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On the embedding of 1-convex manifolds with 1-dimensional exceptional set

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Introduction

Let X be a 1-convex manifold and $S \subset X$ its exceptional set. X is called embeddable if there exists a holomorphic embedding of X into $\mathbb{C}^k \times \mathbb{P}^l$ for suitable $k, l \in \mathbb{N}$. When X has dimension 2 a result of C. Bănică [1], proved also by Vo Van Tan [13c], asserts that X is embeddable (in fact in this case we may allow X to have singularities).

The purpose of the present paper is to generalize this result to higher dimensions. We consider a 1-convex manifold X such that its exceptional set S is an irreducible curve. Under the assumption that S is not rational (i.e. its normalization is not $\mathbb{P}^1$) we prove that X is embeddable. A similar result holds if we assume that $S \cong \mathbb{P}^1$ and $\dim X \neq 3$ (see Theorem 5).

The technique of proof enables us to obtain also the following result:

If X is a complex manifold (not necessarily 1-convex) and $S \subset X$ is an irreducible exceptional curve with the above properties then the fundamental class of S in X does not vanish (see Theorem 6).

1. Preliminaries

Throughout this paper we shall not distinguish between holomorphic line bundles and invertible sheaves.

If X is a complex manifold and L is a holomorphic line bundle on X given by transition functions $\{g_{kl}\}$ corresponding to an open covering $\{U_k\}$ of X, a hermitian metric on L is a system $\{h_k\}$ of C^∞ functions $h_k : U_k \to (0, \infty)$ such that $h_k/h_l = |g_{kl}|^2$ on $U_k \cap U_l$.

L is said to be Nakano semipositive if there exists a hermitian metric $h = (h_k)$ on L such that $-\log h_k$ is plurisubharmonic on U_k for any k.

Let now X be a 1-convex manifold and $S \subset X$ its exceptional set. X is said to be embeddable if it can be realized as a closed analytic submanifold of some $\mathbb{C}^k \times \mathbb{P}^l$.

The following theorem of M. Schneider [12], proved also by Vo Van Tan [13a], gives sufficient and necessary conditions for a 1-convex manifold to be embeddable.

THEOREM 1. *Let X be a 1-convex manifold and $S \subset X$ its exceptional set. Then X is embeddable iff there exists a holomorphic line bundle L on X such that $L|_S$ is ample.*

If X is a complex manifold we denote by $K = K_X$ the canonical line bundle on X. In order to prove our results we shall need also the following "precise vanishing theorems":

THEOREM 2 [10] [13b]. *Let X be a 1-convex manifold with exceptional set S and let L be a holomorphic line bundle on X such that $L|_S$ is ample. Then $H^q(X, K \otimes L) = 0$ for $q \geq 1$.*

THEOREM 3 [5]. *Let X be a Kählerian manifold and L a Nakano semipositive line bundle on X. If $D \subset X$ is a relatively compact strongly pseudoconvex domain with smooth boundary then $H^q(D, K \otimes L) = 0$ for $q \geq 1$.*

2. Main results

DEFINITION. Let S be an irreducible curve and $\pi : \tilde{S} \to S$ its normalization. S is called a rational curve iff $\tilde{S} = \mathbb{P}^1$.

The following theorem explains us the behaviour of the canonical bundle in the neighbourhood of an exceptional irreducible curve.

THEOREM 4. *Let X be a 1-convex manifold and assume that its exceptional set S is an irreducible curve. Suppose that*:
a) *S is not a rational curve or*
b) *$S \cong \mathbb{P}^1$ and $\dim X \geq 4$*
Then $K|_S$ is ample.

The proof of Theorem 4 is based on several lemmas.

LEMMA 1. *Let X be a 1-convex manifold, $S \subset X$ its exceptional set and $k = \dim S$. Then for every $\mathcal{F} \in \mathrm{Coh}(X)$ it follows that $H^q(X, \mathcal{F}) = 0$ for $q > k$.*

Proof. By a theorem of Narasimhan [9] $H^q(X, \mathcal{F}) \cong H^q(S, \mathcal{F}|_S)$ for any $q > 0$.

Here $\mathscr{F}|_S$ denotes the topological restriction of $\mathscr{F}$ to S, hence $\mathscr{F}|_S$ is not a coherent sheaf on S. However, by a result of Reiffen [11 Satz 2] the cohomology groups $H^q(S, \mathscr{F}|_S)$ vanish for $q > k$ and the lemma is proved.

LEMMA 2. *Let X be a 1-convex manifold such that its exceptional set S is 1-dimensional. Then S has a Kählerian neighbourhood.*

A proof of this lemma can be found in [10 p. 165]. In fact it is shown that S has an embeddable neighbourhood.

If S is an irreducible curve we denote by $\pi : \tilde{S} \to S$ its normalization. There is an injective morphism of sheaves $\mathcal{O}_S \overset{i}{\hookrightarrow} \pi_*\mathcal{O}_{\tilde{S}}$ where $\pi_*\mathcal{O}_{\tilde{S}}$ is the 0-direct image of $\mathcal{O}_{\tilde{S}}$ (i.e. the sheaf of weakly holomorphic functions on S). Let $\mathbb{R}_S$ be the sheaf on S of locally constant real valued functions and similarly define $\mathbb{R}_{\tilde{S}}$ on $\tilde{S}$. If $\mathbb{R}_S \overset{j}{\hookrightarrow} \mathcal{O}_S$ is the natural inclusion map then $k = i \circ j$ is an injective morphism of sheaves. Let $k^* : H^1(S, \mathbb{R}_S) \to H^1(S, \pi_*\mathcal{O}_{\tilde{S}})$ denote the induced map on cohomology.

LEMMA 3. *The map k^* is surjective.*

Proof. Consider first the commutative diagram

$$\begin{array}{ccc} H^1(\tilde{S}, \mathbb{R}_{\tilde{S}}) & \xrightarrow{\alpha} & H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) \\ \gamma \uparrow & & \uparrow \delta \\ H^1(S, \pi_*\mathbb{R}_{\tilde{S}}) & \xrightarrow{\beta} & H^1(S, \pi_*\mathcal{O}_{\tilde{S}}) \end{array}$$

Remark that:

the map δ is bijective since $R^q\pi_*(\mathcal{O}_{\tilde{S}}) = 0$ for $q > 0$ (π is a finite morphism).

the map γ is bijective since $R^q\pi_*(\mathbb{R}_{\tilde{S}}) = 0$ for $q > 0$

(if $U \subset S$ is contractible it follows easily that $H^q(\pi^{-1}(U), \mathbb{R}_{\tilde{S}}) = 0$ for $q > 0$; since any point in S has a fundamental system of contractible open neighbourhoods we deduce that $R^q\pi_*(\mathbb{R}_{\tilde{S}}) = 0$ for $q > 0$).

the map α is bijective since $\tilde{S}$ is Kählerian.

It follows from the the commutativity of this diagram that β is bijective.

Consider now the commutative diagram:

$$\begin{array}{ccc} H^1(S, \pi_*\mathbb{R}_{\tilde{S}}) & \xrightarrow{\beta} & H^1(S, \pi_*\mathcal{O}_{\tilde{S}}) \\ v \uparrow & \nearrow k^* & \uparrow i^* \\ H^1(S, \mathbb{R}_S) & \xrightarrow[j^*]{} & H^1(S, \mathcal{O}_S) \end{array}$$

The map v is surjective because $\operatorname{supp}(\pi_* \mathbb{R}_{\tilde{S}}/\mathbb{R}_S)$ is a finite set. Hence k^* is surjective and Lemma 3 is proved.

LEMMA 4. *Let S be an irreducible curve and $\pi:\tilde{S} \to S$ its normalization. Let L be a holomorphic line bundle on S which is topologically trivial. Then there exists a holomorphic line bundle L' on S which can be given by constant transition functions $\{g_{kl}\}$ with $|g_{kl}| = 1$ and such that $\pi^*(L \otimes L')$ is the trivial line bundle on $\tilde{S}$.*

Proof. Let $\mathcal{U} = \{U_i\}$ be a finite open covering of S such that $L|_{U_i}$ is trivial and all intersections $U_{i_0} \cap \cdots \cap U_{i_r}$ are connected and contractible. Let $h_{kl} \in \mathcal{O}^*(U_k \cap U_l)$ denote the transition functions for L. Since L is topologically trivial and the covering $\mathcal{U}$ is topologically acyclic we can find holomorphic functions $\lambda_{kl} \in \mathcal{O}(U_k \cap U_l)$ such that $\exp(2\pi i \lambda_{kl}) = h_{kl}$ and $\lambda_{kl} + \lambda_{ls} + \lambda_{sk} = 0$ on $U_k \cap U_l \cap U_s$ for any k, l, s. Hence $\{\lambda_{kl}\}$ defines a cocycle in $Z^1(\mathcal{U}, \mathcal{O}_S)$. Set: $\hat{U}_i = \pi^{-1}(U_i)$, $\hat{\mathcal{U}} = \{\hat{U}_i\}$ and $\hat{\lambda}_{kl} = \lambda_{kl} \circ \pi \cdot \{\hat{\lambda}_{kl}\}$ is a cocycle in $Z^1(\mathcal{U}, \pi_* \mathcal{O}_{\tilde{S}})$. Consider now the commutative diagram:

$$\begin{array}{ccc} H^1(\mathcal{U}, \mathbb{R}_S) & \xrightarrow{p} & H^1(\mathcal{U}, \pi_* \mathcal{O}_{\tilde{S}}) \\ \downarrow & & \downarrow \\ H^1(S, \mathbb{R}_S) & \xrightarrow[k^*]{} & H^1(S, \pi_* \mathcal{O}_{\tilde{S}}) \end{array}$$

Note that:

the map k^* is surjective by Lemma 3

the map m is bijective because $\mathcal{U}$ is topologically acyclic

the map n is injective

It follows that p is surjective. This implies that one can find a cocycle $\{c_{kl}\} \in Z^1(\mathcal{U}, \mathbb{R}_S)$ and holomorphic functions $f_k \in \mathcal{O}(\hat{U}_k)$ such that $\hat{\lambda}_{kl} - f_k + f_l = c_{kl}$ on $\hat{U}_k \cap \hat{U}_l$ for any k, l.

If L' is the holomorphic line bundle on S with transition functions $g_{kl} = \exp(-2\pi i c_{kl})$ it follows from our construction that $\{\exp(2\pi i f_k)\}$ defines a nonvanishing section in $\pi^*(L \otimes L')$, hence $\pi^*(L \otimes L')$ is the trivial line bundle and Lemma 4 is completely proved.

LEMMA 5. *Let S be an irreducible curve and $\pi:\tilde{S} \to S$ its normalization. Suppose that there exists a holomophic line bundle L on S such that $H^1(S, L) = 0$ and π^*L is the trivial line bundle on $\tilde{S}$. Then S is a rational curve.*

Proof. There is a canonical morphism of sheaves $L \xrightarrow{\phi} \pi_* \pi^* L$. If we set $\mathcal{F}_1 = \ker \phi$ and $\mathcal{F}_2 = \operatorname{Im} \varphi$ we get an exact sequence

$$0 \to \mathcal{F}_1 \to L \to \mathcal{F}_2 \to 0$$

Since $H^1(S, L) = 0$ by hypothesis and $H^2(S, \mathscr{F}_1) = 0$ because $\dim S = 1$ it follows from the long exact sequence of cohomology that $H^1(S, \mathscr{F}_2) = 0$.

Consider now the exact sequence

$$0 \to \mathscr{F}_2 \to \pi_* \pi^* L \to \frac{\pi_* \pi^* L}{\mathscr{F}_2} \to 0$$

Since $\operatorname{supp}(\pi_* \pi^* L/\mathscr{F}_2)$ is a finite set it follows that $H^1(S, \pi_* \pi^* L/\mathscr{F}_2) = 0$, hence $H^1(S, \pi_* \pi^* L) = 0$. But $H^1(S, \pi_* \pi^* L) \cong H^1(\tilde{S}, \pi^* L)$ because π is a finite morphism. We deduce that $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$ and consequently $\tilde{S} \cong \mathbb{P}^1$, i.e. S is a rational curve. Lemma 5 is completely proved.

We are now in a position to prove Theorem 4.

a) Suppose first that S is an irreducible curve which is not rational. We prove that $K|_S$ is ample.

It is easy to verify that $H^2(S, \mathbb{Z}) \cong H^2(\tilde{S}, \mathbb{Z}) \cong \mathbb{Z}$ for any irreducible curve and if F is a holomorphic line bundle on S then F is ample iff $c(F)$ (the Chern class of F) corresponds under the above isomorphisms to a strictly positive integer. Consequently we have to prove that $c(K|_S) > 0$.

We remark first that $c(K|_S) \geq 0$. Indeed, if $c(K|_S) < 0$ then K^{-1} (the dual of K) is ample when restricted to S. By Theorem 2 we obtain $H^1(X, K \otimes K^{-1}) = 0$, hence $H^1(X, \mathcal{O}_X) = 0$. If $\mathscr{T}$ denotes the ideal sheaf of S there is an exact sequence of sheaves on X:

$$0 \to \mathscr{T} \to \mathcal{O}_X \to \mathcal{O}_X/\mathscr{T} \to 0$$

Since $H^1(X, \mathcal{O}_X) = 0$ and $H^2(X, \mathscr{T}) = 0$ (by Lemma 1) we deduce from the long exact sequence of cohomology that $H^1(S, \mathcal{O}_S) = 0$ which implies $S \cong \mathbb{P}^1$. This contradicts our assumption that S is not a rational curve. So we must have $c(K|_S) \geq 0$.

In order to prove Theorem 4 in case a) we have only to verify that $c(K|_S) \neq 0$.

Suppose that $c(K|_S) = 0$, hence $L := K|_S$ is topologically trivial. If $\pi : \tilde{S} \to S$ denotes the normalization of S from Lemma 4 there exists a holomorphic line bundle L' on S which can be given by constant transition functions $\{g_{kl}\}$ with $|g_{kl}| = 1$ and such that $\pi^*(L \otimes L')$ is the trivial line bundle on $\tilde{S}$.

By Lemma 2 S has an open neighbourhood U which is Kählerian and shrinking U if necessary we may assume that there exists a continuous retract $\rho : U \to S$. Let $S \subset U' \Subset U$ be a strongly pseudoconvex neighbourhood of S with smooth boundary and let $\mathscr{V} = \{V_i\}$ be an open covering of S such that L' is given on $V_k \cap V_l$ by the constants g_{kl} with $|g_{kl}| = 1$. Set $\tilde{V}_k := \rho^{-1}(V_k) \subset U$ and on $\tilde{V}_k \cap \tilde{V}_l$ consider the transition functions $\tilde{g}_{kl} := g_{kl}$. Since g_{kl} are constants it follows that the cocycle $\{\tilde{g}_{kl}\}$ defines a holomorphic line bundle $\tilde{L}'$ on U and $\tilde{L}'|_S = L'$. Moreover $\tilde{L}'$ is Nakano semipositive because $|\tilde{g}_{kl}| = 1$ for any k, l. From Theorem 3 of Grauert and Riemenschneider we get $H^1(U', K \otimes \tilde{L}') = 0$.

Now consider the exact sequence on U':

$$(*)\ 0 \to \mathcal{T} \to \mathcal{O}_{U'} \to \mathcal{O}_{U'}/\mathcal{T} \to 0$$

where $\mathcal{T}$ is the ideal sheaf of S. From $(*)$ we get the exact sequence on U':

$$(**)\ 0 \to K \otimes \tilde{L}' \otimes \mathcal{T} \to K \otimes \tilde{L}' \to K \otimes \tilde{L}' \otimes \mathcal{O}/\mathcal{T} \to 0.$$

By Lemma 1 $H^2(U', K \otimes \tilde{L}' \otimes \mathcal{T}) = 0$. Since $\tilde{L}'|_S = L'$ the long exact sequence of cohomology implies that $H^1(S, K|_S \otimes L') = 0$. But $\pi^*(K|_S \otimes L')$ is the trivial line bundle on $\tilde{S}$ and from Lemma 5 it follows that S is a rational curve which contradicts our hypothesis. Consequently a) is proved.

b) Assume that $S \cong \mathbb{P}^1$ and $n = \dim X \geq 4$. We shall prove that $K|_S$ is ample. Let $N_{S|X}$ denote the normal bundle of S in X and K_S the canonical line bundle of S. If we use the adjunction formula $K|_S = K_S \otimes \det(N^*_{S|X})$ we obtain the following formula for the Chern class of $K|_S$:

$$c(K|_S) = c(K_S) - c(\det(N_{S|X}))$$

Since $S \cong \mathbb{P}^1$ we have $c(K_S) = -2$. On the other hand a result of Laufer [6] gives the following estimation: $c(\det(N_{S|X})) \leq -n+1$. Hence we obtain $c(K|_S) \geq n-3 > 0$ and Theorem 4 is completely proved.

Remark. If $\dim X = 3$ and $S \cong \mathbb{P}^1$ it may happen that K is trivial in the neighbourhood of S. If $N_{S|X} = \mathcal{O}(c_1) \otimes \mathcal{O}(c_2)$, $c_1 \leq c_2$, is the decomposition of $N_{S|X}$ into line bundles and K is trivial in the neighbourhood of S then $(c_1, c_2) \in \{(-1,-1), (-2,0), (-3,1)\}$ (see Laufer [6]). Hence Theorem 4 does not hold if $\dim X = 3$ and $S \cong \mathbb{P}^1$. If $\dim X = 2$ and $S \cong \mathbb{P}^1$ easy examples show us that $K|_S$ may even be negative.

THEOREM 5. *Let X be a 1-convex manifold such that its exceptional set S is an irreducible curve. Assume that:*

a) *S is not a rational curve*

or

b) *$S \cong \mathbb{P}^1$ and* $\dim X \neq 3$.

Then X is embeddable.

Proof. In case a) it follows from Theorem 4 that $K|_S$ is ample. By Theorem 1 X is embeddable. A similar argument shows us that X is embeddable if $S \cong \mathbb{P}^1$

and $\dim X \geq 4$. If X has dimension 2 then S is a divisor and if we denote by $[S]$ the corresponding line bundle it follows that $[S]^{-1}$ (the dual of $[S]$) is ample when restricted to S. Again by Theorem 1 we deduce that X is embeddable.

Remark. It seems very likely that Theorem 5 should hold for any curve S.

Let now X be a complex manifold, $S \subset X$ an irreducible, compact curve and $\pi : \tilde{S} \to S$ its normalization. The image of the fundamental class of $\tilde{S}$ in $H_2(X, \mathbb{Z})$ is called the fundamental class of S in X. A straightforward consequence of Theorem 4 is the following topological result:

THEOREM 6. *Let X be a complex manifold and $S \subset X$ an irreducible exceptional curve such that:*
a) *S is not a rational curve*
or
b) *$S \cong \mathbb{P}^1$ and* $\dim X \neq 3$
Then the fundamental class of S in X does not vanish.

Remarks. i) In [13b] Vo Van Tan has proved that any 1-convex manifold with 1-dimensional exceptional set is Kählerian. Unfortunately, as we shall see, there is a gap in a main step of his proof.

According to his notations let $\pi : X \to Y$ be the Remmert reduction of X. We assume also that the exceptional set S is a smooth curve and let T be any point of S and set $Z := X \setminus T$, $\check{S} := S \setminus T$. If $\hat{E}$ is a holomorphic line bundle on Y we set $E := \pi^*(\hat{E})$ and $L := E|_Z$. The author asserts that if $\hat{E}$ is positive then there exists a metric $\{h_i\}$ on L such that:

$$(*) \quad \begin{cases} -\partial\bar{\partial} \log h_i(x) > 0 & \text{on } T_{\check{S},x} \\ -\partial\bar{\partial} \log h_i(x) \geq 0 & \text{on } N_{\check{S},x} \\ -\partial\bar{\partial} \log h_i(z) > 0 & \text{on } T_{Z,z} \text{ if } z \in Z \setminus \check{S} = X \setminus S \end{cases}$$

where $T_{\check{S},x}$ is the tangent space to $\check{S}$ at x and $N_{\check{S},x}$ is the complement space of $T_{\check{S},x}$ in $T_{Z,x}$.

We shall show that $(*)$ does not hold. We take $\hat{E}$ to be the trivial line bundle on Y which is positive since Y is Stein. It follows that L is also the trivial line bundle on Z and $(*)$ implies the existence of a C^∞ function $h : Z \to (0, \infty)$ such that $-\log h$ is strongly plurisubharmonic on $Z \setminus \check{S}$ and $-\log h|_{\check{S}}$ is strongly plurisubharmonic. Since $-\log h$ is strongly plurisubharmonic on $Z \setminus \check{S}$ it follows from the continuity of second derivatives that $-\log h$ is plurisubharmonic on Z. By a well known result concerning the extension of plurisubharmonic functions (see Grauert–Remmert [4]) there exists a plurisubharmonic function p on X such that $p|_Z = -\log h$. The maximum principle for plurisubharmonic functions implies that $p|_S = \text{constant}$, hence $-\log h|_{\check{S}} = \text{constant}$. This contradicts the fact that $-\log h|_{\check{S}}$ is strongly plurisubharmonic.

The gap in the proof of Vo Van Tan is the following: since $\check{S} := S \setminus T$ is Stein the metric $\{h_i\}$ can be suitably modified such that $L|_{\check{S}}$ is Nakano positive [8] but this can be done only on $\check{S}$ and there is no control outside $\check{S}$.

ii) Under the assumptions of Lemma 5 it follows that S is a rational curve with $\dim_{\mathbb{C}} H^1(S, \mathcal{O}_S) \leq 1$. This can easily be deduced from Riemann–Roch theorem for singular curves. Consequently all our theorems hold if we assume that S is a rational curve with $\dim_{\mathbb{C}} H^1(S, \mathcal{O}_S) \geq 2$.

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