phi(x) = 0\}$ divides Ω into two subdomains. Two case occur: Either the nodal line of ϕ enclosed a region in Ω or the nodal line intersects with the boundary $\partial\Omega$ at exactly two points. If the first case happens, Q could be chosen to be any interior point of Ω. Then, $\forall p = (x, y) \in \partial\Omega, w = (x - x_0)\frac{\partial u}{\partial x} + (y - y_0)\frac{\partial u}{\partial y} = \overrightarrow{Qp} \cdot \nabla u < 0$, and $\frac{\partial\phi}{\partial\nu}$ has only one sign on $\partial\Omega$. Then (6) implies that $\frac{\partial\phi}{\partial\nu} \equiv 0$ on $\partial\Omega$, which is impossible by Hopf's boundary lemma.

Suppose the second case happens. Let $p_i \in \partial\Omega, i = 1, 2$, be these two points at which the nodal line of ϕ intersects with $\partial\Omega$. Suppose that the tangent line L_i at p_i are not parallel. Then we choose Q to be the intersection point of L_1 and L_2. Therefore w and $\frac{\partial\phi}{\partial\nu}$ both simultaneously change sign at $p_i, i = 1, 2$, and $w\frac{\partial\phi}{\partial\nu}$ has only one sign on $\partial\Omega$. Again, (6) implies $\frac{\partial\phi}{\partial\nu} \equiv 0$ on $\partial\Omega$, which is a contradiction to the Hopf's boundary lemma. If L_i are parallel, we may assume that the direction is in the x_1-direction. Set $w = \frac{\partial u}{\partial x_1}$, and repeat the same argument as before. We can obtain

$$\int_{\partial\Omega} w\frac{\partial\phi}{\partial x_1} = 0$$

which implies $\frac{\partial\phi}{\partial x_1} \equiv 0$ on $\partial\Omega$, a contradiction to the Hopf's boundary Lemma again. Hence the proof of Lemma 2 is complete.

Q.E.D.

To prove Theorem 1, we need another lemma.

Lemma 3. *Suppose that $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain, then there exists $p_0 > 1$ such that (1) has a unique solution for $1 < p \leq p_0$.*

Proof. The Lemma will be proved by several steps.

Step 1. Suppose that u_1, u_2 are two distinct solutions of (1.1), then $u_1 - u_2$ must change sign.

By (1), we have,

$$0 = \int_\Omega u_2\Delta u_1 - u_1\Delta u_2 = \int_\Omega u_1u_2(u_1^{p-1} - u_2^{p-1}).$$

If $u_1 \geq u_2$, then it implies $u_1 \equiv u_2$.

Now, let u_n be solution of (1) with $p = p_n$, and p_n tends to 1. And let $M_n = \sup_{\overline{\Omega}} u_n = u_n(Q_n)$, for some $Q_n \in \Omega$.

Step 2. $M_n^{p_n-1}$ tends to λ_1 as $p_n \to 1$, where λ_1 is the first eigenvalue of Δ with respect to Dirichlet's problem.

By a blowing-up argument, we can show that $M_n^{p_n-1}$ is bounded. Suppose that $M_n^{p_n-1}$ tends to $+\infty$, Set

$$u_n^*(x) = u_n(\epsilon_n x + Q_n)/M_n, \quad \text{where} \quad \epsilon_n^2 = \frac{1}{M_n^{p_n-1}}.$$

By the method of moving plane, Q_n is always away from the boundary $\partial\Omega$. Applying standard estimates of elliptic equations, u_n^* is uniformly convergent to a function u in $C^2(K)$ for any compact set K in $\mathbb{R}^n$, and u satisfies

$$(7)\qquad \begin{cases} \Delta u + u = 0, \quad u > 0 \text{ in } \Omega \\ \\ u(0) = 1 \end{cases}$$

Let $\lambda_{\mathbb{R}}$ and $\phi_{\mathbb{R}}$ be respectively the first eigenvalue and eigenfunction of Δ for the ball $B_{\mathbb{R}}(0)$. If $\mathbb{R}$ is large, we have,

$$\begin{aligned} 0 &> \int_{\partial B_R(0)} u\frac{\partial\phi_{\mathbb{R}}}{\partial\nu}ds = \int_{B_{\mathbb{R}}(0)} u\Delta\phi_{\mathbb{R}} - \phi_{\mathbb{R}}\Delta u \\ &= (1-\lambda_{\mathbb{R}})\int_{B_{\mathbb{R}}(0)} u\phi_{\mathbb{R}} > 0, \end{aligned}$$

a contradiction. Therefore, $M_n^{p_n-1}$ is bounded.

Now, suppose that λ is any accumulation value of $M_n^{p_n-1}$. Let $\overline{u}_n = u_n/M_n$, and

$$\begin{cases} \Delta\overline{u}_n + M_n^{p_n-1}\overline{u_n}^{p_n} = 0 \quad \text{in } \Omega, \\ \\ \overline{u}_n \mid_{\partial\Omega} = 0. \end{cases}$$

By elliptic estimates, $\overline{u}_n$ uniformly converges to $\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$, and $\overline{u}$ satisfies

$$\begin{cases} \Delta\overline{u} + \lambda\overline{u} = 0, \quad \overline{u} > 0 \quad \text{in } \Omega, \\ \\ \overline{u} \mid_{\partial\Omega} = 0. \end{cases}$$

Hence, λ and $\overline{u}$ must be respectively the first eigenvalue and the first eigenfunction of Δ. And the proof of step 2 is complete.

The final step. Suppose that Lemma 3 is false. u_n and v_n are two solutions of (1) with $p = p_n$ and p_n tends to 1. From step 2, we know that both $u_n^{p_n-1}$ and $v_n^{p_n-1}$ uniformly converges to λ_1 in any compact set in Ω. Set

$$\phi_n = \frac{u_n - v_v}{\|u_n - v_n\|_{L^\infty}}.$$

Then ϕ_n satisfies

$$\begin{cases} \Delta\phi_n + V_n(x)\phi_n = 0 \quad \text{in } \Omega, \\ \\ \phi_n = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

where $V_n(x) = \dfrac{u_n^{p_n} - v_n^{p_n}}{u_n - v_n}$ which converges to λ_1 by Step 2. It is easy to see that ϕ_n uniformly converges to the first eigenfunction of Δ. But, by Step 1, ϕ_n is

k-th eigenfunction for some $k \geq 2$. This leads to a contradiction. Therefore, we have finished our proof of Lemma 3.

Q.E.D.

Now we are in the position to prove Theorem 1.

Proof of Theorem 1. Suppose there are two functions $v_i \in H_0^1(\Omega)$, $i = 1, 2$, to achieve the infinimum of (2) with $p_o > 1$. Let $u_i = (m_{p_0})^{-\frac{1}{p-1}} v_i$. Since (1) is superlinear, the first eigenvalue of the linearized operator is always negative. By Lemma 2, the linearized equation of (1) at u_i is non-singular. By implicit function's theorem, we can construct two solutions $u_i(p)$ of 1 for $1 < p < p_0$, and by Lemma 2, the linearized equation at $u_i(p)$ are always nonsingular. But, by Lemma 2, there exists $p_1 > 1$ such that $u_1(p_1) = u_2(p_1)$, and the linearized equation is singular. This leads to a contradiction. Therefore, the proof of Theorem 1 is complete.

Q.E.D.

Our nex theorem is

Theorem 2. *Let v be the unique solution stated in Main Theorem, Then $v^{-\frac{p-1}{2}}$ is convex in Ω.*

Proof. Without loss of generality, we may assume that Ω is smooth and strictly convex. Set $M_p = \sup_{\overline{\Omega}} v_p$, where v_p denote the unique minimizing solution. By the step 2 of Lemma 3, v_p/M_p uniformly converges to the first eigenfunction v_1 of Δ. By a well-known theorem, $-\log v_1$ is strictly convex. Hence, $-\log v_p$ is strictly convex and then $v_p^{-\frac{p-1}{2}}$ is strictly convex for all p close to 1. Suppose that Theorem 2 fails for some $p = p_0$. Hence, there exists $p \leq p_0$ such that $v_p^{-\frac{p-1}{2}}$ is convex but is not strictly convex. Let $u = v^{-\frac{p-1}{2}}$, then u satisfies

$$\Delta u = f(u, \nabla u) \equiv \frac{\alpha + (1 + \frac{1}{\alpha})|\nabla u|}{u}$$

where $\alpha = \dfrac{p-1}{2}$. Obviously $\dfrac{1}{f}$ is convex in u. By a theorem of Korevaar and Lewis [KL], we know that the rank of the Hessian of $\partial^2 u$ is constant throughout Ω. But the rank $(\partial^2 u)(x)$ is two for x near $\partial\Omega$. Therefore, u is strictly convex at p. This is a contradiction. Therefore, the proof of Theorem 2 is complete.

Q.E.D

References

[BC] V. Benci and G. Cerami, The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems. Arch. Rational Mech. Anal. 114(1991), 79-93

[D1] E. N. Dancer, The effect of domain shape on the number of positive solutions of certain nonlinear equations. J. of Differential Equations 74(1988), 120-156

[D2] E. N. Dancer, The effect of domain shape on the number of positive solutions of certain nonlinear equations, (II). J. of Differential Equations 87(1990), 316-339

[KL] N. J. Korevaar and J. L. Lewis, Convex solutions of certain elliptic equations have constant rank Hessian. Arch. Rational Mech. Anal. 97(1987), 19-32

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