Comment. Math. Helvetici 52 (1977) 561–572 Birkhäuser Verlag, Basel

Univalent functions and the Schwarzian derivative

F. W. GEHRING[1]

Dedicated to Professor A. Pfluger on his seventieth birthday

1. Introduction

This paper is concerned with the problem of extending to an arbitrary simply connected plane domain D the following two well known results relating the univalence of a function f analytic in the unit disk B with the magnitude of its Schwarzian derivative

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

THEOREM 1. *If f is analytic and univalent in B, then*

$$|S_f(z)| \leq 6(1-|z|^2)^{-2}$$

in B. The constant 6 is sharp.

THEOREM 2. *If f is analytic with*

$$|S_f(z)| \leq 2(1-|z|^2)^{-2}$$

in B, then f is univalent in B. The constant 2 is best possible.

Theorem 1 is due to Kraus [7] and Theorem 2 to Nehari [10].

Suppose next that D is a simply connected proper subdomain of the finite

[1] This research was supported in part by grants from the U.S. National Science Foundation, Grant MPS 7507940, and from the Samuel Neaman Fund, Special Year in Complex Analysis, Technion, Haifa, 1975–76.

complex plane $\mathbf{C}$. Then the hyperbolic metric in D is given by

$$\rho_D(z)=\frac{|g'(z)|}{1-|g(z)|^2},$$

where g is any conformal mapping of D onto B. The inequality

$$\tfrac{1}{4}\operatorname{dist}(z,\partial D)^{-1}\le\rho_D(z)\le\operatorname{dist}(z,\partial D)^{-1} \tag{1}$$

follows immediately from well known results due to Koebe and Schwarz. (See, for example, page 22 in [12].)

A Jordan curve γ in the extended complex plane $\bar{\mathbf{C}}$ is said to be a *K-quasiconformal circle*, $1\le K<\infty$, if there exists a K-quasiconformal mapping f of $\bar{\mathbf{C}}$ onto $\bar{\mathbf{C}}$ which maps the unit circle onto γ. The curve γ is said to be a quasiconformal circle if it is a K-quasiconformal circle for some K.

The following analogues of Theorems 1 and 2 for simply connected subdomains D of $\mathbf{C}$ are due to Lehto [8] and Ahlfors [1], respectively. See also [3].

THEOREM 3. *If f is analytic and univalent in D, then*

$$|S_f(z)|\le 12\rho_D(z)^2$$

in D. The constant 12 *is sharp.*

THEOREM 4. *Suppose that ∂D is a K-quasiconformal circle. Then there exists a positive constant a which depends only on K such that f is univalent in D whenever f is analytic with*

$$|S_f(z)|\le a\rho_D(z)^2 \tag{2}$$

in D.

Remark. Ahlfors actually proved more than the conclusion given above, namely that one can choose $a=a(K)$ so that f has a quasiconformal extension to $\bar{\mathbf{C}}$ whenever f is analytic and satisfies (2) in D.

In view of the above remark, it is natural to ask if the hypothesis that ∂D be a quasiconformal circle is necessary in Theorem 4. We shall show that this is indeed the case by establishing the following result.

THEOREM 5. *Suppose there exists a positive constant a such that f is* *univalent in D whenever f is analytic with*

$$|S_f(z)| \le a\rho_D(z)^2$$

in D. Then ∂D is a K-quasiconformal circle where K depends only on a.

2. Schwarzian univalence criterion

We obtain Theorem 5 as a corollary of an analogous result for proper subdomains D of $\mathbf{C}$ with arbitrary connectivity. For such domains D we have the following consequence of Theorem 1.

COROLLARY 1. *If f is analytic and univalent in D, then*

$$|S_f(z)| \le 6 \operatorname{dist}(z, \partial D)^{-2} \tag{3}$$

in D. The constant 6 is best possible.

Proof. Fix $z_0 \in D$, choose r so that $0 < r < \operatorname{dist}(z_0, \partial D)$ and let $g(z) = f(rz + z_0)$. Then g is analytic and univalent in B,

$$|S_f(z_0)| = |S_g(0)| r^{-2} \le 6r^{-2}$$

by Theorem 1, and we obtain (3) for $z = z_0$ by letting $r \to \operatorname{dist}(z_0, \partial D)$. There is equality in (3) when f is the Koebe function $z(1-z)^{-2}$, $D = B$ and $z = 0$.

Corollary 1 and inequality (1) suggest that $\operatorname{dist}(z, \partial D)^{-1}$ is a reasonable substitute for the hyperbolic metric $\rho_D(z)$ in the case where D is multiply connected.

DEFINITION. *Suppose that D is an arbitrary proper subdomain of $\mathbf{C}$. We say that D satisfies the Schwarzian univalence criterion if there exists a positive constant a such that f is univalent in D whenever f is analytic with*

$$|S_f(z)| \le a \operatorname{dist}(z, \partial D)^{-2}$$

in D.

The purpose of this paper is to establish the following result.

THEOREM 6. *If D satisfies the Schwarzian univalence criterion with constant a, then each component of ∂D is either a point or a K-quasiconformal circle where K depends only on a.*

Proof of Theorem 5. Suppose that D is a simply connected proper subdomain of $\mathbf{C}$ which satisfies the hypotheses of Theorem 5. Then by inequality (1), D satisfies the Schwarzian univalence criterion with constant $a/16$. Since ∂D is connected and contains at least two points, Theorem 6 implies that ∂D is a K-quasiconformal circle where K depends only on a.

COROLLARY 2. *Suppose that D is a simply connected proper subdomain of $\mathbf{C}$. Then D satisfies the Schwarzian univalence criterion if and only if ∂D is a quasiconformal circle.*

Proof. Theorem 4 and inequality (1) imply that D satisfies the Schwarzian univalence criterion whenever ∂D is a quasiconformal circle. The converse follows from Theorem 6.

3. Proof of Theorem 6

The proof of Theorem 6 depends on five lemmas given below. In what follows we let D denote an arbitrary domain in $\bar{\mathbf{C}}$, $B(z_0, r)$ the open disk with center $z_0 \in \mathbf{C}$ and radius $r \in (0, \infty)$, and b a constant in $(1, \infty)$. Next we say that two points z_1, z_2 can be *joined* in a set $E \subset \bar{\mathbf{C}}$ if there exists an arc $\alpha \subset E$ with z_1, z_2 as its endpoints. Finally for each set $E \subset \bar{\mathbf{C}}$ we let ∂E, $\bar{E}$ and $C(E)$ denote respectively the boundary, closure and complement of E in $\bar{\mathbf{C}}$.

LEMMA 1. *Suppose that for some z_0 and r there exist two points in $D \cap \bar{B}(z_0, r)$ which cannot be joined in $D \cap \bar{B}(z_0, br)$. Then there exist finite points z_1, z_2 in D and w_1, w_2 in $C(D)$ such that*

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

is analytic in D with

$$|h(z_1) - h(z_2) - 2\pi i| \le \frac{4}{b-1}. \tag{4}$$

Proof. By hypothesis there exist two points z_1', z_2' in $D \cap \bar{B}(z_0, r)$ which cannot be joined in $D \cap \bar{B}(z_0, br)$. Let α' denote the closed segment from z_1' to z_2'

and let $B_0 = B(z_0, br)$. Since $z_1', z_2' \in D$, there exists an open polygonal arc β' from z_2' to z_1' in D which meets α' in at most a finite set of points; when $z_1', z_2' \neq z_0$, we choose β' so that it lies in $D-\{z_0\}$. Then $\beta' - (\alpha' \cap \beta')$ is the union of a finite number of open subarcs β with endpoints in α'. Since z_1', z_2' cannot be joined in $D \cap \bar{B}_0$, we can choose a β whose endpoints cannot be joined in $D \cap \bar{B}_0$. Let z_1 and z_2 denote respectively the terminal and initial points of β, and let α denote the closed segment from z_1 to z_2. Note that $z_1, z_2 \neq z_0$ whenever $z_1', z_2' \neq z_0$.

We want next to find finite points $w_1, w_2 \in C(D)$ so that the function h is analytic in D and satisfies (4). Now z_1 and z_2 are separated in $\bar{B}_0$ by the closed set $C(D)$. Using Theorem VI.7.1 in [11] it is easy to show that z_1 and z_2 are separated in $\bar{B}_0$ by a component C_0 of $C(D)$. Let $D_0 = C(C_0)$. Then D_0 is a simply connected domain by Theorem IV.3.3 in [11], $D \subset D_0$, and the points z_1, z_2 cannot be joined in $D_0 \cap \bar{B}_0$. Hence by replacing D by D_0, we may assume without loss of generality that D is simply connected.

Now $\gamma = \alpha \cup \beta$ is a Jordan curve. Let D_1 and D_2 denote respectively the bounded and unbounded components of $C(\gamma)$. We shall show that there exist points w_1, w_2 such that

$$w_i \in C(D) \cap \partial B_0 \cap D_i \tag{5}$$

for $i = 1, 2$. Fix i. Since z_1, z_2 cannot be joined in $D \cap \bar{B}_0$, β and hence γ must meet ∂B_0 in at least two points. From Kerékjártó's theorem it follows that each component of

$$C(\gamma) \cap C(\partial B_0) = C(\gamma \cup \partial B_0)$$

is a Jordan domain, and hence that each component of $D_i \cap B_0$ is bounded by a Jordan curve. (See page 168 in [11].) Next since D_i is a Jordan domain and since $z_1 \in \partial D_i \cap B_0$, there exists a neighborhood U of z_1 such that points of $D_i \cap U$ can be joined in $D_i \cap B_0$. Hence $D_i \cap U$ is contained in a component D^* of $D_i \cap B_0$,

$$D^* \cap U = D_i \cap U, \tag{6}$$

and ∂D^* is a Jordan curve γ^*.

Choose $z \in \alpha - \{z_1\}$. Since α lies at a positive distance from ∂B_0, we can choose an open crosscut δ of D_i from z_1 to z which lies in B_0. Then (6) implies that $\delta \subset D^*$, that $z \in \gamma^*$, and hence that $\alpha \subset \gamma^*$. Thus $\beta^* = \gamma^* - \alpha$ is an open arc joining z_2 to z_1 in $\bar{B}_0$, and there exists a point

$$w_i \in \beta^* \cap C(D). \tag{7}$$

Since

$$\gamma^* \subset \partial(D_i \cap B_0) \subset \gamma \cup (\partial B_0 \cap D_i),$$

we have

$$\beta^* \subset \beta \cup (\partial B_0 \cap D_i) \subset D \cup (\partial B_0 \cap D_i), \tag{8}$$

and (5) follows from (7) and (8).

Since D is simply connected, we can define an analytic branch of

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

in D. Then

$$\begin{aligned} h(z_1) - h(z_2) &= \int_\beta \frac{dz}{z - w_1} - \int_\beta \frac{dz}{z - w_2} \\ &= 2\pi i(n(\gamma, w_1) - n(\gamma, w_2)) - \int_\alpha \frac{dz}{z - w_1} + \int_\alpha \frac{dz}{z - w_2}, \end{aligned}$$

where $n(\gamma, w_i)$ is the winding number of γ with respect to w_i. Since D_1 is the bounded component of $C(\gamma)$,

$$n(\gamma, w_1) = n = \pm 1, \qquad n(\gamma, w_2) = 0,$$

and we have

$$|h(z_1) - h(z_2) - 2n\pi i| \leq \int_\alpha \frac{|dz|}{|z - w_1|} + \int_\alpha \frac{|dz|}{|z - w_2|}. \tag{9}$$

(See [2].) Then

$$\int_\alpha \frac{|dz|}{|z - w_i|} \leq \frac{|z_1 - z_2|}{(b-1)r} \leq \frac{2}{b-1} \tag{10}$$

for $i = 1, 2$, and (4) follows from (9) and (10) when $n = 1$. When $n = -1$, we obtain (4) by interchanging w_1 and w_2.

LEMMA 2. *Suppose that for some z_0 and r there exist two points in $D - B(z_0, r)$ which cannot be joined in $D - B(z_0, r/b)$. Then the conclusion of Lemma* 1 *again holds.*

Proof. By hypothesis there exist two points z_1', z_2' in $D-B(z_0, r)$ which cannot be joined in $D-B(z_0, r/b)$; we may assume without loss of generality that $z_1', z_2' \neq \infty$. Next let Δ and ζ_i' denote the images of D and z_i' under

$$f(z)=\frac{1}{z-z_0}+z_0.$$

Then ζ_1', ζ_2' are points in $\Delta \cap \bar{B}(z_0, 1/r)$ which cannot be joined in $\Delta \cap \bar{B}(z_0, b/r)$. By the argument for Lemma 1, there exist finite points

$$\zeta_1, \zeta_2 \in \Delta-\{z_0\}, \qquad \omega_1, \omega_2 \in C(\Delta) \cap \partial B(z_0, b/r)$$

such that

$$g(\zeta)=\log\frac{\zeta-\omega_1}{\zeta-\omega_2}$$

is analytic in Δ with

$$|g(\zeta_1)-g(\zeta_2)-2\pi i| \leq \frac{4}{b-1}.$$

Let z_i, w_i denote the images of ζ_i, ω_i under f^{-1}. Then

$$h(z)=g\circ f(z)+\log\frac{z_0-w_1}{z_0-w_2}=\log\frac{z-w_1}{z-w_2}$$

is analytic in D and satisfies (4).

DEFINITION. *A set E in $\bar{\mathbf{C}}$ is said to be b-locally connected if for all z_0 and r, points in $E \cap \bar{B}(z_0, r)$ can be joined in $E \cap \bar{B}(z_0, br)$ and points in $E-B(z_0, r)$ can be joined in $E-B(z_0, r/b)$.*

See [5] and [6] for other applications of this concept.

LEMMA 3. *Suppose that D is a proper subdomain of $\mathbf{C}$. If D satisfies the Schwarzian univalence criterion for some constant a, then D is b-locally connected where*

$$b=\max\left(\frac{5}{a}+1, 3\right). \tag{11}$$

Proof. Suppose that D is not b-locally connected. Then there exist $z_0 \in \mathbf{C}$, $r \in (0, \infty)$ and two points in D for which the hypotheses of Lemma 1 or Lemma 2 hold. In either case, we obtain finite points z_1, $z_2 \in D$ and w_1, $w_2 \in C(D)$ such that

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

is analytic in D and satisfies (4). Since $b \geq 3$, inequality (4) implies that

$$|h(z_1) - h(z_2)| \geq 2\pi - \frac{4}{b-1} > 4. \tag{12}$$

Now set

$$f(z) = \exp(ch(z)), \qquad c = \frac{2\pi i}{h(z_1) - h(z_2)}.$$

Then f is analytic with

$$S_f(z) = \frac{1-c^2}{2}\left(\frac{1}{z-w_1} - \frac{1}{z-w_2}\right)^2$$

in D. Next (4), (11) and (12) imply that

$$2|1-c^2| < \frac{5}{b-1} \leq a,$$

and hence that

$$|S_f(z)| \leq 2|1-c^2| \operatorname{dist}(z, \partial D)^{-2} \leq a \operatorname{dist}(z, \partial D)^{-2}$$

in D. Since D satisfies the univalence criterion, it follows that f must be univalent in D. But

$$\frac{f(z_1)}{f(z_2)} = \exp(c(h(z_1) - h(z_2))) = 1,$$

and we have a contradiction.

LEMMA 4. *Suppose that D is b-locally connected and that ∂D is connected and contains at least two points. Then ∂D is a K-quasiconformal circle where K depends only on b.*

Proof. Suppose that p is a point in $\bar{D}$. With each neighborhood U of p we associate a second neighborhood V as follows. If $p = z_0 \in \mathbf{C}$, choose $r \in (0, \infty)$ so that $\bar{B}(z_0, br) \subset U$ and let $V = B(z_0, r)$; if $p = \infty$ choose $r \in (0, \infty)$ so that $C(B(0, r/b)) \subset U$ and let $V = C(\bar{B}(0, r))$. In each case, the fact that D is b-locally connected implies that points in $D \cap V$ can be joined in $D \cap U$. Thus D is uniformly locally connected and ∂D is a Jordan curve γ by Theorem VI.16.2 in [11].

We show next that for any pair of finite points $z_1, z_2 \in \gamma$,

$$\min(\text{dia}(\gamma_1), \text{dia}(\gamma_2)) \le b^2 |z_1 - z_2|, \tag{13}$$

where γ_1, γ_2 denote the components of $\gamma - \{z_1, z_2\}$. By a theorem of Ahlfors, inequality (13) will then imply that γ is a K-quasiconformal circle, where K depends only on b, thus completing the proof. (See, for example, Theorem II.8.6 in [9].)

To this end fix $z_1, z_2 \in \gamma$, set

$$z_0 = \tfrac{1}{2}(z_1 + z_2), \qquad r = \tfrac{1}{2}|z_1 - z_2|,$$

and suppose that (13) does not hold. Then there exist $t \in (r, \infty)$ and finite points w_1, w_2 such that

$$w_i \in \gamma_i - B(z_0, b^2 t) \tag{14}$$

for $i = 1, 2$. Choose $s \in (r, t)$. Since $z_1, z_2 \in \gamma \cap B(z_0, s)$, we can find for $i = 1, 2$ an endcut α_i of D joining z_i to $z_i' \in D$ in $\bar{B}(z_0, s)$. Next since D is b-locally connected, we can find an arc α_3 joining z_1' to z_2' in $D \cap \bar{B}(z_0, bs)$. Then $\alpha_1 \cup \alpha_2 \cup \alpha_3$ contains a crosscut α of D from z_1 to z_2 with

$$\alpha \subset \bar{B}(z_0, bs). \tag{15}$$

By virtue of (14), the same argument can be applied to obtain a crosscut β of D from w_1 to w_2 with

$$\beta \subset C(B(z_0, bt)). \tag{16}$$

But (15) and (16) imply that $\alpha \cap \beta = \varnothing$, contradicting the fact that z_1 and z_2 separate w_1 and w_2 in γ. Thus (13) holds and the proof of Lemma 4 is complete.

LEMMA 5. *Suppose that D is b-locally connected. Then each component of ∂D is either a point or a K-quasiconformal circle where K depends only on b.*

Proof. Let B_0 be a component of ∂D, let C_0 denote the component of $C(D)$ which contains B_0, and let $D_0 = C(C_0)$. Then D_0 is a domain with $\partial D_0 = B_0$. (See, for example, the proof of Theorem VI.16.3 in [11].) To complete the proof we need only show that D_0 is b-locally connected. For then by Lemma 4, ∂D_0 will be a point or a K-quasiconformal circle where $K = K(b)$.

Fix $z_0 \in \mathbf{C}$ and $r \in (0, \infty)$. Given z_1, $z_2 \in D_0 \cap \bar{B}(z_0, r)$ we must find an arc γ joining these points in $D_0 \cap \bar{B}(z_0, br)$. For this let α be any arc joining z_1 and z_2 in $\bar{B}(z_0, r)$. If $\alpha \subset D_0$, we may take $\gamma = \alpha$. Suppose that $\alpha \not\subset D_0$ and for $i = 1, 2$ let α_i denote the component of $\alpha \cap D_0$ which contains z_i. Then for each i there exists a point w_i such that

$$w_i \in \alpha_i \cap D. \tag{17}$$

If $z_i \in D$, we may take $w_i = z_i$; otherwise $z_i \in C_i$, a component of $C(D)$ different from C_0, and the fact that

$$\bar{\alpha}_i \cap C_0 \neq \varnothing, \qquad \alpha_i \cap C_i \neq \varnothing$$

implies that α_i must meet D and hence contain a point w_i satisfying (17). Since D is b-locally connected and since

$$w_1,\ w_2 \in \alpha \cap D \subset D \cap \bar{B}(z_0, r),$$

we can join w_1 and w_2 by an arc β in $D \cap \bar{B}(z_0, br)$. Then $\alpha_1 \cup \beta \cup \alpha_2$ will contain an arc γ joining z_1 and z_2 in $D_0 \cap \bar{B}(z_0, br)$.

Next the same argument shows that each pair of points in $D_0 - B(z_0, r)$ can be joined in $D_0 - B(z_0, r/b)$. Hence D_0 is b-locally connected and the proof is complete.

Proof of Theorem 6. Suppose that D is a proper subdomain of $\mathbf{C}$ which satisfies the Schwarzian univalence criterion with constant a. Lemma 3 implies that D is b-locally connected, where b is as in (11). Then Lemma 5 implies that each component of ∂D is either a point or a K-quasiconformal circle, where K depends only on b, and hence only on a.

4. Universal Teichmüller space

We conclude this paper with an application of Theorem 5 to Teichmüller theory.

Let $B_2 = B_2(L, 1)$ denote the Banach space of functions φ analytic in the lower half plane L with norm

$$\|\varphi\| = \sup_{z \in L} \rho_L(z)^{-2} |\varphi(z)| < \infty,$$

where $\rho_L(z) = \frac{1}{2}|y|^{-1}$ is the hyperbolic metric in L. Next let S denote the family of $\varphi = S_g$ where g is conformal in L, and let $T = T(1)$ denote the subfamily of those $\varphi = S_g$ for which g has a quasiconformal extension to $\bar{\mathbf{C}}$. From Theorem 1 it follows that $\|\varphi\| \leq 6$ for all $\varphi \in S$ and hence that $T \subset S \subset B_2$. The set T is the universal Teichmüller space. (See, for example, [4].)

Suppose that $\varphi \in \operatorname{int}(S)$. Then $\varphi = S_g$ where g maps L conformally onto a simply connected subdomain D of $\mathbf{C}$. In addition, there exists a constant $a > 0$ such that $\psi \in S$ whenever $\|\psi - \varphi\| \leq a$. If f is analytic with

$$|S_f(z)| \leq a \rho_D(z)^2$$

in D, then $\psi = S_{f \circ g}$ is analytic in L, $\|\psi - \varphi\| \leq a$, and hence f is univalent in D. Thus ∂D is a quasiconformal circle by Theorem 5, g has a quasiconformal extension to $\bar{\mathbf{C}}$, and $\varphi \in T$. Hence

$$\operatorname{int}(S) \subset T. \tag{18}$$

Next using the Remark following Theorem 4, Ahlfors showed in [1] that

$$T = \operatorname{int}(T). \tag{19}$$

Combining (18) and (19) we obtain the following result.

COROLLARY 3. *T is the interior of S in B_2.*

Unfortunately Corollary 3 neither implies nor is implied by the truth of the following interesting conjecture due to Bers. (See, for example, [4].)

CONJECTURE. *S is the closure of T in B_2.*

Lehto observed in [8] that one would settle the Bers conjecture in the negative if one could find a Jordan domain D and a positive constant a such that ∂D is not a quasiconformal circle and such that f has a quasiconformal extension to $\bar{\mathbf{C}}$

whenever f is analytic with

$$|S_f(z)| \leq a\rho_D(z)^2$$

in D. Theorem 5 shows, however, that no such domain D exists.

REFERENCES

[1] AHLFORS, L. V., Quasiconformal reflections, *Acta Math. 109* (1963) 291–301.
[2] ——, *Complex analysis*, McGraw-Hill, New York 1966.
[3] BERS, L., A non-standard integral equation with applications to quasiconformal mappings, *Acta Math. 116* (1966) 113–134.
[4] ——, Uniformization, moduli, and Kleinian groups, *Bull. London Math. Soc. 4* (1972) 257–300.
[5] GEHRING, F. W., Extension of quasiconformal mappings in three space, *J. d'Analyse Math. 14* (1965) 171–182.
[6] ——, Quasiconformal mappings of slit domains in three space, *J. Math. Mech. 18* (1969) 689–703.
[7] KRAUS, W., Über den Zusammenhang einiger Charakteristiken eines einfach zusammenhängenden Bereiches mit der Kreisabbildung, *Mitt. Math. Sem. Giessen 21* (1932) 1–28.
[8] LEHTO, O., Quasiconformal mappings in the plane, *Lecture Notes 14*, Univ. of Maryland 1975.
[9] LEHTO, O. and VIRTANEN, K. I., *Quasiconformal mappings in the plane*, Springer-Verlag, Berlin-Heidelberg-New York 1973.
[10] NEHARI, Z., The Schwarzian derivative and schlicht functions, *Bull. Amer. Math. Soc. 55* (1949) 545–551.
[11] NEWMAN, M. H. A., *Elements of the topology of plane sets of points*, Cambridge Univ. Press 1954.
[12] POMMERENKE, C., *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen 1975.

University of Michigan
Ann Arbor, Michigan

University of Paris
Orsay, France

Received April 26, 1977