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The topology of a moduli space for linear dynamical systems

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1. Introduction

Several basic questions in linear control theory are related to problems concerning the topology of spaces of linear dynamical systems as e.g. the orbit space $\Sigma_{n,m,p}(\mathbb{F})$ of controllable linear systems given by

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

(with m inputs, p outputs and state space $\mathbb{F}^n$) or the space $\mathrm{Rat}_{n,m,p}(\mathbb{F})$ of all $p \times m$ proper rational transfer matrices

$$G(s) = C(sI - A)^{-1}B \in \mathbb{F}^{p\times m}(s)$$

with McMillan degree n.

To illustrate this point a bit, we recall (Hermann and Martin [17]) that any rational transfer matrix $G \in \mathrm{Rat}_{n,m,p}(\mathbb{F})$ defines a unique holomorphic map $\varphi_G : P_1(\mathbb{C}) \to G_m(\mathbb{C}^{m+p})$ into the Grassmann manifold $G_m(\mathbb{C}^{m+p})$ which sends each $s \in \mathbb{C}$ to the graph of the linear map $G(s): \mathbb{C}^m \to \mathbb{C}^p$. Moreover, in this way the space $\mathrm{Rat}_{n,m,p}(\mathbb{C})$ is identified with the complex manifold of all holomorphic maps

$$\varphi : P_1(\mathbb{C}) \to G_m(\mathbb{C}^{m+p}),$$

of degree n, which satisfy the base point condition $\varphi(\infty) = \mathbb{C}^m$.

By means of this construction, Hermann and Martin [17] proved that the system theoretically defined McMillan degree of a transfer matrix $G(s)$ is equal to the first Chern class of a certain holomorphic vector bundle ξ_G on $P_1(\mathbb{C})$, hence a topological invariant. Here ξ_G is defined as the pull back of the dual bundle U^* of the universal vector bundle U on $G_m(\mathbb{C}^{m+p})$ via the Hermann–Martin map

$\varphi_G : P_1(\mathbb{C}) \to G_m(\mathbb{C}^{m+p})$. Moreover, the Birkhoff–Grothendieck decomposition

$$\xi_G \cong \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_m)$$

turns out to be equivalent with Brunovsky's canonical form [5], which is of well known importance in systems theory; see [15], [17], [24].

We further note that the space $\mathrm{Rat}_{n,m,p}(\mathbb{C})$ of linear systems and in particular the manifold $\mathrm{Rat}_{n,m,1}(\mathbb{C})$ of based holomorphic maps from $P_1(\mathbb{C})$ to $P_m(\mathbb{C})$ arises also naturally in physics, namely in the so-called "nonlinear σ-models" of two-dimensional Yang–Mills theory; see e.g. Atiyah [1], Atiyah and Jones [2].

Despite the great importance of the moduli spaces $\mathrm{Rat}_{n,m,p}(\mathbb{F})$ their topology is still not sufficiently understood. Partial results have been obtained by e.g. Brockett [4], Byrnes and Duncan [9], Delchamps [11], Segal [25]; see also [16] and section 5 of this paper.

In this paper another natural class of linear dynamical systems is studied: the orbit space $\Sigma_{n,m}(\mathbb{F})$ of all controllable linear systems. This space $\Sigma_{n,m}(\mathbb{F})$ has the advantage to be easier to analyse than $\mathrm{Rat}_{n,m,p}(\mathbb{F})$, furthermore the vector bundle $\Sigma_{n,m,p}(\mathbb{F})$ on $\Sigma_{n,m}(\mathbb{F})$ (defined in section 5) may serve as a "partial compactification" for $\mathrm{Rat}_{n,m,p}(\mathbb{F})$.

To define $\Sigma_{n,m}(\mathbb{F})$, recall that a linear dynamical system

$$(A, B) : \dot{x}(t) = Ax(t) + Bu(t)$$

with $x(t) \in \mathbb{F}^n$, $u(t) \in \mathbb{F}^m$, $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$, ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is *controllable* iff the generic rank condition $\mathrm{rk}\,(B, AB, \ldots, A^{n-1}B) = n$ holds. This condition implies that for any states x_0, x_1 in $\mathbb{F}^n$ and times $t_0 < t_1$ there exists a control function u on $[t_0, t_1]$ and a solution $x(t)$ of (A, B) with $x(t_0) = x_0$, $x(t_1) = x_1$.

Let $\tilde{\Sigma}_{n,m}(\mathbb{F}) := \{(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \mid \mathrm{rk}\,(B, AB, \ldots, A^{n-1}B) = n\}$ denote the Zariski-open set of all controllable systems (A, B). Any linear change of coordinates $z = Sx$ in the state space $\mathbb{F}^n$ transforms (A, B) into the equivalent system

$$(SAS^{-1}, SB) : \dot{z}(t) = SAS^{-1}z(t) + SBu(t).$$

This defines an algebraic group action on $\tilde{\Sigma}_{n,m}(\mathbb{F})$

$$\alpha : GL_n(\mathbb{F}) \times \tilde{\Sigma}_{n,m}(\mathbb{F}) \to \tilde{\Sigma}_{n,m}(\mathbb{F})$$

$$(S, A, B) \mapsto (SAS^{-1}, SB),$$

called the similarity action on $\tilde{\Sigma}_{n,m}(\mathbb{F})$. Each two similar systems (A, B), (SAS^{-1}, SB) have the same systemtheoretic properties. Therefore the orbit space

$$\Sigma_{n,m}(\mathbb{F}) := \tilde{\Sigma}_{n,m}(\mathbb{F})/GL_n(\mathbb{F})$$

of the similarity action should be viewed as the true space of all controllable linear systems.

We always endow $\Sigma_{n,m}(\mathbb{F})$ with the quotient topology.

Previous work of Hazewinkel, Kalman [13], [14], Byrnes, Hurt [7], [10] has shown that $\Sigma_{n,m}(\mathbb{F})$ is a connected algebraic manifold of dimension mn; $\Sigma_{n,m}(\mathbb{F})$ is non-compact and for $n = 1$ or $m = 1$ there are diffeomorphisms

$$\Sigma_{n,1}(\mathbb{F}) \cong \mathbb{F}^n \tag{1}$$

$$\Sigma_{1,m}(\mathbb{F}) \cong \mathbb{F} \times P_{m-1}(\mathbb{F}) \tag{2}$$

In particular $\Sigma_{n,m}(\mathbb{F})$ is a generalization of projective spaces.

Byrnes [7] has shown that $\Sigma_{n,m}(\mathbb{F})$ is homologically nontrivial for $m > 1$ by finding lower bounds for the Betti numbers. In [6] the author constructed a cell decomposition of $\Sigma_{n,m}(\mathbb{F})$ to determine the Betti numbers. By a direct calculation it was found that $\Sigma_{n,m}(\mathbb{C})$ has the same homology groups as the Grassmann manifold $G_n(\mathbb{C}^{m+n-1})$. However the method of [16] worked only over the field of complex numbers $\mathbb{F} = \mathbb{C}$. In this paper a different cell decomposition of $\Sigma_{n,m}(\mathbb{F})$ is constructed which will enable us to compute also the mod 2 Betti numbers of $\Sigma_{n,m}(\mathbb{R})$. By combining these calculations with [16] we will show that again the mod 2 Betti numbers of $\Sigma_{n,m}(\mathbb{R})$ coincide with those of the Grassmann manifolds $G_n(\mathbb{R}^{m+n-1})$.

One should perhaps remark that besides these computational coincidences of the Betti numbers of $\Sigma_{n,m}(\mathbb{F})$ with those of $G_n(\mathbb{F}^{m+n-1})$, no *direct* relation to the Grassmann manifold $G_n(\mathbb{F}^{m+n-1})$ is known so far. Nevertheless it appears that the orbit space $\Sigma_{n,m}(\mathbb{F})$ of controllable linear systems shares many interesting topological properties with the Grassmann manifold.

This paper is organized as follows: In section 2 we show that a well known set of arithmetic invariants for the similarity action α – the Kronecker indices of (A, B) – *define a Whitney stratification of* $\tilde{\Sigma}_{n,m}(\mathbb{F})$. The main technical result of this paper appears in section 3 where we explicitly characterize those Kronecker strata which are contained in the closure of a given one. These are described by an ordering on the set of combinations. To prove our main result Theorem 3.1 we need an explicit description of the covers of this ordering; this is done in Appendix A. The Whitney stratification of $\tilde{\Sigma}_{n,m}(\mathbb{F})$ induces a cellular decomposition of the orbit space $\Sigma_{n,m}(\mathbb{F})$. Using a result of Borel and Haefliger we compute the Betti numbers of $\Sigma_{n,m}(\mathbb{F})$ and then prove that the mod 2 Betti numbers of $\Sigma_{n,m}(\mathbb{R})$ are equal to those of the real Grassmannian $G_n(\mathbb{R}^{m+n-1})$. Section 5 deals with the Betti numbers of $\mathrm{Rat}_{n,m,p}(\mathbb{R})$. Using a result of [16] (where it is shown that the spaces $\mathrm{Rat}_{n,m,p}(\mathbb{F})$ and $\Sigma_{n,m}(\mathbb{F})$ are homotopy equivalent up to a certain degree), we apply our previous results on $\Sigma_{n,m}(\mathbb{R})$ to determine the first $\max(m,p)-1$ mod 2 Betti numbers of $\mathrm{Rat}_{n,m,p}(\mathbb{R})$.

This work was part of the author's doctoral thesis [16a] written at the University of Bremen.

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2. Kronecker indices

We start by describing a well known class of arithmetic invariants for the similarity $\alpha : GL_n(\mathbb{F}) \times \tilde{\Sigma}_{n,m}(\mathbb{F}) \to \tilde{\Sigma}_{n,m}(\mathbb{F})$, introduced by Brunovsky [5], Popov [21].

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Let $(A, B) \in \tilde{\Sigma}_{n,m}(\mathbb{F})$ be a controllable linear system and let $b_1, \ldots, b_m$ denote the column vectors of the $n \times m$-matrix B. Consider the following *deletion procedure* on mn vectors of $\mathbb{F}^n$:

> Delete in the list $(b_1, \ldots, b_m, Ab_1, \ldots, Ab_m, \ldots, A^{n-1}b_1, \ldots, A^{n-1}b_m)$, while going from the left to the right, all vectors A^ib_j which are linear dependent on the predecessors.

Symbolically:

$$
\begin{array}{l}
b_1 \to b_2 \to \cdots \to b_m \,\urcorner \\
\hookrightarrow Ab_1 \to Ab_2 \to \cdots \to Ab_m \,\urcorner \\
\hookrightarrow \quad \vdots \qquad\qquad \vdots \,\urcorner \\
\hookrightarrow A^{n-1}b_1 \to A^{n-1}b_2 \to \cdots \to A^{n-1}b_m
\end{array}
$$

After a suitable permutation of the remaining vectors one obtains a basis

$$(b_1, Ab_1, \ldots, A^{K_1-1}b_1, \ldots, b_m, Ab_m, \ldots, A^{K_m-1}b_m) \in GL_n(\mathbb{F})$$

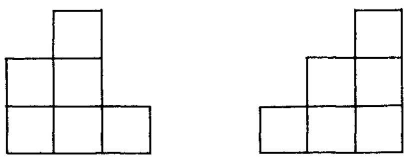
with certain non-negative integers $K_1, \ldots, K_m$ satisfying $K_1 + \cdots + K_m = n$. The m-tuple $K = K(A, B) = (K_1, \ldots, K_m)$ is called the list of *Kronecker indices* of (A, B). By construction, the Kronecker indices are invariant with respect to the similarity action, i.e. for all $S \in GL_n(\mathbb{F})$

$$K(SAS^{-1}, SB) = K(A, B) \tag{2.1}$$

Any m-tuple of non-negative integers $(K_1, \ldots, K_m)$ with sum equal to n is called a *combination of* n with length m; let $K_{n,m}$ denote the set of all such combinations. The number of these combinations is equal to the binomial coefficient

$$\text{card } K_{n,m} = \binom{n+m-1}{n}.$$

A combination $(K_1, \ldots, K_m)$ of n can be visualized by a Young diagram of appropriate size; for example the Young diagrams for (2, 3, 1) resp. (1, 2, 3) are



(2, 3, 1) (1, 2, 3)

Figure 1. Young diagrams.

Remark 2.1. The *set* $\{K_1, \ldots, K_m\}$ of Kronecker indices of (A, B) coincides with the set of minimal indices for the singular matrix pencil $(sI_n - A, B)$. These minimal indices were studied by Kronecker [20], extending earlier work of Weierstrass [25] on regular matrix pencils. The system theoretic interpretation of the minimal indices is due to Kalman [19].

The following lemma is proved in [16]:

LEMMA 2.2. *For any combination* $K \in K_{n,m}$, *the set*

$\text{Kro}_K(\mathbb{F}) := \{(A, B) \in \tilde{\Sigma}_{n,m}(\mathbb{F}) \mid (A, B)$ *has Kronecker indices* $K\}$ *is an analytic* *submanifold of* $\tilde{\Sigma}_{n,m}(\mathbb{F})$ *of dimension*

$$n^2 + \sum_{i,j=1}^{m} \min(K_i, K_j) + \sum_{i>j} K_{ij}, \qquad K_{ij} = \begin{cases} 1 & K_i < K_j \\ 0 & K_i \geq K_j \end{cases}.$$

We call the submanifolds $\mathrm{Kro}_K(\mathbb{F})$ the *Kronecker strata* of $\tilde{\Sigma}_{n,m}(\mathbb{F})$. They form a decomposition

$$\tilde{\Sigma}_{n,m}(\mathbb{F}) = \bigcup_{K \in \mathbf{K}_{n,m}} \mathrm{Kro}_K(\mathbb{F}) \tag{2.2}$$

of $\tilde{\Sigma}_{n,m}(\mathbb{F})$ into non-empty disjoint submanifolds and each Kronecker stratum is invariant under the similarity action on $\tilde{\Sigma}_{n,m}(\mathbb{F})$.

There is a system theoretic interpretation of the Kronecker indices which is useful in order to understand the decomposition (2.2) further.

In both systems theory and its applications to automatic control, the concept of *feedback* plays a central role in controlling the dynamics of a given dynamical system.

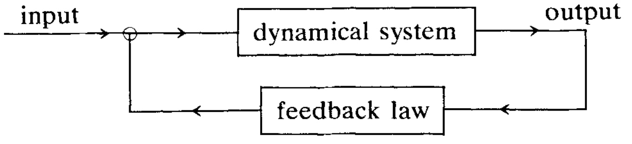


Figure 2. Feedback loop.

In linear system theory, state feedback is defined by a certain algebraic group action on the space $\tilde{\Sigma}_{n,m}(\mathbb{F})$ of controllable systems. More precisely, the *state feedback group* $\mathcal{F}_{n,m}$ is the subgroup of $GL_{n+m}(\mathbb{F})$, consisting of all $(n+m)\times(n+m)$-matrices

$$\begin{bmatrix} S & O \\ F & U \end{bmatrix}$$

where $S \in GL_n(\mathbb{F})$, $F \in \mathbb{F}^{m\times n}$, $U \in GL_m(\mathbb{F})$.

The *state feedback action* is defined as the algebraic group action

$$\Phi : \mathcal{F}_{n,m} \times \tilde{\Sigma}_{n,m}(\mathbb{F}) \to \tilde{\Sigma}_{n,m}(\mathbb{F})$$
$$((S, F, U), (A, B)) \mapsto (S(A + BF)S^{-1}, SBU^{-1}).$$

Brunovsky's theorem [5] classifies the orbits of this action Φ: they are in one to one correspondence with the *partitions* $\mathcal{K}_1 \geq \cdots \geq \mathcal{K}_m \geq 0$ of n; see e.g. Brunovsky [5], Byrnes [8], Hazewinkel and Martin [15] for this result and more information on the feedback theory for linear dynamical systems.

Instead of dealing with the full state feedback group $\mathcal{F}_{n,m}$ and the corresponding feedback action Φ we consider the *restricted feedback group* $F_{n,m}$. $F_{n,m}$ is defined as the set of all $(n+m)\times(n+m)$-matrices $\begin{bmatrix} S & O \\ F & U \end{bmatrix}$, with $S \in GL_n(\mathbb{F})$, $F \in \mathbb{F}^{m\times n}$, $U \in GL_m(\mathbb{F})$ upper triangular. $F_{n,m}$ is a parabolic subgroup of $GL_{n+m}(\mathbb{F})$ and the *restricted feedback action*

$$\phi : F_{n,m} \times \tilde{\Sigma}_{n,m}(\mathbb{F}) \to \tilde{\Sigma}_{n,m}(\mathbb{F})$$
$$((S, F, U), (A, B)) \mapsto (S(A+BF)S^{-1}, SBU^{-1})$$

is an algebraic group action.

It is easy to check that the Kronecker indices $K(A, B)$ are invariant under the restricted feedback action:

$$K(S(A+BF)S^{-1}, SBU^{-1}) = K(A, B)$$

for all $S \in GL_n(\mathbb{F})$, $F \in \mathbb{F}^{m\times n}$, $U \in GL_m(\mathbb{F})$ upper triangular. Even more, these are the only invariants.

THEOREM 2.3. *The orbits of the restricted feedback action ϕ are precisely the Kronecker strata* $\text{Kro}_K(\mathbb{F})$ *of* $\tilde{\Sigma}_{n,m}(\mathbb{F})$.

The proof is by a straightforward modification of the proof for Brunovsky's theorem. We omit the details.

Let $\bar{A}$ denote the relative topological closure of a subset $A \subset \tilde{\Sigma}_{n,m}(\mathbb{F})$.

COROLLARY 2.4. *For* $K, L \in K_{n,m}$:

$$\text{Kro}_K(\mathbb{F}) \subset \overline{\text{Kro}_L(\mathbb{F})} \Leftrightarrow \text{Kro}_K(\mathbb{F}) \cap \overline{\text{Kro}_L(\mathbb{F})} \neq \emptyset.$$

By the closed orbit lemma, the topological closure $\overline{\text{Kro}_K(\mathbb{C})}$ of any Kronecker stratum is an algebraic subvariety of $\tilde{\Sigma}_{n,m}(\mathbb{C})$. Since the orbits of a semialgebraic group action are semialgebraic again, Theorem 2.3 implies

COROLLARY 2.5. *The decomposition of* $\tilde{\Sigma}_{n,m}(\mathbb{R})$ *into Kronecker strata* $\text{Kro}_K(\mathbb{R})$, $K \in K_{n,m}$, *is a semialgebraic Whitney stratification of* $\tilde{\Sigma}_{n,m}(\mathbb{R})$.

3. Combinatorics of Kronecker strata

In order to compute the Betti numbers of the orbit space $\Sigma_{n,m}(\mathbb{F})$, we need an explicit characterization of those Kronecker strata $\mathrm{Kro}_K(\mathbb{F})$ which form the boundary of a given Kronecker stratum $\mathrm{Kro}_L(\mathbb{F})$. To do this we study the partial order on combinations $K_{n,m}$, defined by the adherence property

$$K \subseteq L \Leftrightarrow \mathrm{Kro}_K(\mathbb{F}) \subset \overline{\mathrm{Kro}_L(\mathbb{F})} \tag{3.1}$$

Let $\leq$ denote the lexicographic order on $\bar{n} \times \underline{m} := \{0, \ldots, n\} \times \{1, \ldots, m\}$. For any combination $K \in K_{n,m}$ define

$$Y_K := \{(i, j) \in \bar{n} \times \underline{m} \mid 0 \leq i \leq K_j - 1\}$$

and

$$r_{ij}(K) := \mathrm{card}\,\{(k, l) \in Y_K \mid (k, l) \leq (i, j)\}, \qquad (i, j) \in \bar{n} \times \underline{m}.$$

Define the *Kronecker order* $\subseteq$ on $K_{n,m}$ by

$$K \subseteq L \Leftrightarrow r_{ij}(K) \leq r_{ij}(L) \quad \text{for all} \quad (i, j) \in \bar{n} \times \underline{m}.$$

THEOREM 3.1. *The Kronecker order* $\subseteq$ *on* $K_{n,m}$ *is the adherence order for Kronecker strata:*

$$\mathrm{Kro}_K(\mathbb{F}) \cap \overline{\mathrm{Kro}_L(\mathbb{F})} \neq \varnothing \Leftrightarrow \mathrm{Kro}_K(\mathbb{F}) \subset \overline{\mathrm{Kro}_L(\mathbb{F})} \Leftrightarrow K \subseteq L$$

for $K, L \in K_{n,m}$.

In order to prove this, we need to know the covers of a combination K with respect to the Kronecker order $\subseteq$. Recall that for any partially ordered set $(P, \leq)$ an element y is called a cover for $x \in P$ whenever $x < y$ and $x < z < y$ holds for no $z \in P$.

The covers for the Kronecker order have been explicitly characterized by H. F. Münzner in an unpublished manuscript, see Appendix A, Theorem A. It follows from Theorem A, that any combination $L \in K_{n,m}$ with $K \subset L$ *can be obtained from the combination* K by a sequence of successive transpositions $K_1, \ldots, K_r$:

$$K \subset K_1 \subset \cdots \subset K_r = L.$$

Here a transposition is defined as follows:

Let $K \in K_{n,m}$, $i, j \in \underline{m}$ with $i<j$, $K_i \neq K_j+1$. A combination $T_{ij}K := \bar{K}$ is called a *transposition of* K iff:

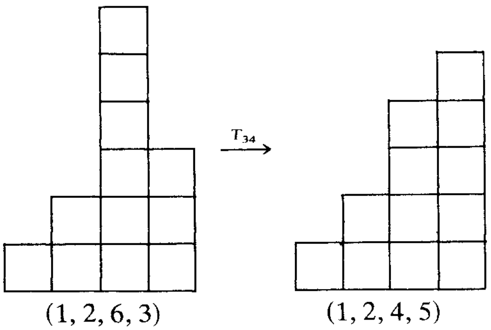
(1) If $K_i = K_j$, $T_{ij}K := K$;

(2) If $K_i < K_j$:
$\bar{K}_l := K_l$ for $l \neq i, j$
$\bar{K}_i := K_j$
$\bar{K}_j := K_i$;

(3) If $K_i > K_j + 1$:
$\bar{K}_l := K_l$ for $l \neq i, j$.
$\bar{K}_i := K_j + 1$
$\bar{K}_j := K_i - 1$.

For $j<i$ define $T_{ij}K := T_{ji}K$.

EXAMPLE



Proof of Theorem 3.1. We have already seen that

$$\mathrm{Kro}_K(\mathbb{F}) \subset \overline{\mathrm{Kro}_L(\mathbb{F})} \Leftrightarrow \mathrm{Kro}_K(\mathbb{F}) \cap \overline{\mathrm{Kro}_L(\mathbb{F})} \neq \varnothing$$

holds.

(a) "$\mathrm{Kro}_K(\mathbb{F}) \cap \overline{\mathrm{Kro}_L(\mathbb{F})} \neq \phi \Rightarrow K \subseteq L$". Let $(A, B) \in \mathrm{Kro}_K(\mathbb{F}) \cap \overline{\mathrm{Kro}_L(\mathbb{F})}$. Obviously for any $(i, j) \in \bar{n} \times \underline{m}$

$$r_{ij}(K) = \mathrm{rk}\,(B, \ldots, A^{i-1}B, A^i b_1, \ldots, A^i b_j).$$

Since the rank function is upper semicontinuous, any $(\tilde{A}, \tilde{B}) \in \mathrm{Kro}_L(\mathbb{F})$ sufficiently near to (A, B) satisfies:

$$\begin{aligned} \mathrm{rk}\,(\tilde{B}, \ldots, \tilde{A}^{i-1}\tilde{B}, \tilde{A}^i\tilde{b}_1, \ldots, \tilde{A}^i\tilde{b}_j) &= r_{ij}(L) \\ &\geq r_{ij}(K). \end{aligned}$$

Therefore $r_{ij}(K) \le r_{ij}(L)$ for all $(i, j) \in \bar{n} \times \underline{m}$. q.e.d.

(b) "$K \subseteq L \Rightarrow \mathrm{Kro}_K(\mathbb{F}) \subset \overline{\mathrm{Kro}_L(\mathbb{F})}$". Without loss of generality we can assume that L is a cover of K, i.e. by Theorem A there exists $(i, j) \in \bar{n} \times \underline{m}$ with $L = T_{ij}K$. It is enough to find a pair $(A, B) \in \mathrm{Kro}_K(\mathbb{F}) \cap \overline{\mathrm{Kro}_L(\mathbb{F})}$.

CONSTRUCTION OF (A, B). There exists a unique $(A, B) \in \mathrm{Kro}_K(\mathbb{F})$ satisfying:

(1) $A^i b_j = e_{r_{ij}(K)}$ for all $(i, j) \in Y_k$, where e_r denotes the r-th standard basis vector of $\mathbb{F}^n$.

(2) $A^{K_j} b_j = 0$ for all $j \in \underline{m}$.

We show that for any $\varepsilon > 0$ there exists $(\tilde{A}, \tilde{B}) \in \mathrm{Kro}_L(\mathbb{F})$ which is ε-near to (A, B).

CONSTRUCTION OF $(\tilde{A}, \tilde{B})$. $L = T_{ij}K$ for $i < j$, $K_i \neq K_j + 1$; w.l.o.g. we may assume $K_i \neq K_j$.

Case 1. $K_i < K_j$. Thus

$$\begin{aligned} &L_l = K_l \quad \text{for} \quad l \neq i, j \\ &L_i = K_j \\ &L_j = K_i \end{aligned}$$

For $K_i = 0$ set

$$\begin{aligned} &\tilde{A} := A \quad \text{and} \quad \tilde{B} := (\tilde{b}_1, \ldots, \tilde{b}_m), \quad \text{where} \\ &\tilde{b}_s := b_s \quad \text{for} \quad s \neq i \\ &\tilde{b}_i := b_i + \varepsilon b_j. \end{aligned}$$

For $K_i \ge 1$ set

$$\begin{aligned} &\tilde{B} := B, \qquad \tilde{A}^r \tilde{b}_s = A^r b_s \quad \text{for all} \quad r \ge 0, \qquad s \neq i \\ &\tilde{A} b_i := A b_i, \ldots, \tilde{A}(A^{K_i-2} b_i) := A^{K_i-1} b_i, \\ &\tilde{A}(A^{K_i-1} b_i) := A^{K_i} b_i + \varepsilon A^{K_i} b_j = \varepsilon A^{K_i} b_j. \end{aligned}$$

In both cases $(\tilde{A}, \tilde{B})$ is well-defined and ε-near to (A, B). One easily verifies that $(\tilde{A}, \tilde{B}) \in \tilde{\Sigma}_{n,m}(\mathbb{F})$ has Kronecker indices $(L_1, \ldots, L_m)$, i.e. $(\tilde{A}, \tilde{B}) \in \mathrm{Kro}_L(\mathbb{F})$.

Case 2. $K_i > K_j + 1$. Here

$$\begin{aligned} &L_l = K_l \quad \text{for} \quad l \neq i, j \\ &L_i = K_j + 1 \\ &L_j = K_i - 1. \end{aligned}$$

For $K_j = 0$ define

$\tilde{A} := A$ and $\tilde{B} := (\tilde{b}_1, \ldots, \tilde{b}_m)$ with
$\tilde{b}_s := b_s$ for $s \neq j$
$\tilde{b}_j := b_j + \varepsilon A b_i = \varepsilon A b_i$.

For $K_j \geq 1$ define

$$\tilde{B} := B, \qquad \tilde{A}^r \tilde{b}_s := A^r b_s \quad \text{for all} \quad r \geq 0,\ s \neq j,$$
$$\tilde{A} b_j := A b_j, \ldots, \tilde{A}(A^{K_j - 2} b_j) := A^{K_j - 1} b_j,$$
$$\tilde{A}(A^{K_j - 1} b_j) := A^{K_j} b_j + \varepsilon A^{K_j + 1} b_i = \varepsilon A^{K_j + 1} b_i.$$

$(\tilde{A}, \tilde{B})$ is well-defined, ε-near to (A, B) and has the right Kronecker indices $(L_1, \ldots, L_m)$. This shows that

$$\mathrm{Kro}_K(\mathbb{F}) \cap \overline{\mathrm{Kro}_L(\mathbb{F})} \neq \varnothing. \quad \text{q.e.d.}$$

It follows from Theorem 3.1 that the topological closure of $\mathrm{Kro}_K(\mathbb{F})$ in $\tilde{\Sigma}_{n,m}(\mathbb{F})$ is given by

$$\overline{\mathrm{Kro}_K(\mathbb{F})} = \{(A, B) \in \tilde{\Sigma}_{n,m}(\mathbb{F}) \mid \mathrm{rk}\,(B, \ldots, A^{i-1}B, A^i b_1, \ldots, A^i b_j) \leq r_{ij}(K) \quad \text{for all} \quad (i, j) \in \bar{n} \times \underline{m}.\}$$

Therefore

COROLLARY 3.2. *The topological closure* $\overline{\mathrm{Kro}_K(\mathbb{F})}$ *is an algebraic subvariety of* $\tilde{\Sigma}_{n,m}(\mathbb{F})$.

EXAMPLE. The Kronecker strata of $\tilde{\Sigma}_{n,m}(\mathbb{F})$ are linearly ordered by adherence:

$$(0, n) \subseteq (n, 0) \subseteq (1, n-1) \subseteq (n-1, 1) \subseteq (2, n-2) \subseteq \cdots.$$

4. Kronecker cells

The spaces $\tilde{\Sigma}_{n,m}(\mathbb{F})$ and $\Sigma_{n,m}(\mathbb{F})$ are related by the principal fibre bundle

$$\pi : \tilde{\Sigma}_{n,m}(\mathbb{F}) \to \Sigma_{n,m}(\mathbb{F})$$
$$(A, B) \mapsto [A, B] = \text{similarity orbit of } (A, B).$$

Since the similarity action $\alpha : GL_n(\mathbb{F}) \times \tilde{\Sigma}_{n,m}(\mathbb{F}) \to \tilde{\Sigma}_{n,m}(\mathbb{F})$ restricts to a free action with a closed graph on each Kronecker stratum $\text{Kro}_K(\mathbb{F})$ of $\tilde{\Sigma}_{n,m}(\mathbb{F})$, each quotient

$$\text{Kro}(K) := \pi(\text{Kro}_K(\mathbb{F})) = \text{Kro}_K(\mathbb{F})/GL_n(\mathbb{F})$$

is an analytic submanifold of $\Sigma_{n,m}(\mathbb{F})$ of dimension

$$n(K) = \sum_{i,j=1}^{m} \min(K_i, K_j) + \sum_{i>j} K_{ij}, \qquad K_{ij} = \begin{cases} 1 & K_i < K_j \\ 0 & K_i \geq K_j \end{cases}; \tag{4.1}$$

by Lemma 2.2.

LEMMA 4.1. *For each combination* $K \in K_{n,m}$, Kro (K) *is an* analytic cell, *i.e. analytically isomorphic to affine space* $\mathbb{F}^{n(K)}$.

Proof. Let $\leq$ denote the lexicographic order on $\bar{n} \times \underline{m}$. For each $1 \leq l \leq m$ and $(A, B) \in \text{Kro}_K(\mathbb{F})$ there are uniquely determined $c^l_{ij}(A, B) \in \mathbb{F}$ with

$$A^{K_l} b_l = \sum_{\substack{(i,j)<(K_l,l) \\ i<K_j}} c^l_{ij}(A, B) A^i b_j.$$

By uniqueness, $c^l_{ij}(SAS^{-1}, SB) = c^l_{ij}(A, B)$ for all $S \in GL_n(\mathbb{F})$. Let $c_l(A, B) \in \mathbb{F}^{n_l(K)}$ denote the vector consisting of the $n_l(K) = \text{card}\{(i, j) \in \bar{n} \times \underline{m} \mid i < K_j, (i, j) < (k_l, l)\}$ components $c^l_{ij}(A, B)$; $c(A, B) := (c_1(A, B), \ldots c_m(A, B))$. Since $n(K) = n_1(K) + \cdots + n_m(K)$, $c(A, B) \in \mathbb{F}^{n(K)}$.

The map

$$\tilde{t} : \text{Kro}_K(\mathbb{F}) \to GL_n(\mathbb{F}) \times \mathbb{F}^{n(K)}$$

$$\tilde{t}(A, B) := ((b_1, \ldots, A^{K_1-1}b_1, \ldots, b_m, \ldots, A^{K_m-1}b_m), c(A, B))$$

$$= (R_K(A, B), c(A, B))$$

is an $\mathbb{F}$-analytic diffeomorphism. Since

$$\tilde{t}(SAS^{-1}, SB) = (SR_K(A, B), c(A, B)),$$

$\tilde{t}$ induces the $\mathbb{F}$-analytic diffeomorphism

$$t : \text{Kro}(K) \to \mathbb{F}^{n(K)}$$

$$[A, B] \mapsto c(A, B). \quad \text{q.e.d.}$$

We call Kro (K) a *Kronecker cell* and its topological closure $\overline{\text{Kro}(K)}$ a *Kronecker variety* of $\Sigma_{n,m}(\mathbb{F})$. By Corollary 2.5 the decomposition of $\Sigma_{n,m}(\mathbb{F})$ into Kronecker cells $(\text{Kro}(K))_{K \in K_{n,m}}$ is a finite cellular decomposition.

It is in general a difficult problem to compute topological invariants like the Betti numbers of a space X from a given cellular decomposition $(X_i)_{i \in I}$. Often one has to impose additional assumptions on the cell decomposition, e.g. that $(X_i)_{i \in I}$ defines a *CW* cell complex; but even then the calculations can be quite complicated.

Unfortunately the Kronecker cell decomposition of $\Sigma_{n,m}(\mathbb{F})$ is not a *CW* cell complex, since $\Sigma_{n,m}(\mathbb{F})$ is non-compact. Therefore we have to look for a different concept. A decomposition $(X_i)_{i \in I}$ of a real analytic manifold X into disjoint submanifolds X_i is called an *analytic cellular decomposition*, if the following conditions are satisfied:

(a) $(X_i)_{i \in I}$ is locally finite and each X_i is diffeomorphic to some affine space $\mathbb{R}^{n_i}$,
(b) the boundary of X_i in X is contained in the union of cells X_j of strictly smaller dimensions,
(c) the topological closure $\overline{X_i}$ of X_i is a locally analytic subvariety of X.

Here a closed subset $A \subset X$ is called a *locally analytic subvariety* if for any $a \in A$ there is an open neighbourhood U of a in X and finitely many analytic functions $f_j : U \to \mathbb{R}$, $j \in J$, such that

$$A \cap U = \{x \in U \mid f_j(x) = o \text{ for all } j \in J\}.$$

The *adherence order* on the set of cells X_i, $i \in I$, is defined by

$$i \leq j :\Leftrightarrow X_i \subset \overline{X_j}, \qquad i, j \in I.$$

$(X_i)_{i \in I}$ is said to satisfy the *frontier condition*, if for all $i, j \in I$:

$$X_i \cap \overline{X_j} \neq \varnothing \Leftrightarrow X_i \subset \overline{X_j}.$$

Not all analytic manifolds admit an analytic cellular decomposition. The following example has been suggested to me by D. Fried and F. Takens: For coprime integers p, $q \in \mathbb{N}$ let $L(p, q)$ denote the 3-dimensional *lens space*. $L(p, q)$ is a compact analytic manifold which has no analytic cellular decomposition, provided p is odd. The reason is that in this case there is odd torsion in the integral homology of $L(p, q)$.

A classical example of a space with an analytic cellular decomposition is the Grassmann manifold $G_r(\mathbb{F}^n)$ of r-dimensional linear subspaces of $\mathbb{F}^n$. Recall that the *Schubert cells* $S_0(a)$ resp. the *Schubert varieties* $S(a)$ are defined for any sequence $a := (a_1, \ldots, a_r)$ of integers a_i with

$$0 \leq a_1 \leq \cdots \leq a_r \leq n - r$$

by

$$S_0(a) := \{X \in G_r(\mathbb{F}^n) \mid \dim(X \cap V_{a_i+i}) = i, \dim(X \cap V_{a_i+i-1}) = i - 1 \text{ for all } 1 \leq i \leq r\}$$

resp.

$$S(a) := \{X \in G_r(\mathbb{F}^n) \mid \dim(X \cap V_{a_i+i}) \geq i \text{ for all } i \leq i \leq r\},$$

where $0 \subset V_1 \subset \cdots \subset V_n = \mathbb{F}^n$, $\dim V_i = i$, denotes a fixed flag of subspaces of $\mathbb{F}^n$.

$S_0(a)$ is a cell of dimension $a_1 + \cdots + a_r$ and dense in the algebraic subvariety $S(a)$ of $G_r(\mathbb{F}^n)$. The adherence order on the Schubert cells is the product order

$$S_0(a) \subset \overline{S_0(b)} \Leftrightarrow a_1 \leq b_1, \ldots, a_r \leq b_r;$$

Stoll [24].

It is well known that the set of Schubert cells, endowed with this ordering, is a graded lattice which is rank symmetric and unimodal. In fact, this lattice of Schubert cells is isomorphic to the lattice of integer partitions; see Brylawski [6].

THEOREM 4.2. *The decomposition of the orbit space $\Sigma_{n,m}(\mathbb{F})$ into Kronecker cells* Kro (K), $K \in K_{n,m}$, *is a finite analytic cellular decomposition which satisfies the frontier condition. The adherence order is the Kronecker order on combinations.*

Proof. By Corollary 3.2, the closure $\overline{\text{Kro}_K(\mathbb{F})}$ of a Kronecker stratum is an analytic subvariety of $\tilde{\Sigma}_{n,m}(\mathbb{F})$. Therefore the closure $\overline{\text{Kro}(K)} = \pi(\overline{\text{Kro}(\mathbb{F})})$ is a locally analytic subvariety of $\Sigma_{n,m}(\mathbb{F})$. The rest follows immediately from Theorem 3.1.

Let $(X_i)_{i \in I}$ denote a finite analytic cellular decomposition of an analytic manifold X, $\dim X = n$.

Borel and Haefliger [3] have shown the existence of a mod 2 fundamental class $[\overline{X_i}] \in H_*^{BM}(X; \mathbb{Z}_2)$ in the Borel–Moore homology of X. By Poincaré duality, $H_q^{BM}(X; \mathbb{Z}_2)$ is isomorphic to $H_{n-q}(X; \mathbb{Z}_2)$, the $(n-q)$-th singular homology group of X (with coefficients in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$).

For any non-negative integer q let

$$c_q := \operatorname{card}\{i \in I \mid \operatorname{codim} X_i = q\}$$

denote the number of cells of (real) codimension q. The following result is due to Borel and Haefliger [3]:

THEOREM 4.3. (Borel–Haefliger). *Let X be a real analytic manifold and $(X_i)_{i\in I}$ a finite analytical cellular decomposition. Then for any $q \geq 0$, the set $\{[\overline{X_i}] \mid \operatorname{codim} X_i = q\}$ of fundamental classes is a basis for $H_q(X; \mathbb{Z}_2)$ and consequently*

$$H_q(X; \mathbb{Z}_2) \cong \mathbb{Z}_2^{c_q}.$$

It follows that c_q is a *topological invariant* for X: the q-th mod 2 Betti number of X.

Let $c_q(n, m)$ denote the number of Kronecker cells Kro (K) of $\Sigma_{n,m}(\mathbb{F})$ of real codimension q. By the Borel–Haefliger Theorem, Theorem 4.2 implies

COROLLARY 4.4. *For any $q \geq 0$:*

$$H_q(\Sigma_{n,m}(\mathbb{F}); \mathbb{Z}_2) \cong \mathbb{Z}_2^{c_q(n,m)}.$$

More precisely, we have the following result which is analogous to the basis theorem in the Schubert calculus for Grassmann manifolds [24].

For any Kronecker variety $\overline{\text{Kro (K)}}$ of $\Sigma_{n,m}(\mathbb{F})$ with codimension q, its fundamental class $[\overline{\text{Kro}\,(K)}] \in H_q(\Sigma_{n,m}(\mathbb{F}); \mathbb{Z}_2)$ is called a *q-Kronecker cycle.*

COROLLARY 4.5. *The q-th Kronecker cycles form a basis of $H_q(\Sigma_{n,m}(\mathbb{F}); \mathbb{Z}_2)$.*

Furthermore, since Kronecker cycles are represented by algebraic subvarieties (see Appendix B), $H_*(\Sigma_{n,m}(\mathbb{F}); \mathbb{Z}_2)$ is totally algebraic.

One would like to have a more explicit formula for the mod 2 Betti numbers of $\Sigma_{n,m}(\mathbb{F})$ than the one given by Corollary 4.4 and the dimension formula (4.1). In [16], a different cell decomposition of $\Sigma_{n,m}(\mathbb{F})$ has been constructed by means of "Hermite cells" Her (K), $K \in K_{n,m}$. Unfortunately, the corresponding decomposition of $\Sigma_{n,m}(\mathbb{R})$ into Hermite cells does *not* define an analytic cellular decomposition: the real Hermite varieties $\overline{\text{Her}\,(K)}$ are only semialgebraic subvarieties of $\Sigma_{n,m}(\mathbb{R})$. However, for $\mathbb{F} = \mathbb{C}$, the Hermite cell decomposition can be used to effectively determine the Betti numbers of $\Sigma_{n,m}(\mathbb{C})$. A central result appearing in [16] is:

THEOREM 4.6. *The integral homology groups* $H_*(\Sigma_{n,m}(\mathbb{C}); \mathbb{Z})$ *are isomorphic to the homology groups* $H_*(G_n(\mathbb{C}^{n+m-1}); \mathbb{Z})$ *of the Grassmann manifold.*

By combining Corollary 4.4 with Theorem 4.6 we obtain our main result

THEOREM 4.7. *The* mod 2 *homology groups of* $\Sigma_{n,m}(\mathbb{R})$ *are isomorphic to those of the Grassmann manifold* $G_n(\mathbb{R}^{n+m-1})$:

$$H_*(\Sigma_{n,m}(\mathbb{R}); \mathbb{Z}_2) \cong H_*(G_n(\mathbb{R}^{n+m-1}); \mathbb{Z}_2).$$

Observe that this result is obtained by a pure dimension count; no direct relation between the spaces $\Sigma_{n,m}(\mathbb{F})$ and $G_n(\mathbb{F}^{n+m-1})$ is known till now.

Remark 1. It follows from Theorem 4.7 that the partially ordered set $(K_{n,m}, \subseteq)$ of all combinations of n endowed with the Kronecker ordering is rank symmetric and unimodal. It is in general not a lattice.

Remark 2. As a consequence of Theorem 4.7 we see that the Hermite cycles, introduced in [16], also form a bases for $H_*(\Sigma_{n,m}(\mathbb{R}); \mathbb{Z}_2)$. Therefore there are two different basis for $H_*(\Sigma_{n,m}(\mathbb{R}); \mathbb{Z}_2)$: The *algebraic Kronecker cycles* constructed in this paper and the *semialgebraic Hermite cycles* of [16].

It seems that they correspond to different kinds of a Schubert calculus for the cohomology ring $H^*(\Sigma_{n,m}(\mathbb{R}); \mathbb{Z}_2)$. The cohomology ring of $\Sigma_{n,m}(\mathbb{F})$ will be studied in a subsequent paper (joint work with C. I. Byrnes).

5. Topology of the spaces of rational maps

In this chapter the previous results on the topology of $\Sigma_{n,m}(\mathbb{R})$ are applied to compute some Betti numbers of the space $\mathrm{Rat}_{n,m,p}(\mathbb{R})$ of all real proper rational matrices $G \in \mathbb{R}^{p\times m}(s)$ with McMillan degree n. Recall that this space $\mathrm{Rat}_{n,m,p}(\mathbb{R})$ can be identified with the manifold of all base point preserving holomorphic maps of degree n

$$\varphi : P_1(\mathbb{C}) \to G_m(\mathbb{C}^{m+p})$$

which commute with complex conjugation.

Quite a lot is already known about the topology of $\mathrm{Rat}_{n,m,p}(\mathbb{F})$ for $\min(m, p) = 1$, due to work of Brockett [4], Byrnes and Duncan [9] and Segal [23]. The deepest results have been obtained by Segal [23] who shows that the inclusion map $i:\mathrm{Rat}_{n,m,1}(\mathbb{C}) \to \Omega_n^2(P_m(\mathbb{C}))$ into the loop space of all base point preserving continuous maps $\varphi: S^2 \to P_m(\mathbb{C})$ of degree n is a homotopy equivalence up to dimension $n(2m-1)$. The general multivariable case, $\max(m, p) > 1$, has been quite intensively studied in the theses of Delchamps [11] and Guest [12]. Delchamps [11] uses a Morse theoretic approach to study $\mathrm{Rat}_{n,m,p}(\mathbb{F})$. He computed the Betti numbers of $\mathrm{Rat}_{n,m,p}(\mathbb{R})$ (resp. $\mathrm{Rat}_{n,m,p}(\mathbb{C})$) up to dimension $\min(m, p)-2$ (resp. $2\min(m, p)-2$). Even more, for $n=1$ he computed *all* singular homology groups of $\mathrm{Rat}_{1,m,p}(\mathbb{F})$. However for $n \geq 2$ the necessary Morse theoretic calculations become too involved to be carried out completely. Due to our complete knowledge of the mod 2 Betti numbers of $\Sigma_{n,m}(\mathbb{R})$, the first $\max(m, p)-1$ mod 2 Betti numbers of $\mathrm{Rat}_{n,m,p}(\mathbb{R})$ will be easily obtained. An analogous result concerning $\mathrm{Rat}_{n,m,p}(\mathbb{C})$ is given in [16].

For n, m, $p \geq 1$ let

$$\tilde{\Sigma}_{n,m,p}(\mathbb{R}) := \{(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \mid (A, B) \text{ controllable}\}$$

and

$$\Sigma_{n,m,p}(\mathbb{R}) := \tilde{\Sigma}_{n,m,p}(\mathbb{R})/GL_n(\mathbb{R})$$

denote the orbit space of the similarity action $(A, B, C) \mapsto (SAS^{-1}, SB, CS^{-1})$ on $\tilde{\Sigma}_{n,m,p}(\mathbb{R})$. Byrnes and Hurt [10] have shown that $\Sigma_{n,m,p}(\mathbb{R})$ is a real analytic manifold of dimension $n(m+p)$ and

$$p: \Sigma_{n,m,p}(\mathbb{R}) \to \Sigma_{n,m}(\mathbb{R}), [A, B, C] \mapsto [A, B],$$

an analytic vector bundle on $\Sigma_{n,m}(\mathbb{R})$. In particular $\Sigma_{n,m,p}(\mathbb{R})$ is homotopy equivalent to $\Sigma_{n,m}(\mathbb{R})$.

For $0 \leq r \leq n$ let

$$\tilde{S}^r_{n,m,p} := \{(A, B, C) \in \tilde{\Sigma}_{n,m,p}(\mathbb{R}) \mid \mathrm{rk}\,(C^T, A^TC^T, \ldots, (A^T)^{n-1}C^T) = r\}$$

and $\widetilde{\mathrm{Rat}}_{n,m,p}(\mathbb{R}) := \tilde{S}^n_{n,m,p}$ denote the set of all systems (A, B, C) which are controllable and observable. By [16], Thm. 5.1, $\tilde{S}^r_{n,m,p}$ is an analytic submanifold of $\tilde{\Sigma}_{n,m,p}(\mathbb{R})$ with codimension $p(n-r)$ and the union $\tilde{S} := \bigcup_{r=0}^{n-1} \tilde{S}^r_{n,m,p}$ is a closed analytic subvariety of $\tilde{\Sigma}_{n,m,p}(\mathbb{R})$. Note $\widetilde{\mathrm{Rat}}_{n,m,p}(\mathbb{R}) = \tilde{\Sigma}_{n,m,p}(\mathbb{R}) \backslash \tilde{S}$.

Since the similarity action $(A, B, C) \mapsto (SAS^{-1}, SB, CS^{-1})$ acts freely and with a closed graph on $\tilde{S}^r_{n,m,p}$ and $\widetilde{\mathrm{Rat}}_{n,m,p}(\mathbb{R})$, the corresponding orbit spaces

$$S^r_{n,m,p} := \tilde{S}^r_{n,m,p}/GL_n(\mathbb{R})$$

$$\mathrm{Rat}_{n,m,p}(\mathbb{R}) = \widetilde{\mathrm{Rat}}_{n,m,p}(\mathbb{R})/GL_n(\mathbb{R})$$

are analytic submanifolds of $\Sigma_{n,m,p}(\mathbb{R})$.

Remark. It follows from the main theorem of realization theory for finite dimensional linear dynamical systems (Kalman [18]), Byrnes and Duncan [9]) that the orbit space $\widetilde{\mathrm{Rat}}_{n,m,p}(\mathbb{R})/GL_n(\mathbb{R})$ can in fact be identified with the space of all (strictly) proper rational transfer matrices $G \in \mathbb{R}^{p\times m}(s)$ with McMillan degree n.

$\mathrm{Rat}_{n,m,p}(\mathbb{R})$ is open and dense in $\Sigma_{n,m,p}(\mathbb{R})$ and $S^r_{n,m,p}$ has codimension $p(n-r)$. The set

$$S := \tilde{S}/GL_n(\mathbb{R}) = \Sigma_{n,m,p}(\mathbb{R})\backslash \mathrm{Rat}_{n,m,p}(\mathbb{R})$$

is a closed analytic subvariety of $\Sigma_{n,m,p}(\mathbb{R})$ with codimension p. Thus the inclusion map

$$i : \mathrm{Rat}_{n,m,p}(\mathbb{R}) \to \Sigma_{n,m,p}(\mathbb{R})$$

is a homotopy equivalence up to dimension $p-2$. Since the transposition of transfer matrices $G(s) \mapsto G(s)^T$ defines a diffeomorphism from $\mathrm{Rat}_{n,m,p}(\mathbb{R})$ onto $\mathrm{Rat}_{n,p,m}(\mathbb{R})$, the homology groups $H_q(\mathrm{Rat}_{n,m,p}(\mathbb{R}))$ are isomorphic to $H_q(\mathrm{Rat}_{n,p,m}(\mathbb{R}))$ for all q.

Therefore we get

THEOREM 5.1. *For* $\max(m, p) \geq 2$ *there are isomorphisms of (integral) homology groups*

$$H_q(\mathrm{Rat}_{n,m,p}(\mathbb{R})) \cong H_q(\Sigma_{n,\min(m,p)}(\mathbb{R}))$$

for $0 \leq q \leq \max(m, p) - 2$.

By Theorem 4.7 we conclude

THEOREM 5.2. *Let* $l := \min(m, p)$ *and* $\max(m, p) \geq 2$. *Then* $H_q(\mathrm{Rat}_{n,m,p}(\mathbb{R}); \mathbb{Z}_2)$ *is isomorphic to* $H_q(G_n(\mathbb{R}^{n+l-1}); \mathbb{Z}_2)$ *for* $0 \leq q \leq \max(m, p) - 2$.

Appendix A

In this appendix we prove a technical result concerning a characterization of the covers for the Kronecker order $\subseteq$ on $K_{n,m}$. The material in this section is due to H. F. Münzner; I like to thank him for his help in these matters.

A useful description of combinations $K \in K_{n,m}$ is obtained by means of the counting function $z: \mathbb{Z} \times \underline{m} \to \mathbb{Z}$, $z(i, j) := im + j$. z is monotone increasing

$$(k, l) \leq (i, j) \Rightarrow z(k, l) \leq z(i, j)$$

and shift-invariant, i.e. $z(i \pm 1, j) = z(i, j) \pm m$. To any combination $K = (K_1, \ldots, K_m)$ there is an associated m-tuple $s(K) = (s_1, \ldots, s_m)$ defined by

$$s_j := z(K_j - 1, j) = (K_j - 1)m + j$$

for all $j \in \underline{m}$. s_j satisfies

(a) $s_1 + \cdots + s_m = m(n-1) + \frac{1}{2}m(m+1)$
(b) $1 - m \leq s_j \leq mn$
(c) $s_j \equiv j \pmod{m}$.

Conversely, for any m-tupel $s = (s_1, \ldots, s_m)$ with (a), (b), (c) there exists an unique $K \in K_{n,m}$ with $s = s(K)$. For any real number x, let $[x] := \max\{l \in \mathbb{Z} \mid l \leq x\}$. Let $\subseteq$ be the Kronecker order on $K_{n,m}$ and for $K \in K_{n,m}$ set

$$h_K(r) := \sum_{s_l \leq r} \left[\frac{r - s_l}{m}\right], \qquad r \in \mathbb{N}.$$

Then it is easy to check

$$\boxed{K \subseteq L \Leftrightarrow h_K(r) \geq h_L(r) \quad \text{for all} \quad r \in \mathbb{N}.}$$

We will make use of the following operation on combinations: Given $i, j \in \underline{m}$, $i \neq j$, and $K \in K_{n,m}$ with $K_j \geq 1$. Set $t_{ij}K := \bar{K} \in K_{n,m}$ with

(1) $\bar{K}_l := K_l$ for $l \neq i, j$
(2) $\bar{K}_i := K_i + 1$
(3) $\bar{K}_j := K_j - 1$.

Equivalently, in terms of $(\bar{s}_1, \ldots, \bar{s}_m) := s(\bar{K})$:

(1′) $\bar{s}_l = s_l$ for $l \neq i, j$
(2′) $\bar{s}_i = s_i + m$
(3′) $\bar{s}_j = s_j - m$.

Similarly, the effect of a transposition $T_{ij}K$ (see section 3) can be described as follows:

For $a, b \in \mathbb{Z}$, $a < b$, define

$$d(a, b) := \min \{l \in \mathbb{N}_0 \mid l \equiv b - a \pmod m\}$$

and $d(b, a) := -d(a, b)$.

For $s_i < s_j$ let $T_{ij}s(K) := \bar{s}$ be defined by

(4) $\bar{s}_l := s_l$ *for* $l \neq i, j$
(5) $\bar{s}_i := s_j - d(s_i, s_j)$
(6) $\bar{s}_j := s_i + d(s_i, s_j)$

while for $s_i > s_j$:

$$\bar{s}_l := s_l \text{ for } l \neq i, j$$
$$\bar{s}_i := s_j + d(s_j, s_i)$$
$$\bar{s}_j := s_i - d(s_j, s_i).$$

Then $T_{ji}s = T_{ij}s$ and $T_{ij}s(K) = s(T_{ij}K)$.

Consider for $r \in \mathbb{N}_0$ and $K, L \in K_{n,m}$:

$$\Delta h_K(r) := h_K(r) - h_K(r-1)$$
$$h_{K,L}(r) := h_K(r) - h_L(r)$$
$$\Delta h_{K,L}(r) := h_{K,L}(r) - h_{K,L}(r-1).$$

Since $h(0) = 0$ and $h_{K,L}(r) = 0$ for $r > mn$, we have

$$h_K(r) = \sum_{l=1}^{r} \Delta h_K(l)$$

$$h_{K,L}(r) = \sum_{l=1}^{r} \Delta h_{K,L}(l)$$
$$= -\sum_{l>r} \Delta h_{K,L}(l)$$

Obviously, $K \subseteq L \Leftrightarrow h_{K,L}(r) \geq 0$ for all r.

One easily shows

LEMMA A1. *Let $r = lm + j$ for $l \geq 0$, $j \in \underline{m}$. Then*

(a) $\Delta h_K(r) = \begin{cases} 1 & \text{for } r > s_j(K) \\ 0 & \text{for } r \leq s_j(K) \end{cases}$

(b) $\Delta h_{K,L}(r) = \begin{cases} 1 & \text{for } s_j(K) < r \leq s_j(L) \\ 0 & \text{for } s_j(K),\ s_j(L) < r \text{ or } s_j(K),\ s_j(L) \geq r. \\ -1 & \text{for } s_j(L) < r \leq s_j(K) \end{cases}$

For $K_j \geq s \geq 1$ let $t_{ij}^s K := \underbrace{t_{ij} \circ \cdots \circ t_{ij}}_{s\text{-times}}(K)$

For $L = t_{ij}^s K$, Lemma A1 *specializes to*

$$\Delta h_{K,L}(r) = \begin{cases} 1 & \text{for } r = s_j(K) + lm,\ l = 1, \ldots, s \\ -1 & \text{for } r = s_j(L) + lm,\ l = 1, \ldots, s \\ 0 & \text{otherwise} \end{cases}$$

LEMMA A2. *Let $K \in K_{n,m}$ and $L = t_{ij}K$.*

(a) $s_i(L) < s_j(K) \Rightarrow K \subset L$.

(b) $s_j(K) < s_i(L) \Rightarrow L \subset K$.

Proof. $\Delta h_{K,L}(r) = 1$, resp. -1, resp. 0 for $r = s_j(L)$, resp. $r = s_j(K)$, otherwise. Thus $s_i(L) < s_j(K)$ implies $h_{K,L}(r) \geq 0$, while $s_j(K) < s_i(L)$ implies $h_{K,L}(r) \leq 0$. q.e.d.

Analogously one obtains

LEMMA A3. *Given $K \in K_{n,m}$, $L = T_{ij}K$, $s_i(K) + m < s_j(K)$. Then*

$$h_{K,L}(r) = \begin{cases} 1 & \text{for } r = s_i + lm + t,\ 1 \leq l \leq \left[\dfrac{s_j - s_i}{m}\right], \qquad 0 \leq t \leq d(s_i, s_j) - 1 \\ 0 & \text{otherwise} \end{cases}$$

i.e. $K \subset L$.

Suppose $s_i(K)+m<s_j(K)$, $\bar{K}:=T_{ij}K$. By Lemma A3, the size of the "rectangle"

$$R_{ij}(K):=\{r=s_i(K)+lm+t \mid 0\leq l\leq \bar{K}_i-K_i,\ 0<t<d(s_i(K),s_j(K))\}$$

measures how much the combinations K and $\bar{K}$ differ from each other. Therefore one is led to conjecture that those combinations $\bar{K}=T_{ij}K$ will be covers of K, for which $R_{ij}(K)$ is as small as possible. This is in fact true:

THEOREM A. *Given $K, L\in K_{n,m}$. L is a cover of K for the Kronecker order $\subseteq$ on $K_{n,m}$ iff*:
(1) $L=T_{ij}K$ *for some* (i,j)
(2) $s_i(K)+m<s_j(K)$
(3) $\{s_1(K),\ldots,s_m(K)\}\cap R_{ij}(K)=\emptyset$.

Proof. Clearly these conditions suffice. To prove the necessity, we introduce $s_i:=s_i(K)$, $\hat{s}_i:=s_i(L)$, $\bar{s}_i:=s_i(\bar{K})$, $i\in \underline{m}$. Suppose $K\subset L$. It is enough to find an $(i,j)\in \underline{m}\times\underline{m}$ with $s_i+m<s_j$ and $K\subset T_{ij}K\subseteq L$. In fact, in this case there exists also $(\underline{i},\underline{j})$ with $R_{\underline{ij}}(K)\subset R_{ij}(K)$ such that $K\subset T_{\underline{ij}}K\subseteq L$ and conditions (2), (3) are satisfied for $(\underline{i},\underline{j})$.

Let $\bar{K}:=T_{ij}K$.

CONSTRUCTION OF (i,j). Set

$$r_+:=\max\{r\in\mathbb{N}\mid \Delta h_{KL}(r)=1\}$$
$$r_-:=\max\{r\in\mathbb{N}\mid \Delta h_{KL}(r)=-1\}$$

By Lemma A1(b):

$$r_+=\max\{\hat{s}_l\mid \hat{s}_l>s_l\}=:\hat{s}_i$$
$$r_-=\max\{s_l\mid s_l>\hat{s}_l\}=:s_{\underline{j}}.$$

Suppose $r_+>r_-$. Then for all $r>r_+$:

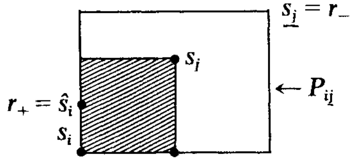
$$h_{KL}(r)=h_{KL}(r-1)+\Delta h_{KL}(r)=h_{KL}(r-1)$$
$$h_{KL}(r_+)=h_{KL}(r_+-1)+1.$$

Thus $h_{KL}(r)\geq 1$, contradiction to $h_{KL}(r)=0$ for $r>mn$. Therefore $s_i+m\leq\hat{s}_i<s_{\underline{j}}$.

Let

$$P_{i\underline{j}} := \left\{ r = s_i + lm + t \mid 0 \le l \le \left[\frac{s_{\underline{j}} - s_i}{m} \right], 0 \le t \le d(s_i, s_{\underline{j}}) \right\}.$$

Let $\underline{j} \in \underline{m}$ denote the uniquely determined index such that
(a) $s_j \in P_{i\underline{j}}$
(b) $d(s_i, s_{\underline{j}}) = \min \{ d(s_i, s_k) \mid s_i + m < s_k \}$.
In particular $s_j \le s_{\underline{j}}$.



Remark 1. For $l \ge 1$ and $1 \le t < d(s_i, s_j)$ given, let $s_i + lm + t = pm + q$ for $1 \le q \le m$. Then either $s_q \le s_i + m$ or $s_q \ge \hat{s}_q$.

Proof. Suppose $s_q > s_i + m$ and $s_q < \hat{s}_q$. It follows from Lemma A1(b) that $s_q \in P_{ij}$ and $d(s_i, s_q) < d(s_i, s_j)$. Contradiction.

By Remark 1, for any $l \ge 1$ and $1 \le t < d(s_i, s_j)$

$$\Delta h_{KL}(s_i + lm + t) \ge 0$$

holds.

Since

$$h_{KL}(s_i + lm) = \Delta h_{KL}(s_i + lm) + h_{KL}(s_i + lm - 1)$$
$$\ge \Delta h_{KL}(s_i + lm) = 1$$

for $1 \le l \le (\hat{s}_i - s_i)/m$,

$$h_{KL}(s_i + lm + t) \ge 1$$

holds for all $1 \le l \le (\hat{s}_i - s_i)/m, 0 \le t \le d(s_i, s_j)$.

Let $s_j > \hat{s}_i = r_+$. Then for $r_+ < r < r_-$

$$h_{KL}(r) = -\sum_{l > r} \Delta h_{KL}(r) \ge -\Delta h_{KL}(r_-) = 1.$$

For $s_i < r_- = \underline{s_j}$ we have

$$h_{KL}(s_i + lm + t) \geq 1 \quad \text{for} \quad 1 \leq 1 \leq \left[\frac{s_j - s_i}{m}\right], \qquad 0 \leq t < d(s_i, s_j).$$

Let $\bar{K} = T_{ij}K$. By Lemma A3, $K \subset T_{ij}K$ and $h_{\bar{K}L}(r) = h_{KL}(r) - h_{\bar{K}K}(r) =$

$$= \begin{cases} h_{KL}(r) - 1 & \text{for } r = s_i + lm + t,\ 1 \leq l \leq \left[\frac{s_j - s_i}{m}\right], \qquad 0 \leq t < d(s_i, s_j) \\ h_{KL}(r) & \text{otherwise.} \end{cases}$$

for all $r \geq 0$. By assumption, $h_{KL}(r) \geq 0$ and the previous estimate gives $h_{\bar{K}L}(r) \geq 0$ for all $r \geq 0$. Thus $\bar{K} \subseteq L$ and Theorem A is proved.

Appendix B

We show that the Kronecker cell decomposition of $\Sigma_{n,m}(\mathbb{F})$ is induced by the Schubert cell decomposition of the Grassmann manifold $G_n(\mathbb{F}^{(n+1)m})$.

Let $R(A, B)$ denote the vectorspace spanned by the rows of the $n \times (n+1)m$-matrix $(B, AB, \ldots, A^n B)$. Then

$$R : \Sigma_{n,m}(\mathbb{F}) \to G_n(\mathbb{F}^{(n+1)m}), [A, B] \to R(A, B)$$

defines an analytic embedding of $\Sigma_{n,m}$, called the *Kalman embedding*; see Byrnes, Hurt [10], Hazewinkel, Kalman [14]. Let e_i denote the i-th standard basis vector of $\mathbb{F}^{(n+1)m}$ and $0 \subset V_1 \subset \cdots \subset V_{(n+1)m} = \mathbb{F}^{(n+1)m}$ the complete flag defined by $F_i := \operatorname{span}\{e_{(n+1)m}, \ldots, e_{(n+1)m-i+1}\}$. For any combination $K \in K_{n,m}$, $Y_K := \{(i, j) \in \bar{n} \times \underline{m} \mid 0 \leq i \leq K_j - 1\}$ has exactly n elements $(i_1, j_1) < \cdots < (i_n, j_n)$, ordered lexicographically. Define $U(K) = (u_1, \ldots, u_n)$, where $u_t := mi_{n-t+1} + j_{n-t+1}$, $1 \leq t \leq n$. Let $a_K := (a_1, \ldots, a_n)$ defined by

$$a_i := (n+1)m - u_i - i + 1 \quad \text{for} \quad 1 \leq i \leq n.$$

Then $0 \leq a_1 \leq \cdots \leq a_n \leq (n+1)m - n$. Therefore a_K is a Schubert symbol for $G_n(\mathbb{F}^{(n+1)m})$ and satisfies:

(1) $a_K = a_L$ iff $K = L$

(2) $K \subseteq L \Leftrightarrow a_1(K) \leq a_1(L), \ldots, a_n(K) \leq a_n(L)$.

Let $S_K := S_0(a_K)$ denote the Schubert cell of $G_n(\mathbb{F}^{(n+1)m})$ corresponding to a_K, for any combination $K \in K_{n,m}$; let further $R : \Sigma_{n,m}(\mathbb{F}) \to G_n(\mathbb{F}^{(n+1)m})$ denote the Kalman embedding.

THEOREM B. *Let* $a=(a_1,\dots,a_n)$, $0\le a_1\le\cdots\le a_n\le(n+1)m-n$.
(1) $R^{-1}(S_0(a))\ne\emptyset$ *iff* $a=a_K$ *for some* $K\in K_{n,m}$.
(2) $\text{Kro}(K)=R^{-1}(S_K)$ *for all* $K\in K_{n,m}$.
(3) $\overline{\text{Kro}(K)}=R^{-1}(\overline{S_K})$ *for all* $K\in K_{n,m}$.

Proof. Suppose $[A,B]\in\Sigma_{n,m}(\mathbb{F})$ and $R([A,B])\in S_0(a)$. Let $K=(K_1,\dots,K_m)$ denote the Kronecker indices for (A,B). Set $R:=R([A,B])$ and $(\tilde a_1,\dots,\tilde a_n):=a_K$. The a_K's are defined in precisely the way so that

$$\dim(R\cap V_{\tilde a_i+i})=i,\qquad \dim(R\cap V_{\tilde a_i+i-1})=i-1$$

holds for $1\le i\le n$. Thus $R\in S_0(a)\cap S_0(a_K)$, i.e. $S_0(a)\cap S_0(a_K)\ne\emptyset$. This shows $a=a_K$ and $\text{Kro}(K)\subset R^{-1}(S_K)$. Suppose $\text{Kro}(K')\cap R^{-1}(S_K)\ne\emptyset$. Then $S_{K'}=S_K$, i.e. $a_{K'}=a_K$. Therefore $\text{Kro}(K)=R^{-1}(S_K)$.

To prove (3) let $\le$ denote the product order on n-tuples $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$:

$$a\le b:\Leftrightarrow a_1\le b_1,\dots,a_n\le b_n.$$

By Theorem 3.1,

$$\begin{aligned}R^{-1}(\overline{S_K})&=\bigcup_{a\le a_K}R^{-1}(S_0(a))=\bigcup_{a_L\le a_K}R^{-1}(S_L)\\&=\bigcup_{L\le K}R^{-1}(S_L)=\bigcup_{L\le K}\text{Kro}(L)\\&=\overline{\text{Kro}(K)}.\end{aligned}$$

The Schubert varieties are irreducible algebraic subvarieties of the projective variety $G_n(\mathbb{F}^{(n+1)m})$. Since the Kalman embedding is algebraic it follows that the Kronecker varieties $\overline{\text{Kro}(K)}$ are algebraic subvarieties of the quasi-projective variety $\Sigma_{n,m}(\mathbb{F})$. It seems interesting to study the singularities of the Kronecker varieties.

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